DM872

Math Opt @ Work

More on Polyhedra and Farkas' Lemma

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Farkas' Lemma
2. Beyond the Simplex

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1. Farkas' Lemma

2. Beyond the Simplex

We look at Farkas' Lemma with two objectives:

- (giving another proof of strong duality)
- understanding a certificate of infeasibility

Farkas' Lemma

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Then,

either	*I.*	$\exists \boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}$ *and* $\boldsymbol{x} \geq 0$
or	*II.*	$\exists \boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{y}^T A \geq 0^T$ *and* $\boldsymbol{y}^T \boldsymbol{b} < 0$

Easy to see that both I and II cannot occur together:

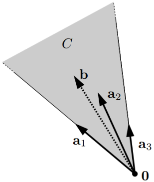
$$(0 \leq) \qquad \boldsymbol{y}^T A\boldsymbol{x} = \boldsymbol{y}^T \boldsymbol{b} \qquad (< 0)$$

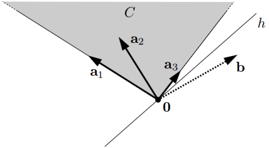
Geometric interpretation of Farkas' Lemma

Linear combination of $\boldsymbol{a}_i$ with nonnegative terms generates a convex cone:

$$\{\lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_n \boldsymbol{a}_n, \mid \lambda_1, \ldots, \lambda_n \geq 0\}$$

Polyhedral cone: $C = \{\boldsymbol{x} \mid A\boldsymbol{x} \leq 0\}$, intersection of many $\boldsymbol{ax} \leq 0$
Conic hull of rays $\boldsymbol{p}_i = \{\lambda_i \boldsymbol{a}_i, \lambda_i \geq 0\}$





Either point $\boldsymbol{b}$ lies in convex cone C
or $\exists$ hyperplane h passing through point 0 $h = \{\boldsymbol{x} \in \mathbb{R}^m : \boldsymbol{y}^T \boldsymbol{x} = 0\}$ for $\boldsymbol{y} \in \mathbb{R}^m$ such that all vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ (and thus C) lie on one side and $\boldsymbol{b}$ lies (strictly) on the other side (ie, $\boldsymbol{y}^T \boldsymbol{a}_i \geq 0, \forall i = 1 \ldots n$ and $\boldsymbol{y}^T \boldsymbol{b} < 0$).

Alternative Formulation

Theorem (Farkas' Lemma)

The inequality $\mathbf{c}^T\mathbf{x} \geq c_0$ is valid for the non-empty polyhedron $P := \{\mathbf{x} \geq 0 \mid A\mathbf{x} = \mathbf{b}\}$ ***if and only if*** *$\mathbf{y} \in \mathbb{R}^m$ exists such that:*

$$\mathbf{c}^T \geq \mathbf{y}^T A$$
$$c_0 \leq \mathbf{y}^T \mathbf{b}$$

$\Longleftarrow$ (sufficiency) (used in Gomory cuts)

$$\mathbf{c}^T\mathbf{x} \geq \qquad \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b} \qquad \geq c_0$$

$\Longrightarrow$ (necessity)
by simplex algorithm similar to our proof of the strong duality theorem

Other Variants of Farkas' Lemma

Corollary

(i) $A\boldsymbol{x} = \boldsymbol{b}$ *has sol* $\boldsymbol{x} \geq 0 \iff \forall \boldsymbol{y} \in \mathbb{R}^m$ *with* $\boldsymbol{y}^T A \geq 0^T, \boldsymbol{y}^T \boldsymbol{b} \geq 0$

(ii) $A\boldsymbol{x} \leq \boldsymbol{b}$ *has sol* $\boldsymbol{x} \geq 0 \iff \forall \boldsymbol{y} \geq 0$ *with* $\boldsymbol{y}^T A \geq 0^T, \boldsymbol{y}^T \boldsymbol{b} \geq 0$

(iii) $A\boldsymbol{x} \leq 0$ *has sol* $\boldsymbol{x} \in \mathbb{R}^n \iff \forall \boldsymbol{y} \geq 0$ *with* $\boldsymbol{y}^T A = 0^T, \boldsymbol{y}^T \boldsymbol{b} \geq 0$

Certificate of Infeasibility

Farkas' Lemma provides a way to certificate infeasibility.

Theorem

Let $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$.
Given a certificate $\mathbf{y}^*$ it is easy to check the conditions (by linear algebra):

$$\begin{aligned} A^T\mathbf{y}^* &\geq 0 \\ \mathbf{b}\mathbf{y}^* &< 0 \end{aligned}$$

Why would $\mathbf{y}^*$ be a certificate of infeasibility?
Proof (by contradiction)
Assume, $A^T\mathbf{y}^* \geq 0$ and $\mathbf{b}\mathbf{y}^* < 0$.
Moreover assume $\exists \mathbf{x}^*$: $A\mathbf{x}^* = \mathbf{b}$, $\mathbf{x}^* \geq 0$,then:

$$(\geq 0) \qquad (\mathbf{y}^*)^T A\mathbf{x}^* = (\mathbf{y}^*)^T\mathbf{b} \qquad (< 0)$$

Contradiction

General form:

$$\begin{aligned} \max\ & c^T x \\ A_1 x &= b_1 \\ A_2 x &\leq b_2 \\ A_3 x &\geq b_3 \\ x &\geq 0 \end{aligned}$$

infeasible $\Leftrightarrow \exists y^*$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 + b_3^T y_3 &> 0 \\ A_1^T y_1 + A_2^T y_2 + A_3^T y_3 &\leq 0 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \end{aligned}$$

Example

$$\begin{aligned} \max\ & c^T x \\ x_1 &\leq 1 \\ x_1 &\geq 2 \end{aligned}$$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 &> 0 \\ A_1^T y_1 + A_2^T y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 &> 0 \\ y_1 + y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$y_1 = -1, y_2 = 1$ is a valid certificate.

- Observe that it is not unique!
- It can be reported in place of the dual solution because same dimension.
- To repair infeasibility we should change the primal at least so much as that the certificate of infeasibility is no longer valid.
- Only constraints with $y_i \neq 0$ in the certificate of infeasibility cause infeasibility

Duality: Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
  - Farkas' Lemma:
    Strong duality + Infeasibility certificate
- Dual Simplex
- Economic interpretation
- Geometric Interpretation
- Sensitivity analysis

Resume

Advantages of considering the dual formulation:

- proving optimality (although the simplex tableau can already do that)
- gives a way to check the correctness of results easily
- alternative solution method (ie, primal simplex on dual)
- sensitivity analysis
- solving P or D we solve the other for free
- certificate of infeasibility

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Interior Point Algorithms

- Ellipsoid method: cannot compete in practice but weakly polynomial time (Khachyian, 1979)
- Interior point algorithm(s) (Karmarkar, 1984) competitive with simplex and polynomial in some versions
  - affine scaling algorithm (Dikin)
  - logarithmic barrier algorithm (Fiacco and McCormick) $\equiv$ Karmakar's projective method

1. Start at an interior point of the feasible region
2. Move in a direction that improves the objective function value at the fastest possible rate while ensuring that the boundary is not reached
3. Transform the feasible region to place the current point at the center of it

- because of patents reasons, now mostly known as barrier algorithms
- one single iteration is computationally more intensive than the simplex (matrix calculations, sizes depend on number of variables)
- particularly competitive in presence of many constraints (eg, for $m = 10,000$ may need less than 100 iterations)
- bad for post-optimality analysis $\rightsquigarrow$ crossover algorithm to convert a solution of barrier method into a basic feasible solution for the simplex