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October 29, 2024
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DM545/DM871 – Linear and integer programming

Sheet 4.5, Autumn 2024

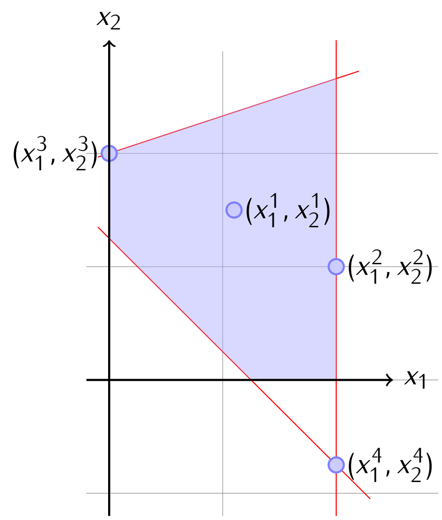
The text in this page is taken from the front page of the Obligatory Assignment that is coming.
The sheet consists of a number of tasks subdivided into subtasks. The answers must be collected in a unique PDF document and are to be handed in electronically in ItsLearning.

- **The test is individual. You are not allowed to collaborate by any means with other persons or with chatbots.**
- Keep your answers anonymous but make sure that you specify your **SDU username** (the part in your SDU email address before the @ symbol). Use the Latex or Word templates provided in the external web page to see where to specify the SDU username (Tutorials → Preparation for the Take-Home Assignments).
- Your answers will be assessed by the teacher and an internal censor. Moreover, the answers will be grouped by subtasks by an automated parser tool. Therefore, it is very important that you follow the instructions below.
- **In the PDF document make sure that you start a new page for every SUBTASK and you write a section title that includes the word "Subtask", specifying which SUBTASK you are addressing. See the examples in the Latex and Word templates provided in the external web page.**
- You can write your answers in Danish or in English.
- *Justify all your statements!* It is not sufficient to present an answer, you must also briefly argue how you found it. You may refer to results from the lecture notes, the slides or the books listed at the course web page. References to other books (outside the course material) or to internet links are not accepted as valid answers to a task.
- You are allowed to use tools such as Python to assist you in the calculations. If you report source code in Python or other languages, you must also report the output it produces when executed.
- Make sure you take security copies of your documents while the test is in progress. It is your own responsibility in case of technical issues.
- Tools and tutorials for typesetting your answers are available from the Public Web Page: Tutorials → Preparation for the Take-Home Assignments.
- The test consists of 4 tasks distributed on 9 pages.

Task 1 Simplex

Consider the following LP problem:

$$
\begin{array}{llrcrl}
(P) & \max & z = x_1 & + & 5x_2 & \\
 & \text{s.t.} & -x_1 & + & 3x_2 & \leq 6 \\
 & & 4x_1 & + & 4x_2 & \geq 5 \\
 & & 0 & \leq & x_1 & \leq 2 \\
 & & & & x_2 & \geq 0
\end{array}
$$



(Note: the following subtasks can be carried out independently; use fractional mode for numerical calculations)

Subtask 1.a

The polyhedron representing the feasibility region is shown in the figure. Indicate for each of the four marked points whether they are feasible and/or basic solutions.[1] Justify your answer.

Subtask 1.b

Write the initial tableau or dictionary for the simplex method. Write the corresponding basic solution and its value. State whether the solution is feasible or not and whether it is optimal or not.

Subtask 1.c

Write the dual of (P), its standard form and the first tableau.

Subtask 1.d

Explain:

- why you cannot apply the *primal simplex* to neither the tableau from Subtask 1.b nor to the tableau from Subtask 1.c;
- why you cannot apply the *dual simplex* to neither the tableau from Subtask 1.b nor to the tableaux from Subtask 1.c;

[1] In the figure, a point i is described by its coordinates (x_1^i, x_2^i).

- how you can modify the original problem such that you can then apply the primal simplex in either the primal problem or the dual problem.

Provide the tableau of the modified problem and comment how you would proceed from there. (Note, you can address this task also if you did not correctly solve Subtask 1.b and Subtask 1.c. In this case, answer in hypothetical terms.)

Subtask 1.e

Consider the following tableau:

	x1	x2	x3	x4	x5	-z	b
I	0	4	1	-1/4	0	0	29/4
II	1	1	0	-1/4	0	0	5/4
III	0	-1	0	1/4	1	0	3/4
IV	0	4	0	1/4	0	1	-5/4

and the following three pivoting rules:

- largest coefficient
- largest increase
- steepest edge.

Which entering and leaving variables would each of them indicate? In this specific case, which rule would be convenient to follow? Report the details of the computations for the first two rules and deduce the result from the application of the third rule by visual inspection using the plot in the figure above (for those working with LaTeX, the tikz code to reproduce the figure is available on the next page.)

```
\begin{center}
\begin{tikzpicturehevea}[domain=-0.1:2.2,xscale=1.5,yscale=1.5,
basis/.style={circle,draw=blue!50,fill=blue!20,thick,inner sep=0pt,minimum size=2mm},
opt/.style={circle,draw=red!50,fill=red!50,thick,inner sep=0pt,minimum size=2mm},
point/.style={rectangle,draw=black!50,fill=black!20,thick,inner sep=0pt,minimum size=4mm}]

\draw[very thin,color=gray] (-0.2,-1.2) grid (2.9,2.9);
\draw[color=red] plot[id=c0]  function{2 +0.33*x} node[right] {};

\draw[color=red] (2,-1.2) -- (2,3) node[right] {};

\draw[color=red] plot[id=c2, domain=-0.1:2.3]  function{1.2500 -x} node[right] {};

\fill[blue!30,opacity=0.5]
(2,2.666667) -- %
(0,2) -- %
(0,1.25) -- %
(1.25,0) -- %
(2,0);


\node (p1) at (1.1,1.5) [basis] {};
\node at (p1) [right] {($x_1^1,x_2^1$)};
\node (p2) at (2,1) [basis] {};
\node at (p2) [right] {($x_1^2,x_2^2$)};
\node (p3) at (0,2) [basis] {};
\node at (p3) [left] {($x_1^3,x_2^3$)};
\node (p4) at (2,-3/4) [basis] {};
\node at (p4) [right] {($x_1^4,x_2^4$)};

\draw[->,thick] (-0.2,0) -- (2.500000,0) node[right] {$x_1$};
\draw[->,thick] (0,-1.2) -- (0,3) node[above] {$x_2$};
\end{tikzpicturehevea}
\end{center}
```

Task 2 Revised Simplex

Consider the following LP problem

$$\begin{array}{rrcrcrcrcrcl} \text{maximize} & x_1 & + & 2x_2 & + & 4x_3 & + & 8x_4 & + & 16x_5 & & \\ \text{st} & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & \leq & 5 \\ & 7x_1 & + & 5x_2 & - & 3x_3 & - & 2x_4 & & & \leq & 0 \\ & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0 \end{array}$$

and call x_6, x_7 the two slack variables to bring the problem in standard form.

[Use Python to carry out the calculations. You can copy and paste the data from below. There is no requirement to use the fraction mode in this Task.]

```
# %%
import numpy as np
np.set_printoptions(precision=3,suppress=True)

c = np.array([1, 2, 4, 8, 16])
A = np.array([[1, 2, 3, 4, 5],
              [7, 5, -3, -2, 0]])
b = np.array([5, 0])
```

Subtask 2.a

In the optimal simplex tableau the basic variables are $x_B = [x_3, x_5]$ and the non basic variables are $x_N = [x_1, x_2, x_4, x_6, x_7]$. Using only matrix computations, like in the revised simplex method:

- determine the value of the variables $[x_1, x_2, x_3, x_4, x_5]$
- determine the value of the two dual variables associated with the two constraints
- verify that the solution is indeed optimal.

Subtask 2.b

Knowing that $[x_3, x_5]$ and $[x_5, x_7]$ are both optimal bases and that they both give the same optimal solution, what can you conclude about the state of the simplex tableau? How many optimal *dual* solutions are there?

Subtask 2.c

Which constraints are "binding" (or active) and which have slack in the optimal solution?

Subtask 2.d

If we wanted to make x_4 enter the basis in the optimal solution (and hence possibly have a value different from zero), how should we update its cost coefficient in the objective function?

Task 3 Duality

Consider the following problem.

$$\begin{array}{lrcrl} \max & z = 2x_1 & - & 4x_2 & \\ \text{s.t.} & x_1 & - & x_2 & \leq 1 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

Subtask 3.a

Construct the dual problem using the Lagrangian method and then find its optimal solution by inspection.

Subtask 3.b

Use the complementary slackness property and the optimal solution for the dual problem to find the optimal solution for the primal problem.

Subtask 3.c

Suppose that the coefficient of x_1 in the primal objective function is a free parameter, c_1. For which values of c_1 does the dual problem have no feasible solution? For these values, what does duality theory then imply about the primal problem?

Task 4 Crop planning

The following are two cases of crop planning that arise in agriculture. They are two separated cases and do not share information. The first case considers the single period scenario while the second a multi-period scenario. In real-life applications, a combination of the two cases is the most likely needed.

Subtask 4.a

Increasing the efficiency in agriculture by exploring the possibilities of optimal allocation of the scarce resources for the various farms has become a urgent need of society. Crop planning is one of the important stages in crop production and it has a considerable effect on farmer revenue.
Assume we have collected data regarding land allocation, labor days, machine hours, cost of seeds, fertilizers, and pesticides from farmers for four crops: paddy, cotton, chilli and maize, which are grown in a region.
Using averages of these data we want to determine the crop plan that maximizes the net return for an average farmer of a given region. The profit that farmers make per hectare (1 ha =10.000 m^2) of arable area of the four crops, paddy, cotton, chilli and maize are in Danish krone: 15.434, 46.186, 160.607, and 14.088, respectively.
The resource constraints to be considered are land, labor in man days, machine hours, operating cost and alternating crop plans.

- Land. Land is the basic resource of production. All land related activities are expressed in hectares basis. The total land available that we consider is 46 hectares and the total arable area cannot exceed this size.

- Labor. The labor is taken in man days. The man days needed per hectare of cultivation of the four crops are 29, 58, 308 and 24, respectively and the capacity is 5819 days.

- Machine hours. Farmers are hiring the tractor services for land preparation and machine hours are limited. The machine hours required per hectare of area of the four crops are 5, 4, 2, and 4, respectively, and the capacity is 525 hours.

- Operating capital. The operating capital refers to the funds required to meet the cost of seeds, fertilizers, farm yard manure, plant protection chemicals, insurance charges, marketing expenses and wages of the human and bullock labor and tractor power. The operating costs per hectare of area of the four crops are 32.474, 38.897, 111.962, 21.713, respectively, and the maximum budget is 4.343.868 Dkk.

- Alternate crop plans. The alternate crop plans are developed by introducing minimum and maximum land allocation constraints for the crops because in practical situations it is not applicable to allocate the whole land to one crop due to the fluctuations in the demand and risk associated with specialization of one crop. Here, we consider only a limitation on the crop that is cultivated by the majority of the farmers, paddy. The portion of (total available) land that can be allocated to paddy crop must be reduced by 20%.

Formulate the problem as a linear programming problem.

Subtask 4.b

Common Agricultural Policies stipulate a "payment for agricultural practices which are beneficial for the climate and the environment". This is commonly known as "greening payment" and represents a direct aid per hectare rewarding agricultural sustainable practices.
The greening payment may potentially encourage farmers to meet certain environmental requirements in return for governmental support payments. This greening aid rewards farmers complying three basic "greening rules" (or equivalent practices). These are: (1) maintenance of existing permanent grassland (2) establishment of an ecological focus area on arable land and (3) crop diversification.

We consider a farmer who needs to decide whether to adapt his/her cropping plan in order to fulfill the greening rules and therefore receive the greening subsidy. The farmer seeks to maximize his/her revenues over the time horizon of the policy, that is, the farmer seeks to optimize cropping plan decisions for the next five years by maximizing his/her net return.
Let us consider a farmer that owns a piece of arable land of 35 hectares (ha) where $N = 6$ different crops may be grown. We consider a planning period of $T = 5$ years, with one growing season per year. We are given the crop yields, e_i, and the cost of exploitation, c_i, of each crop $i = 1..6$. Moreover, the prices p_i received by farmers for each crop and the value of the greening subsidy payment b_i for each crop are available. These data are shown in Figure 1. The first table above shows the crops considered, their selling price and their exploitation cost, the other two tables below show the yield and the basic payment per crop type. All this data is per unit of hectare and we assume a linear growth with respect to cultivated hectares for the corresponding crop.
The price received by the farmer and the exploitation cost of the crops are capitalized each year by recursively updating the current price and cost of the corresponding crop (p_i and c_i, respectively) at a given inflation rate (i.e., consumer price index) $r = 5\%$:

$$p_{it} = p_i \cdot (1 + r)^t \quad i = 1..N, t = 1..T$$

$$c_{it} = c_i \cdot (1 + r)^t \quad i = 1..N, t = 1..T$$

The farmer's net return NV_t (€) at the end of period t must be actualized to the *net present value* at $t = 0$. The total net present value accumulated per hectare of each crop $i = 1..N$ during the planning horizon can be derived by means of the following formula:

$$NPV_i = \sum_{t=1}^{T} \frac{(e_i p_{it} + b_i - c_{it})}{(1 + a)^t}$$

where a is a discount rate of 5%.
The goal is to determine the optimal cropping plan by choosing the crops to be grown every year and their acreage in order to maximize the total farmer's economic net present value over the time horizon of five years.
The greening subsidy b_i is however only received if the three basic greening rules of the policy are fulfilled: (1) maintenance of permanent grassland (2) establishment of ecological focus areas and (3) crop diversification.
More precisely, the restrictions imposed by the greening rule are as follows:

- Arable land. The land allocated to the crops, the permanent grassland area and the ecological focus area must not exceed the total farm arable land each year.

i	Crop	Price (€t^{-1})	Cost (€ha^{-1})
1	Wheat	230.1	604.1
2	Barley	194.8	598.1
3	Rye	181.5	377.6
4	Oat	183.6	585.7
5	Dried Pea	223.6	455.6
6	Sunflower	380.1	430.9

Yield (t ha^{-1})					
Wheat	**Barley**	**Rye**	**Oat**	**Dried Pea**	**Sunflower**
2.396	2.334	1.468	1.642	0.811	0.774

Basic payment (€ha^{-1})					
Wheat	**Barley**	**Oat**	**Rye**	**Dried peas**	**Sunflower**
150.96	147.05	92.46	103.45	181.00	48.73

Figure 1: Above: Crops considered, selling prices and production costs per crops. Below left: Crop yields. Below right: Basic payment for crops.

- Maintenance of permanent grassland. The permanent grassland has to be at least 5% of the total arable land every year.
- Establishment of ecological focus areas. The ecological focus area has to be greater than 5% of the total arable land in the previous year up to year $t = 2$, and greater than 7% of the previous year from $t = 3$.
- Crop diversification. The farmer must grow at least three different crops. The main crop (i.e., the crop occupying the largest area) must not cover more than 75% of the total arable land in each period t. The two largest crops together must not cover more than 95% of the arable land.
- Crop rotations. The same crop cannot be grown in the same area in sequential seasons. To allow this it must be enforced that the arable land occupied by one crop in period t is lower or equal to the surface not occupied by this crop during the previous period. For the first planning period we assume the area occupied by the crops in the previous year was 0 ha.

A mathematical model of the problem and its solution can provide valuable insight to both farmers and Governments: farmers can optimize their revenue and Governments can determine the minimum subsidy value that would make the implementation of greening rules economically appealing (eg, by comparing farmer's revenue with and without the greening rules).
Formulate the problem as a linear programming problem.