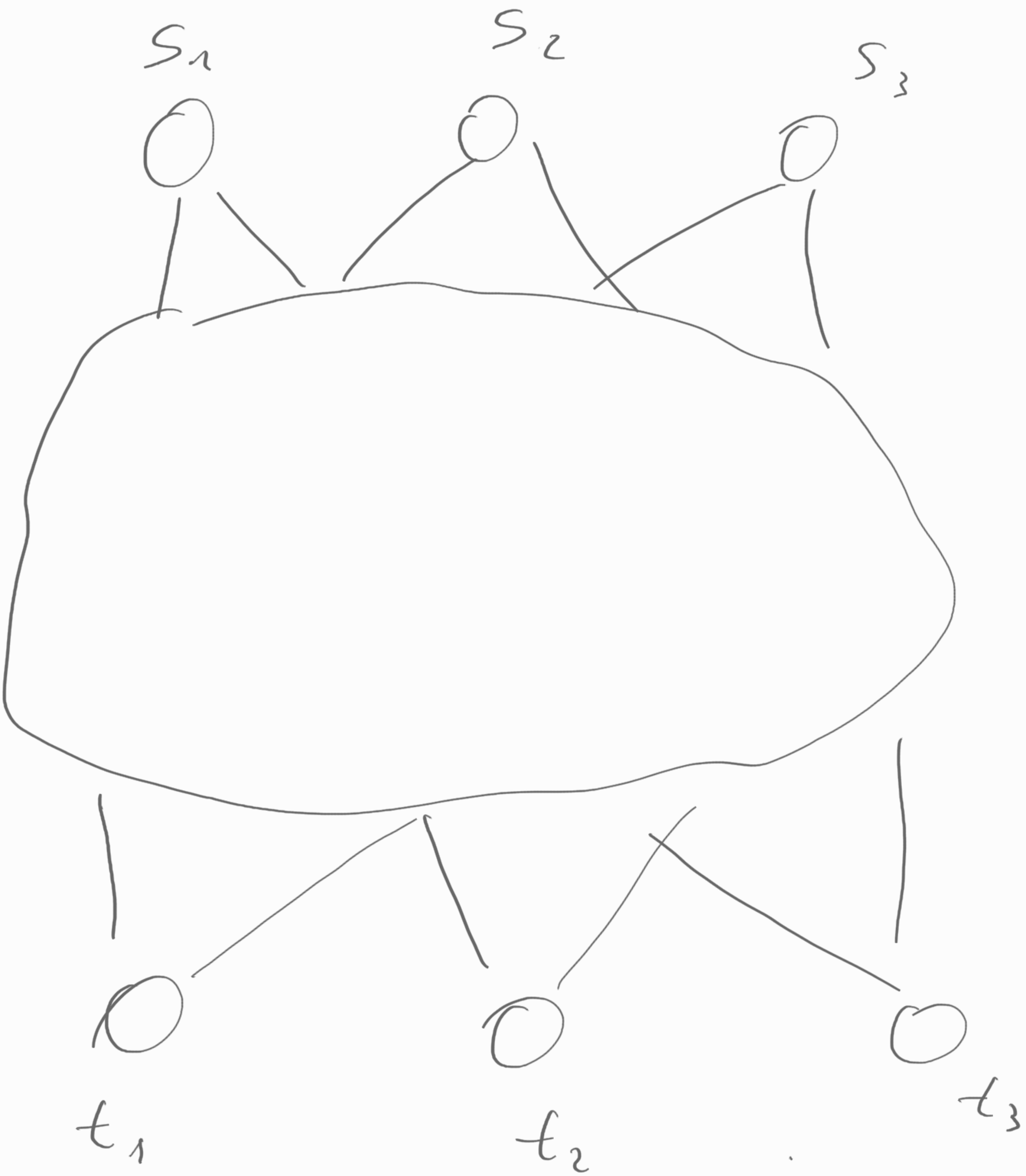
MULTICOMMODITY NETWORK FLOW



$$\min \; c^T x$$

$$N x = b$$

$$0 \le x \le u$$

$$\min \sum c^{kT} x^k$$

$$z = \min \sum_k c^k x^k$$

$$N x^k = b \qquad \forall\, k = 1..3$$

$$\sum_k x_{ij}^k \le u_{ij} \qquad \forall\, k = 1..3 \quad ij \in A$$

$$0 \le x_{ij}^k \le u_{ij}^k \qquad \forall\, ij \in A \quad k = 1..3$$

(x_{ij}^k is integral)

$$\sum_k x_{ij}^k + s_{ij} = u_{ij}$$

Motivations for Lagrangian Relaxation

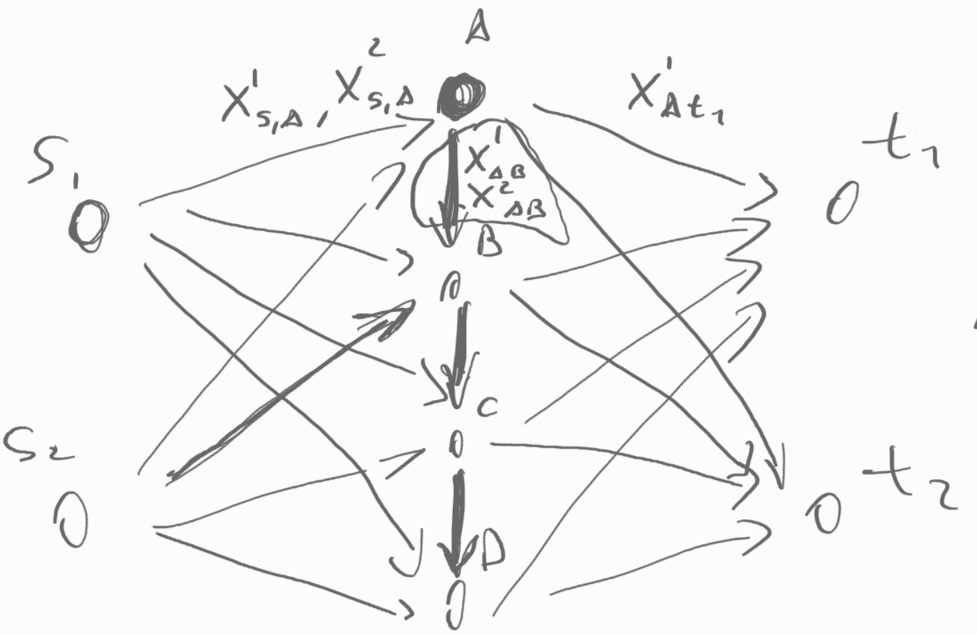
Divisible goods $\Rightarrow$ Linear Progr Problem

- Solve the problem without simplex and possibly faster

Indivisible goods $\Rightarrow$ Integer Lin Prog. Problem

- As a bounding procedure in B2B
- To heuristic solutions

Application example: Warehouse problem



$$N = \begin{array}{c} s_1A \quad s_1B \;\ldots \\ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$A: \quad X^1_{S_2A} + X^1_{S_1A} - X_{AB} - X_{At_1} - X_{At_2} = 0$

$AB: \quad X^1_{AB} + X^2_{AB} \le u_{AB}$

$X^1_{S_2A} + X^2_{S_2A} \le u_{S_1A}$

- $LR(\lambda)$

Lagrangian Relaxation problem (on x vars)

$$z_{LR1} = \min_x \; d^T x^k$$

$$\sum_k x^k_{ij} \le u_{ij}$$

$$0 \le x^k_{ij} \le u^k_{ij}$$

$$z_{LR2} = \min_x \; e^T x^k$$

$$N x^k = b$$

$$0 \le x^k_{ij} \le u^k_{ij}$$

Two alternative relaxations. The second one leads to k shortest paths problems and hence seems easier to solve

$\lambda \ge 0$

$$c^T x^k + \lambda\left(\sum x^k - u\right)$$

$$x^k \underbrace{(c^T + \lambda)}_{e^T} - \underbrace{\lambda u}_{\text{const}}$$

- LD

$$z_{LD} = \max_{\lambda \ge 0} LR(\lambda)$$

Lagrangian Dual Problem (on λ vars)

Relations between z, z_{LR}, z_{LD}

z

$z_{LR} \le z_{LD}$ (since it is the max of $z_{LR}(\lambda)$)

z_{LD}

$z_{LR}(\lambda)$

$z_{LD} \leq z$

Proof:

Assume x^* optimal sol for the original problem. Then

$\sum_u (x^*_{ij})^u \leq u_{ij}$

and

$z_{LR}(\lambda) = c^T(x^*)^u + \underset{\geq 0}{\lambda}\, \Big(\underbrace{\textstyle\sum (x^*)^u - u}_{\leq 0}\Big) \leq c^T(x^*)^u = z$

So $z_{LR}(\lambda) \leq z$ for all $\lambda \geq 0$ and hence also

$z_{LD} \leq z$

The relationship above holds always.

In some cases $z_{LD} = z$. Two of these cases are:

1) when the original problem is an LP problem

2) when the original problem is an ILP problem and the Lagrangian relaxation problem has the integrality property.

Proof of 1) (AMO Th 16.6)

Assume x^* is opt sol for original problem

Opt conditions: complementary slackness th.

$$\pi^* \left(\sum_k x^{*^k} - u \right) = 0$$

dual vars associated with linking constr.

Consider $LR(\lambda)$ with $\lambda = \pi^*$. x^* is feasible in LR.

$$z_{LR}(\pi^*) = c^T x^* + \pi^* \left(\sum_k x^{*^k} - u \right) = c^T x^* = z.$$

Proof of 2)

A consequence of 1).