DM872

Math Optimization at Work

Dantzig-Wolfe Decomposition and Delayed Column Generation

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Outline

1. Solving the Linear Master Problem

2. Solving the Master Problem: Branch and Price

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Solving the Linear Master Problem

Integer Programming Problem with block structure:

$$
\begin{aligned}
z_{MP} = \max \quad & c^1 \sum_{t=1}^{T_1} \lambda_{1,t} x^{1,t} + \quad c^2 \sum_{t=1}^{T_2} \lambda_{2,t} x^{2,t} + \ldots + \quad c^K \sum_{t=1}^{T_K} \lambda_{K,t} x^{K,t} \\
& A^1 (\textstyle\sum_{t=1}^{T_1} \lambda_{1,t} x^{1,t}) + \quad A^2 (\textstyle\sum_{t=1}^{T_2} \lambda_{2,t} x^{2,t}) + \ldots + A^K (\textstyle\sum_{t=1}^{T_K} \lambda_{K,t} x^{K,t}) = b \\
& \textstyle\sum_{t=1}^{T_k} \lambda_{k,t} = \quad 1 \qquad k = 1, \ldots, K \\
& \lambda_{k,t} \in \{0,1\} \qquad t \in T_k, k = 1, \ldots, K
\end{aligned}
$$

Let's consider the case $K = 1$

$$
\begin{aligned}
z_{MP} = \max \quad & \textstyle\sum_{t=1}^{T} (cx^t) \lambda_t \\
& \textstyle\sum_{t=1}^{T} (Ax^t) \lambda_t = b \\
& \textstyle\sum_{t=1}^{T} \lambda_t = 1 \\
& \lambda_t \in \{0,1\} \qquad t \in T
\end{aligned}
\qquad\qquad
\begin{aligned}
z_{LMP} = \max \quad & \textstyle\sum_{t=1}^{T} (cx^t) \lambda_t \\
& \textstyle\sum_{t=1}^{T} (Ax^t) \lambda_t = b \\
& \textstyle\sum_{t=1}^{T} \lambda_t = 1 \\
& \lambda_t \geq 0 \qquad t \in T
\end{aligned}
$$

Restricted LMP and Dual

$$
\begin{aligned}
z_{LMP} = \max \ & \textstyle\sum_{t=1}^{T} (cx^t)\lambda_t \\
& \textstyle\sum_{t=1}^{T} (Ax^t)\lambda_t = b \\
& \textstyle\sum_{t=1}^{T} \lambda_t = 1 \\
& \lambda_t \geq 0 \qquad t \in T
\end{aligned}
$$

$$
\begin{aligned}
z_{RLMP} = \max \ & \textstyle\sum_{t=1}^{p} (cx^t)\lambda_t \\
& \textstyle\sum_{t=1}^{p} (Ax^t)\lambda_t = b \\
& \textstyle\sum_{t=1}^{p} \lambda_t = 1 \\
& \lambda_t \geq 0 \qquad t = 1, \ldots, p
\end{aligned}
$$

$$
\begin{aligned}
z_{DLMP} = \min \ & \pi b + \pi_0 \\
& \pi A^T x^t + \pi_0 \geq cx^t, \ t = 1, \ldots, T \\
& \pi \in \mathbb{R}^m \\
& \pi_0 \in \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
z_{DRLMP} = \min \ & \pi b + \pi_0 \\
& \pi A^T x^t + \pi_0 \geq cx^t, \ t = 1, \ldots, p \\
& \pi \in \mathbb{R}^m \\
& \pi_0 \in \mathbb{R}
\end{aligned}
$$

Column Generation Process and Dual Bound

- $z_{LMP} \geq z_{MP}$ because linear relaxation
- $z_{LMP} \geq z_{RLMP}$ because of simplex theory (some columns missing)
- subproblem (pricing or constraint violation)
  $\xi^p = \max\{cx^t - \pi A^T x^t - \pi_0 \mid x^t \in X\}$. Solution: $(x^*, (\pi^*, \pi_0^*))$
- $z_{MP} \leq z_{LMP} \leq z_{RLMP} + \xi^p$ hence, valid dual bound on z_{MP}
- if $\xi^p = 0$ then $z_{LMP} = z_{RLMP}$ and stop column generation process
- if $\xi^p > 0$ then
  stop if $\pi^*(Ax^* - b) = 0$
  else add column $(cx^*, Ax^*, 1)$

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Branching constraints

- branch on original variables or on column variables
- disadvantages of branching on column variables: B&B tree unbalanced and subproblem difficult to solve

Solving the LP master at a node

The constraints introduced for branching (and other cutting planes) change the master problem or the subproblem. Where they should be considered is a design choice.

Price and branch

Heuristic solution:
After solving the LMP, start the branch and bound with the existing columns.

Note, it can lead to infeasibility