DM872

Math Optimization at Work

Preprocessing

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Outline

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Preprocessing rules

Consider $S = \{\mathbf{x} : a_0 x_0 + \sum_{j=1}^{n} a_j x_j \leq b, l_j \leq x_j \leq u_j, j = 0..n\}$

- Bounds on variables.
  If $a_0 > 0$ then:

$$x_0 \leq \left(b - \sum_{j:a_j>0} a_j l_j - \sum_{j:a_j<0} a_j u_j \right) / a_0$$

  and if $a_0 < 0$ then

$$x_0 \geq \left(b - \sum_{j:a_j>0} a_j l_j - \sum_{j:a_j<0} a_j u_j \right) / a_0$$

- Redundancy. The constraint $\sum_{j=0}^{n} a_j x_j \leq b$ is redundant if

$$\sum_{j:a_j>0} a_j u_j + \sum_{j:a_j<0} a_j l_j \leq b$$

- Infeasibility: $S = \emptyset$ if (swapping lower and upper bounds from previous case)

$$\sum_{j:a_j>0} a_j l_j + \sum_{j:a_j<0} a_j u_j > b$$

- Variable fixing. For a max problem in the form

$$\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$$

  if $\forall i = 1..m : a_{ij} \geq 0, c_j < 0$ then fix $x_j = l_j$
  if $\forall i = 1..m : a_{ij} < 0, c_j > 0$ then fix $x_j = u_j$

- Integer variables:

$$\lceil l_j \rceil \leq x_j \leq \lfloor u_j \rfloor$$

- Binary variables. Probing: add a constraint, eg, $x_2 = 0$ and check what happens

Example

$$
\begin{aligned}
\max\ & 2x_1 + x_2 - x_3 \\
\mathtt{R1}: & 5x_1 - 2x_2 + 8x_3 \leq 15 \\
\mathtt{R2}: & 8x_1 + 3x_2 - x_3 \geq 9 \\
\mathtt{R3}: & x_1 + x_2 + x_3 \leq 6 \\
& 0 \leq x_1 \leq 3 \\
& 0 \leq x_2 \leq 1 \\
& x_3 \geq 1
\end{aligned}
$$

$\mathtt{R1}: 5x_1 \leq 15 + 2x_2 - 8x_3 \leq 15 + 2 \cdot \overbrace{1}^{u_2} - 8 \cdot \overbrace{1}^{l_3} = 9 \quad \rightsquigarrow x_1 \leq 9/5$

$8x_3 \leq 15 + 2x_2 - 5x_1 \leq 15 + 2 \cdot 1 - 5 \cdot 0 = 17 \quad \rightsquigarrow x_3 \leq 17/8$

$2x_2 \geq 5x_1 + 8x_3 - 15 \geq 5 \cdot 0 + 8 \cdot 1 = -7 \quad \rightsquigarrow x_2 \geq -7/2, x_2 \geq 0$

$\mathtt{R2}: 8x_1 \geq 9 - 3x_2 + x_3 \geq 9 - 3 + 1 = 7 \quad \rightsquigarrow x_1 \geq 7/8$

$\mathtt{R1}: 8x_3 \geq 15 + 2x_2 - 5x_1 \leq 15 + 2 - 5 \cdot 7/8 = 101/8 \quad \rightsquigarrow x_3 \leq 101/64$

$\mathtt{R3}: x_1 + x_2 + x_3 \leq 9/5 + 1 + 101/64 < 6$ Hence $\mathtt{R3}$ is redundant

Example

$$\begin{aligned}
&\max\ 2x_1 + x_2 - x_3 \\
&\texttt{R1}: 5x_1 - 2x_2 + 8x_3 \leq 15 \\
&\texttt{R2}: 8x_1 + 3x_2 - x_3 \geq 9 \\
&\quad 7/8 \leq x_1 \leq 9/5 \\
&\quad 0 \leq x_2 \leq 1 \\
&\quad 1 \leq x_3 \leq 101/64
\end{aligned}$$

Increasing x_2 makes constraints satisfied $\rightsquigarrow$ $x_2 = 1$
Decreasing x_3 makes constraints satisfied $\rightsquigarrow$ $x_3 = 1$

We are left with:

$$\max\{2x_1 : 7/8 \leq x_1 \leq 9/5\}$$

Preprocessing for Set Covering/Partitioning

1. if $e_i^T A = 0$ then the ith row can never be satisfied

$$\begin{bmatrix} 0 & 0 & \dots & 1 & \dots & 0 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ 0 \dots 0 & \dots & 0 \\ & & \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

2. if $e_i^T A = e_k$ then $x_k = 1$ in every feasible solution

$$\begin{bmatrix} 0 & 0 & \dots & 1 & \dots & 0 \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ 0 & \dots & 1 & \dots & 0 \\ & & & \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

In SPP can remove all rows t with $a_{tk} = 1$ and set $x_j = 0$ (ie, remove cols) for all cols that cover t

3. if $e_t^T A \geq e_p^T A$ then we can remove row t, row p dominates row t (by covering p we cover t)

$$\begin{array}{c} t \\ \\ p \\ \end{array} \left[\begin{array}{cccccc} & 1 & 1 & & 1 & \\ & & & & & \\ & 1 & & & 1 & \\ \end{array} \right]$$

In SPP we can remove all cols j: $a_{tj} = 1, a_{pj} = 0$

4. if $\sum_{j \in S} Ae_j = Ae_k$ and $\sum_{j \in S} c_j \leq c_k$ then we can cover the rows by Ae_k more cheaply with S and set $x_k = 0$
(Note, we cannot remove S if $\sum_{j \in S} c_j \geq c_k$)

$$\left[\begin{array}{c:ccc:c:c:c} & & 1 & & & 1 & \\ & 1 & & & & 1 & \\ & & & 1 & & 1 & \\ & 0 & 0 & 0 & & 0 & \\ & 1 & & & & 1 & \\ & 0 & 0 & 0 & & 0 & \\ \end{array} \right]$$

Summary

1. Preprocessing