Chapter 5: Maximum Satisfiability

SAT: For a given boolean formula φ in CNF, does there exist a truth assignment satisfying φ?

Conjunctive normal form (CNF): the formula is a conjunction ($\wedge$) of disjunctions ($\vee$). Each disjunction is called a clause.

Ex: $\varphi \equiv \underbrace{(x_1 \vee \overline{x}_2 \vee x_3)}_{\text{clause } C_1} \wedge \underbrace{\overline{x}_3}_{C_2} \wedge \underbrace{(x_1 \vee x_2)}_{C_3}$

positive literal

negative literal

x_1, x_2, x_3 are variables

C_j has length / size l_j:

$l_1 = 3, \; l_2 = 1, \; l_3 = 2$

$x_1 \leftarrow T, \; x_3 \leftarrow F$ will satisfy φ

MAX SAT

Input: Boolean formula φ in CNF
with variables $x_1, x_2, \dots, x_n$
and clauses $C_1, C_2, \dots, C_m$
Each clause, C_j, has a weight w_j

Output: Truth assignment maximizing the total weight of satisfied clauses

Ex: $(x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$w_1 = 2 \quad w_2 = 2 \quad w_3 = 1 \quad w_4 = 3$

$x_1 \leftarrow T$, $x_2 \leftarrow F$, $x_3 \leftarrow T$ satisfies C_1, C_2, C_4 with a total weight of 7.

This is optimal, since we cannot satisfy all clauses:

C_2 requires $x_3 \leftarrow T$
C_3 then requires $x_2 \leftarrow T$
C_1 then requires $x_1 \leftarrow T$
But then C_4 is false.

SAT, and hence, MAX SAT is NP-hard.
How can we approximate?

Section 5.1 : A simple randomized alg.

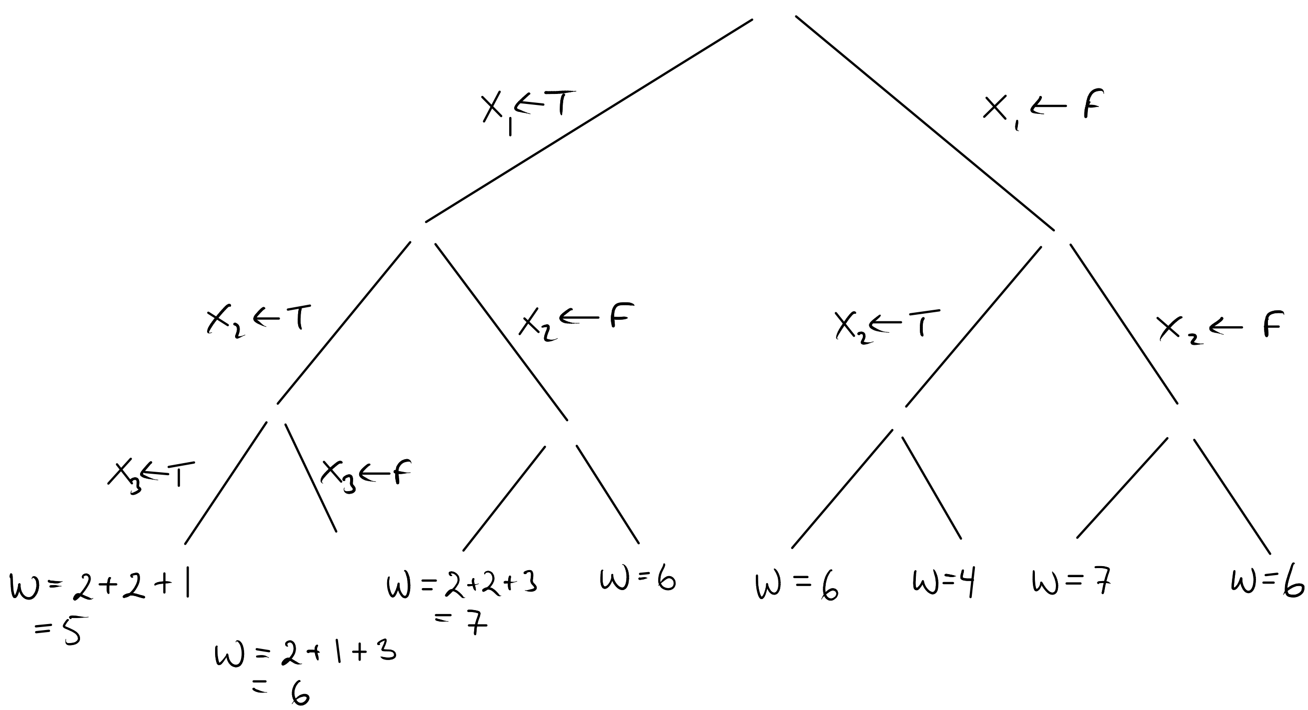
Consider the following alg:

Rand

For $i \leftarrow 1$ to n
  With prob. $1/2$ set x_i true

This corresponds to choosing a solution uniformly at random.

Ex: $(x_1 \vee \overline{x}_2) \wedge x_3 \wedge (x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$



Thus, for this example,

$$E[\text{Rand}] = \frac{1}{8}(5+6+7+6+6+4+7+6) = 5\tfrac{7}{8}$$

We don't need to calculate the weight of each possible output...

Instead, we can calculate the exp. weight of each clause:

$$(x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

satisfied, unless $x_1 \equiv F$ and $x_2 \equiv T$, i.e., satisfied with prob. $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

satisfied w. prob. $1 - \frac{1}{2} = \frac{1}{2}$

satisfied w. prob. $1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{8}$

Thus,

$$E[\text{Rand}] = \tfrac{3}{4} \cdot 2 + \tfrac{1}{2} \cdot 2 + \tfrac{3}{4} \cdot 1 + \tfrac{7}{8} \cdot 3 = 5\tfrac{7}{8}$$

In general, clause C_j is satisfied with prob. $1 - \left(\frac{1}{2}\right)^{l_j}$.

We let $W = \sum_{j=1}^{m} w_j$.

$\boxed{\text{Theorem 5.1: Rand is a } \tfrac{1}{2}\text{-approx. alg}}$

Proof:

$OPT \leq W$

By linearity of expectation:

$$E[Rand] = \sum_{j=1}^{m} \left(1 - \left(\tfrac{1}{2}\right)^{l_j}\right) w_j$$

$$\geq \tfrac{1}{2} W \quad , \text{ since } l_j \geq 1 \qquad \square$$

In Section 5.1 we got a simple algorithm with a guarantee on the expected performance. We can turn it into a guarantee on the worst-case performance:

Section 5.2: Derandomization

Ex from before:

$$\phi : (x_1 \vee \overline{x}_2) \wedge x_3 \wedge (x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$$

$$w_1 = 2 \qquad w_2 = 2 \qquad w_3 = 1 \qquad w_4 = 3$$

If we let $\boxed{x_1 \leftarrow T}$, the formula becomes

$$\phi_T : \overline{\top} \wedge x_3 \wedge (x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3)$$

and

$$E[Rand(\phi_T)] = 2 + \tfrac{1}{2} \cdot 2 + \tfrac{3}{4} \cdot 1 + \tfrac{3}{4} \cdot 3 = 6$$

Similarly, if we let $x_1 \leftarrow F$, the formula becomes

$$\phi_F : \overline{x}_2 \wedge x_3 \wedge (x_2 \vee \overline{x}_3) \wedge \top$$

and

$$E[Rand(\phi_F)] = \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 2 + \tfrac{3}{4} \cdot 1 + 3 = 5\tfrac{3}{4}$$

Note that $E[Rand]$ is the average of 6 and $5\frac{3}{4}$:

$$E[Rand] = \tfrac{1}{2} \cdot E[Rand(\phi_T)] + \tfrac{1}{2} \cdot E[Rand(\phi_F)]$$

Thus,

$$\max\{E[Rand(\phi_T)], E[Rand(\phi_F)]\} \geq E[Rand]$$

$$\phi: (x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

$$E[Rand(\phi)] = 5\tfrac{7}{8}$$

$x_1 \leftarrow T$ $\qquad$ $x_1 \leftarrow F$

$$\phi_T: T \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$$

$$E[Rand(\phi_T)] = 6$$

$$\phi_F: \bar{x}_2 \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge T$$

$$E[Rand(\phi_F)] = 5\tfrac{3}{4}$$

$x_2 \leftarrow T$ $\qquad$ $x_2 \leftarrow F$

$$\phi_{TT}: T \wedge x_3 \wedge T \wedge \bar{x}_3$$

$$E[Rand(\phi_{TT})] = 5\tfrac{1}{2}$$

$$\phi_{TF}: T \wedge x_3 \wedge \bar{x}_3 \wedge T$$

$$E[Rand(\phi_{TF})] = 6\tfrac{1}{2}$$

$x_3 \leftarrow T$ $\qquad$ $x_3 \leftarrow F$

$$\phi_{TFT}: T \wedge T \wedge F \wedge T$$

$$Rand(\phi_{TFT}) = 7$$

$$\phi_{TFF}: T \wedge F \wedge T \wedge T$$

$$Rand(\phi_{TFF}) = 6$$

In general:

$$\max\{E[Rand(\phi_T)],\ E[Rand(\phi_F)]\} \geq E[Rand] \geq \tfrac{1}{2}W,$$

The same is true for ϕ_T and ϕ_F:

$$\max\{E[Rand(\phi_{TT})],\ E[Rand(\phi_{TF})]\} \geq E[Rand(\phi_T)]$$

and

$$\max\{E[Rand(\phi_{FT})],\ E[Rand(\phi_{FF})]\} \geq E[Rand(\phi_F)]$$

Inductively, this proves that the following alg. is a $\frac{1}{2}$-approx. alg.:

DeRand(ϕ)

For $i \leftarrow 1$ to n
    If $E[Rand(\phi_{x_1 \cdots x_{i-1} T})] \geq E[Rand(\phi_{x_1 \cdots x_{i-1} F})]$
        $x_i \leftarrow T$
    Else
        $x_i \leftarrow F$

This method of derandomization is sometimes called the method of conditional expectations. (We calculate the conditional exp. of Rand given that $x_i \leftarrow T$ and given that $x_i \leftarrow F$.)

Note that short clauses are „harder" than long clauses:

If all clauses have $\ell \geq 2$, (De)Rand is a $3/4$-approx. alg.

(In Section 5.3, we will pursue the obs. to obtain a ≈ 0.6-approx. alg.)

If all clauses have $\ell \geq 3$, (De)Rand is a $7/8$-approx alg.

In some sense, this is optimal:

MAX E3SAT: The special case of MAX SAT where $\ell = 3$ for all clauses.

Theorem 5.2:

$\exists\, \varepsilon > 0 : \exists\, (\frac{7}{8}+\varepsilon)$-approx alg for MAX E3SAT $\Rightarrow$ $P = NP$

Section 5.3: A biased rand. alg.

Since unit clauses (clauses of exactly one literal) are the „hardest", we should focus on these to obtain a better approx. ratio.

For each i, $1 \le i \le n$, we define:

$$u_i = \begin{cases} \text{weight of unit clause } x_i, & \text{if it exists} \\ 0, & \text{otherwise} \end{cases}$$

$$v_i = \begin{cases} \text{weight of unit clause } \bar{x}_i, & \text{if it exists} \\ 0, & \text{otherwise} \end{cases}$$

Idea: If $u_i \ge v_i$, set x_i true with prob. $> \frac{1}{2}$, and vice versa.

For ease of presentation, assume that

$$u_i \ge v_i, \quad 1 \le i \le n$$

Why is this not a restriction?

Thus, each variable will be set true with prob. $> \frac{1}{2}$:

For any $p > \frac{1}{2}$, we define the following alg:

Rand$_p$

For $i \leftarrow 1$ to n
    With prob. p set x_i true

What is an optimal value of p?

Lemma 5.4

For any clause C_j which does not consist of one negated variable,
Rand$_p$ satisfies C_j with prob $\geq \min\{p, 1-p^2\}$

Proof:

If $l_j = 1$, C_j consists of one unnegated variable.
In this case, C_j is satisfied with prob. p.

If $l_j = 2$, the worst case is if both literals are negated variables, since $p > \frac{1}{2}$. Thus, in this case, C_j is satisfied with prob. $\geq 1-p^2$.

If $l_j \geq 3$, the prob. of C_j being satisfied is at least the worst-case prob. for $l_j = 2$. $\square$

Lemma 5.6: $OPT \leq W - \sum_{i=1}^{n} v_i$

<u>Proof</u>:
By assumption, $u_i \geq v_i$, for all i.
Thus, if $v_i > 0$, there is both an x_i- and an $\bar{x}_i$-clause. Both clauses cannot be satisfied.
Thus, for each $v_i > 0$, there is an unsatisfied clause of weight $\geq v_i$.

$\square$

We can obtain an alg. with approx. ratio $\frac{1}{2}(\sqrt{5}-1) \approx 0{,}618$:

Theorem 5.7:

For $p = \frac{1}{2}(\sqrt{5}-1)$, Rand_p is a p-approx. alg.

Proof:

By Lemma 5.4,

$$\text{Rand}_p \geq \min\{p, 1-p^2\} \cdot \left(W - \sum_{i=1}^{n} v_i\right)$$

$$= p\left(W - \sum_{i=1}^{n} v_i\right), \text{ for } p = \tfrac{1}{2}(\sqrt{5}-1):$$

$$1 - \left(\tfrac{1}{2}(\sqrt{5}-1)\right)^2 = 1 - \tfrac{1}{4}(5+1-2\sqrt{5}) = 1 - \tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}$$

$$= \tfrac{1}{2}(\sqrt{5}-1)$$

By Lemma 5.6, $\text{OPT} \leq W - \sum_{i=1}^{n} v_i$.

Hence, for $p = \frac{1}{2}(\sqrt{5}-1)$, $\dfrac{\text{Rand}}{\text{OPT}} \geq p$. $\square$

Note that Rand_p can be derandomized exactly like Rand.