Vehicle Routing

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Problem Definition

Vehicle Routing: distribution of goods between depots and customers.

Delivery, collection, transportation.

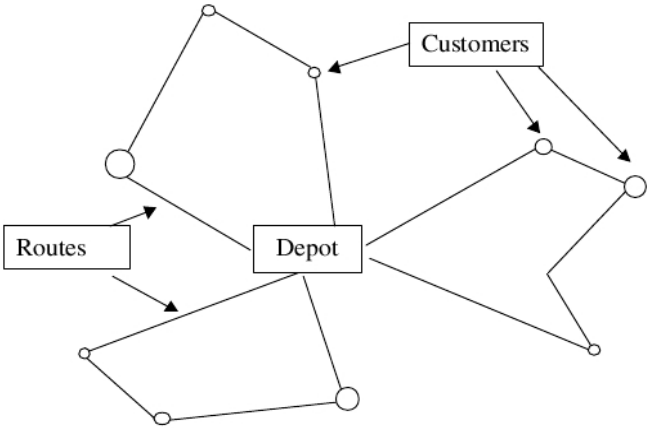
Examples: solid waste collection, street cleaning, school bus routing, dial-a-ride systems, transportation of handicapped persons, routing of salespeople and maintenance unit.

Vehicle Routing Problems

Input: Vehicles, depots, road network, costs and customers requirements.

Output: Set of routes such that:

- requirement of customers are fulfilled,
- operational constraints are satisfied and
- a global transportation cost is minimized.


Customers
Routes
Depot

Refinement

Road Network

- represented by a (directed or undirected) complete graph
- travel costs and travel times on the arcs obtained by shortest paths

Customers

- vertices of the graph
- collection or delivery demands
- time windows for service
- service time
- subset of vehicles that can serve them
- priority (if not obligatory visit)

Vehicles

- capacity
- types of goods
- subsets of arcs traversable
- fix costs associated to the use of a vehicle
- distance dependent costs
- a-priori partition of customers
- home depot in multi-depot systems
- drivers with union contracts

Operational Constraints

- vehicle capacity
- delivery or collection
- time windows
- working periods of the vehicle drivers

Objectives

- minimization of global transportation cost (variable + fixed costs)
- minimization of the number of vehicles
- balancing of the routes
- minimization of penalties for un-served customers

History:
Dantzig, Ramser "The truck dispatching problem", Management Science, 1959
Clark, Wright, "Scheduling of vehicles from a central depot to a number of delivery points". Operation Research. 1964

Vehicle Routing Problems

- Capacitated (and Distance Constrained) VRP (CVRP and DCVRP)
- VRP with Time Windows (VRPTW)
- VRP with Backhauls (VRPB)
- VRP with Pickup and Delivery (VRPPD)
- Periodic VRP (PVRP)
- Multiple Depot VRP (MDVRP)
- Split Delivery VRP (SDVRP)
- VRP with Satellite Facilities (VRPSF)
- Site Dependent VRP
- Open VRP
- Stochastic VRP (SVRP)
- ...

Capacitated Vehicle Routing (CVRP)

Input: (common to all VRPs)

- (di)graph (strongly connected, typically complete) $G(V, A)$, where $V = \{0, \ldots, n\}$ is a vertex set:
  - 0 is the depot.
  - $V' = V \setminus \{0\}$ is the set of n customers
  - $A = \{(i, j) : i, j \in V\}$ is a set of arcs
- C a matrix of non-negative costs or distances c_{ij} between customers i and j (shortest path or Euclidean distance)
  $(c_{ik} + c_{kj} \geq c_{ij} \quad \forall\, i, j \in V)$
- a non-negative vector of costumer demands d_i
- a set of K (identical!) vehicles with capacity Q, $d_i \leq Q$

Task:
Find collection of K circuits with minimum cost, defined as the sum of the costs of the arcs of the circuits and such that:

- each circuit visits the depot vertex
- each customer vertex is visited by exactly one circuit; and
- the sum of the demands of the vertices visited by a circuit does not exceed the vehicle capacity Q.

Note: lower bound on K

- $\lceil d(V')/Q \rceil$
- number of bins in the associated *Bin Packing Problem*

A feasible solution is composed of:

- a partition $R_1, \ldots, R_m$ of V;
- a permutation π^i of $R_i \bigcup \{0\}$ specifying the order of the customers on route i.

A route R_i is feasible if $\sum_{i=\pi_1}^{\pi_m} d_i \leq Q$.

The cost of a given route (R_i) is given by: $F(R_i) = \sum_{j=\pi_0^i}^{\pi_m^i} c_{j,j+1}$

The cost of the problem solution is: $F_{VRP} = \sum_{i=1}^{m} F(R_i)$.

Relation with TSP

- VRP with $K = 1$, no limits, no (any) depot, customers with no demand ➜ TSP
- VRP is a generalization of the Traveling Salesman Problem (TSP) ➜ is NP-Hard.
- VRP with a depot, K vehicles with no limits, customers with no demand ➜ Multiple TSP = one origin and K salesman
- Multiple TSP is transformable in a TSP by adding K identical copies of the origin and making costs between copies infinite.

Variants of CVRP:

- minimize number of vehicles
- different vehicles Q_k, $k = 1, \dots, K$
- Distance-Constrained VRP: length t_{ij} on arcs and total duration of a route cannot exceed T associated with each vehicle
  Generally $c_{ij} = t_{ij}$
  (Service times s_i can be added to the travel times of the arcs: $t'_{ij} = t_{ij} + s_i/2 + s_j/2$)
- Distance constrained CVRP

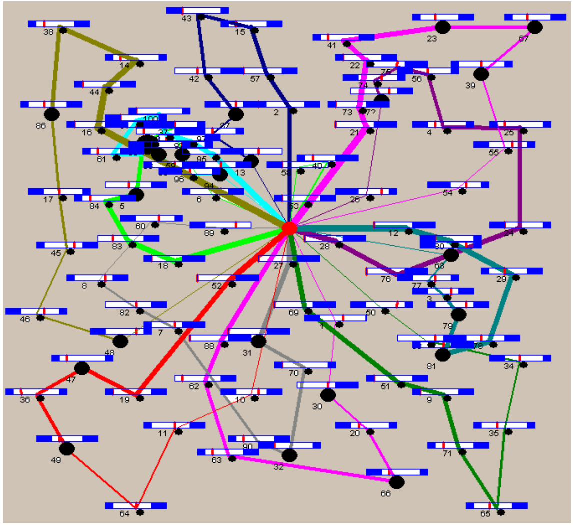
Vehicle Routing with Time Windows (VRPTW)

Further Input:

- each vertex is also associated with a time interval $[a_i, b_j]$.
- each arc is associated with a travel time t_{ij}
- each vertex is associated with a service time s_i

Task:
Find a collection of K simple circuits with minimum cost, such that:

- each circuit visit the depot vertex
- each customer vertex is visited by exactly one circuit; and
- the sum of the demands of the vertices visited by a circuit does not exceed the vehicle capacity Q.
- for each customer i, the service starts within the time windows $[a_i, b_i]$ (it is allowed to wait until a_i if early arrive)



Time windows induce an orientation of the routes.

Variants

- Minimize number of routes
- Minimize hierarchical objective function
- Makespan VRP with Time Windows (MPTW)
  minimizing the completion time
- Delivery Man Problem with Time Windows (DMPTW)
  minimizing the sum of customers waiting times

Solution Techniques for CVRP

- Integer Programming
- Construction Heuristics
- Local Search
- Metaheuristics
- Hybridization with Constraint Programming

Outline

Basic Models

- arc flow formulation

  integer variables on the edges counting the number of time it is traversed
  one, two or three index variables

- set partitioning formulation

- multi-commodity network flow formulation for VRPTW

  integer variables representing the flow of commodities along the paths traveled by the vehicles and
  integer variables representing times

Two index arc flow formulation

$$\min \quad \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \qquad (1)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ij} = 1 \qquad \forall j \in V \setminus \{0\} \quad (2)$$

$$\sum_{j \in V} x_{ij} = 1 \qquad \forall i \in V \setminus \{0\} \quad (3)$$

$$\sum_{i \in V} x_{i0} = K \qquad (4)$$

$$\sum_{j \in V} x_{0j} = K \qquad (5)$$

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq r(S) \qquad \forall S \subseteq V \setminus \{0\}, S \neq \emptyset \quad (6)$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in V \quad (7)$$

$r(S)$ minimum number of vehicles needed to serve set S
(6): capacity-cut constraints

One index arc flow formulation

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e && (8)\\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 2 && \forall i \in V \setminus \{0\} \quad (9)\\
& \sum_{e \in \delta(0)} x_e = 2K && (10)\\
& \sum_{e \in \delta(S)} x_e \geq 2r(S) && \forall S \subseteq V \setminus \{0\}, S \neq \emptyset (11)\\
& x_e \in \{0,1\} && \forall e \notin \delta(0) (12)\\
& x_e \in \{0,1,2\} && \forall e \in \delta(0) (13)
\end{aligned}
$$

$r(S)$ minimum number of vehicles needed to serve set S
$x_e = 2$ if we allow single visit routes

Three index arc flow formulation

$$\min \quad \sum_{i \in V} \sum_{j \in V} c_{ij} \sum_{k=1}^{K} x_{ijk} \tag{14}$$

$$\text{s.t.} \quad \sum_{k=1}^{K} y_{ik} = 1 \qquad \forall i \in V \setminus \{0\} \tag{15}$$

$$\sum_{k=1}^{K} y_{0k} = K \tag{16}$$

$$\sum_{j \in V} x_{ijk} = \sum_{j \in V} x_{jik} = y_{ik} \qquad \forall i \in V, k = 1, \dots, K \tag{17}$$

$$\sum_{i \in V} d_i y_{ik} \leq C \qquad \forall k = 1, \dots, K \tag{18}$$

$$\sum_{i \in S} \sum_{j \notin S} x_{ijk} \geq y_{hk} \qquad \forall S \subseteq V \setminus \{0\}, h \in S, k = 1, \dots, K \tag{19}$$

$$y_{ik} \in \{0, 1\} \qquad \forall i \in V, k = 1, \dots, K \tag{20}$$

$$x_{ijk} \in \{0, 1\} \qquad \forall i, j \in V, k = 1, \dots, K \tag{21}$$

Set Partitioning Formulation

$\mathcal{R} = \{1, 2, \ldots, R\}$ index set of routes

$$a_{ir} = \begin{cases} 1 & \text{if costumer } i \text{ is served by } r \\ 0 & \text{otherwise} \end{cases}$$

$$x_r = \begin{cases} 1 & \text{if route } r \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$\min \quad \sum_{r \in \mathcal{R}} c_r x_r \tag{31}$$

$$\text{s.t.} \quad \sum_{r \in \mathcal{R}} a_{ir} x_r = 1 \qquad \forall i \in V \tag{32}$$

$$\sum_{r \in \mathcal{R}} x_r \leq K \tag{33}$$

$$x_r \in \{0, 1\} \qquad \forall r \in \mathcal{R} \tag{34}$$

What can we do with these integer programs?

- plug them into a commercial solver and try to solve them
- preprocess them
- determine lower bounds
  - solve the linear relaxation
  - combinatorial relaxations
    relax some constraints and get an easy solvable problem
  - Lagrangian relaxation
  - polyhedral study to tighten the formulations
- upper bounds via heuristics
- branch and bound
- cutting plane
- branch and cut
- Dantzig Wolfe decomposition
- column generation (via reformulation)
- branch and price