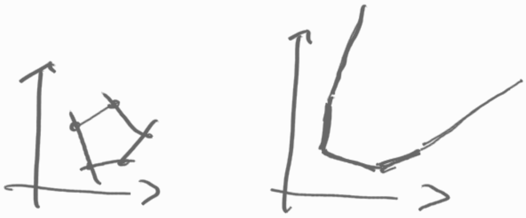
Background to Benders' Algorithm

- Extreme points and extreme rays [DJ, p 36]

$$\min c^T x$$
$$A_1 x \leq b_1$$
$$A_2 x \leq b_2$$
$$x \geq 0$$

$A_2 x \leq b_2$ ⟶ in general it is a polyhedron, which can be <u>bounded</u> (and hence a polytope) or <u>unbounded</u>



In Dantzig Wolfe dec. we rewrite the problem as:

$$\min c^T x$$
$$A_1 x \leq b_1$$
$$x \in X$$

where $X = \{x \mid A_2 x \leq b_2\}$

If X was bounded we have seen that it can be written as a convex combination of its extreme points P.

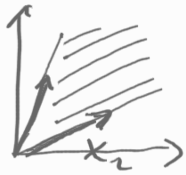
$$X = \{x \mid x = \sum \lambda x \ldots \mid \ldots \sum \lambda = 1\}$$

$$P = \{x \mid x = \sum_{p \in P} \lambda_p x_p,\ \lambda_p \geq 0,\ \sum_{p \in P} \lambda_p = 1\}$$

If we want to account also for the possibility that $A_2 x = b_2$ is unbounded then we need the following:

A <u>convex cone</u> is $\{x \mid Bx \leq 0\}$ that is the intersection of many halfspaces

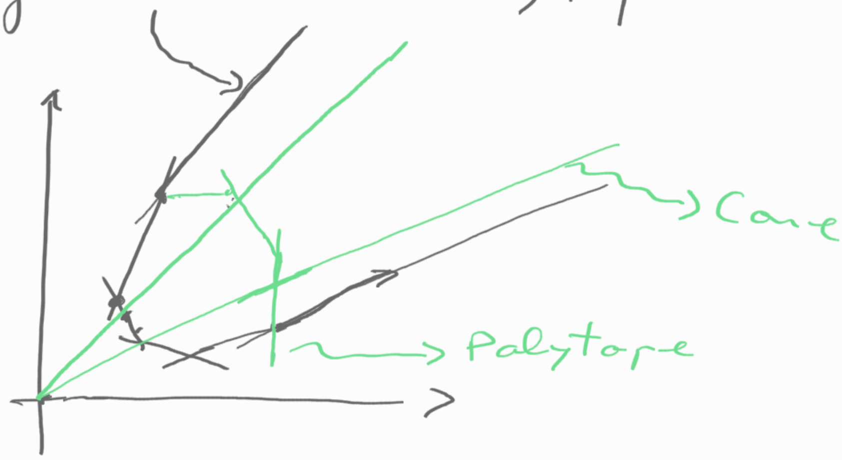
A convex cone is also described by the conic combination of its extreme rays R:



$$C = \{x \mid x = \sum_{r \in R} \delta_r x_r,\ \delta_r \geq 0\}.$$

A result from polyhedral analysis states that:

Polyhedron = Polytope + Cone



Hence, a point of a polyhedron can be described as:

$$X = \{ x \mid x = \sum_{p \in P} \lambda_p x_p + \sum_{r \in R} \delta_r x_r ,$$
$$\lambda_p \geq 0 ,$$
$$\sum_{p \in P} \lambda_p = 1 ,$$
$$\delta_r \geq 0 \}$$

Hence in Dantzig Wolfe decomposition the substitution would be

$$\begin{array}{ll} \min c^T x & \min c^T \sum_{p \in P} \lambda_p x_p + c^T \sum_{r \in R} \delta_r x_r \\ A_1 x \leq b_1 & A_1 \left(\sum_{p \in P} \lambda_p x_p + \sum_{r \in R} \delta_r x_r \right) \leq b_1 \\ x \in X & \sum_{p \in P} \lambda_p = 1 \\ & \lambda_p \geq 0 \quad \forall p \in P \\ & \delta_r \geq 0 \quad \forall r \in R \end{array}$$

We previously ignored the rays and can continue to do so if the pricing problem is bounded and feasible.

- How do we find the extreme rays? From the simplex:

$$\max \quad x_1 + x_2$$
$$-2x_1 + x_2 + x_3 = 1$$

$$x_1 - x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

x_1	x_2	x_3	x_4		
-2	1	1	0	0	1
1	-1	0	1	0	1
1	1	0	0	1	0

$\Rightarrow$

x_1	x_2	x_3	x_4		
-2	1	1	0	0	1
-1	0	1	1	0	2
$+3$	0	-1	0	1	1

basic sol. $[0, 1, 0, 2]$. Trying to bring x_1 in basis unveals that we can increase x_1 arbitrarily

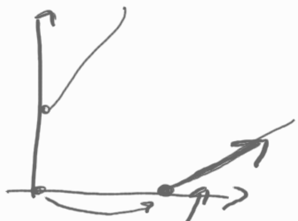
Hence:

$$x_2 = 1 + 2x_1 (-x_3)$$

$$x_4 = 2 + x_1 (-x_3)$$

not in basis stays zero

$$x = \begin{bmatrix} k \\ 1 + 2k \\ 0 \\ 2 + k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + k \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$



extreme ray

Hence, we find it with the simplex and it is returned by solvers when the problem is unbounded.

Note: if a problem is infeasible its dual is unbounded and the extreme ray is a proof of infeasibility for the primal (Farkas Lemma)

- <u>Farkas Lemma</u>

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then:

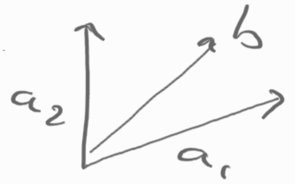
either : $\exists x \in \mathbb{R}^n : Ax \leq b$ and $x \geq 0$

or : $\exists u \in \mathbb{R}^m : u^T A \geq 0$ and $u^T b < 0, u \geq 0$

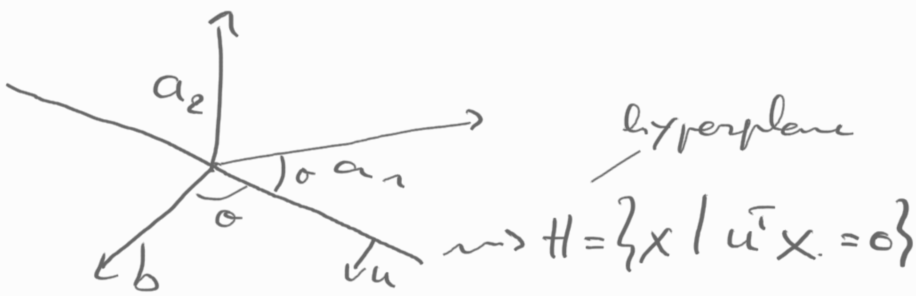
(Consult in case wikipedia for variants)

Meaning:

either the system $Ax \geq b$ admits a solution $x \geq 0$, that is, b can be attained as a linear combination of the column vectors of A :



or the system $Ax \geq b$ does not admit a solution and there is a certificate of this fact: an hyperplane described by $u \geq 0$ that separates b from A :



$u^T b = \|u\| \|b\| \cos\theta$

$u^T A = [\|u\| \|a\| \cos\theta]$

$\Leftarrow$ if different sign then vector b stays on one side and a_i on the other of H

- <u>Benders' reformulation</u>

(OP)

original problem

$$\begin{aligned} \max \quad & cx + hy \\ & Fx + Gy \le d \\ & x \in X \subseteq \mathbb{R}^n_+ \\ & y \in \mathbb{R}^p_+ \end{aligned}$$

(EF)

extended formulation

$$\begin{aligned} \max \quad & cx + \eta \\ & v^r(d - Fx) \ge 0 \qquad r \in R \\ & u^p(d - Fx) \ge \eta \qquad p \in P \\ & x \in X \\ & \eta \in \mathbb{R}^1 \end{aligned}$$

extreme rays of DSP

extreme points of DSP

Derivation of the reformulation

If we fix x to $\bar{x}$ in (OP) then:

(SP)

Subproblem

$$\begin{aligned} \max \quad & \cancel{c\bar{x}} + hy \\ & Gy \le d - F\bar{x} \\ & \cancel{\bar{x} \in X \subseteq \mathbb{R}^n_+} \\ & y \in \mathbb{R}^p_+ \end{aligned}$$

Let $\bar{X} = \{ x \mid \exists\, y \in \mathbb{R}^p_+ \text{ such that } Gy \le d - Fx \}$

if $\bar{X} \cap X = \phi \Rightarrow$ OP is infeasible

if $\bar{X} \cap X \neq \phi \Rightarrow$ OP is solvable — bounded / unbounded

We use Farkas Lemma to derive an operational description of $\bar{X}$:

To avoid infeasibility we negate the Farkas condition for infeasibility:

$$\underbrace{\forall u \geq 0, \quad u^T G \geq 0}: \quad u^T(d - F\bar{x}) \geq 0$$

this a Cone $U: \{u \mid u^T G \geq 0, u \geq 0\}$
hence it can be described by there extreme rays, say, v^r, $r \in R$

Hence, we can write $\bar{X}$ as:

$$\bar{X} = \{x \mid v^r(d - F(x)) \geq 0 \text{ for } r \in R\}$$

We can now rewrite (OP):

$$\max \; cx + hy$$
$$Fx + Gy \leq d$$
$$x \in X \subseteq \mathbb{R}^n_+$$
$$y \in \mathbb{R}^q_+$$

as

$$\max_{x} \; cx + \max_{y} \{hy \mid Gy \le d - Fx, \; y \in \mathbb{R}_+\}$$
$$x \in X \cap \bar{X}$$

Subproblem SP:

$x \in X \cap \bar{X} \neq \emptyset$ then solvable: bounded / unbounded

$x \in X \cap \bar{X} = \emptyset$ then infeasible

by strong duality theorem:

$$\max_{x} \; cx + \min_{u} \{u(d - Fx) \mid uG \ge h, \; u \in \mathbb{R}_+^m\}$$
$$x \in X \cap \bar{X}$$

Dual Subproblem DSP

SP: bounded ------------ DSP: bounded

SP: unbounded $(+\infty)$ ------------ DSP: infeasible

If SP bounded and DSP bounded, then the solution of DSP will be in one of the extreme points by the fundamental th. of lin. programming. Hence:

$$\max_{x} \; cx + \min_{p \in P} \{ u^p(d - Fx) \}$$
$$x \in X \cap \bar{X}$$

u^p are extreme points of $\{uG \ge h, \; u \ge 0\}$

similar to $\min\{2x, 3x, 5x, 6x\}$

that can be linearized as
$$\eta \le 2x$$
$$\eta \le 3x$$
$$\eta \le 5x$$
$$\eta \le 6x$$

After linearization this is equivalent to:

$$
(EF) \quad \begin{aligned}
\max_{x} \quad & cx + \eta \\
& w^p(d - Fx) \geq \eta && \forall p \in P \\
& v^r(d - Fx) \geq 0 && \forall r \in R \\
& x \in X
\end{aligned}
$$

$\rightsquigarrow$ from def. of $\bar{X}$

which is the reformulation we wanted to achieve. Hence

OP has feasible sol $\iff$ EF has feasible sol

OP has unbounded sol $\iff$ EF has unbounded sol

OP is infeasible $\iff$ EF is infeasible

EF has to many constraints to list them all hence we solve it by defining it incrementally:

$z^* = \max \quad cx + \eta$

x

$$w^p(d - Fx) \geq \eta \quad \forall p \in \bar{P} \subseteq P$$

(REF) $$v^r(d - Fx) \geq 0 \quad \forall r \in \bar{R} \subseteq R$$

$$x \in X$$

- Benders' algorithm

Solve (REF) and find (η^*, x^*)

Solve (DSP) $\phi(x^*) = \min \; u(d - Fx^*)$

$$uG \geq h$$

$$u \in \mathbb{R}_+^m$$

if unbounded then

$\exists v^r : v^r(d - Fx^*) < 0 \Rightarrow v^r(d - Fx) \geq 0$ must be added to make the primal feasible (from formulation of $\bar{X}$)

if bounded and $\phi(x^*) < \eta^*$

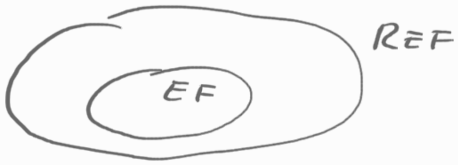
then the solution to DSP, w^p gives a new extreme point; whose relative constr. in EF is violated

$$\phi(x^*) = w^p(d - Fx^*) < \eta^*$$

hence we add

$w^p(d - Fx) \geq \eta$

if bounded and $\phi(x^*) = \eta^*$
then all constr. are satisfied
so the lin. progr EF is solved



REF is a relaxation of EF, hence
if the sol is feasible for EF it
is opt. for EF

Resuming:

if (REF) has no feasible sol $\Rightarrow$ STOP and return infeasible, adding constraints will not remove infeas.

if (REF) has unbounded sol $\Rightarrow \eta^* = +\infty$
guess x^* and solve (DSP)

if (REF) has a bounded sol $\Rightarrow (\eta^*, x^*)$ and solve (DSP)

if (DSP) is infeasible $\Rightarrow$ STOP the (EF) is unbound.

if (DSP) is unbounded $\Rightarrow$ add extreme ray constr.

if (DSP) is bounded $\Rightarrow$ add extreme point constr.
or STOP because opt. found