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DM545/DM871 – Linear and integer programming

Sheet 0, Spring 2024

Solution:

Included.

Review of elements from Linear Algebra that are used in DM545/DM871.

Exercise 1

Consider the matrices:

$$A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

In each part compute the given expression. Where the computation is not possible explain why.

1. $D + E$
2. $D - E$
3. $5A$
4. $2B - C$
5. $2(D + 5E)$
6. $(C^T B)A^T$
7. $2\text{tr}(AB)$
8. $\det(E)$

Solution:

Taken by a former student: andrm17.

1) $D + E$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -2+0 & 1+3 & 8+0 \\ 3+-5 & 0+1 & 2+1 \\ 4+7 & -6+6 & 3+2 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 8 \\ -2 & 1 & 3 \\ 11 & 0 & 5 \end{bmatrix}$$

2) $D - E$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} - E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -2-0 & 1-3 & 8-0 \\ 3--5 & 0-1 & 2-1 \\ 4-7 & -6-6 & 3-2 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 8 \\ 8 & -1 & 1 \\ -3 & -12 & 1 \end{bmatrix}$$

3) $5A$

$$5 \cdot \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 & 5 \cdot 0 \\ 5 \cdot -4 & 5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -20 & 30 \end{bmatrix}$$

4) $2B - C$

$$2 \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot -7 & 2 \cdot 2 \\ 2 \cdot 5 & 2 \cdot 3 & 2 \cdot 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 4 \\ 10 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

We cannot perform a subtraction with the two matrices since they do not have the same dimensions.

5) $2(D + 5E)$

$$2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ n4 & -6 & 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} \right) = 2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 5 \cdot 0 & 5 \cdot 3 & 5 \cdot 0 \\ 5 \cdot -5 & 5 \cdot 1 & 5 \cdot 1 \\ 5 \cdot 7 & 5 \cdot 6 & 5 \cdot 2 \end{bmatrix} \right)$$

$$2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 15 & 0 \\ -25 & 5 & 5 \\ 35 & 30 & 10 \end{bmatrix} \right) = 2 \cdot \left(\begin{bmatrix} -2+0 & 1+15 & 8+0 \\ 3+-25 & 0+5 & 2+5 \\ 4+35 & -6+30 & 3+10 \end{bmatrix} \right)$$

$$2 \cdot \begin{bmatrix} -2 & 16 & 8 \\ -22 & 5 & 7 \\ 39 & 24 & 13 \end{bmatrix} = \begin{bmatrix} 2\cdot -2 & 2\cdot 16 & 2\cdot 8 \\ 2\cdot -22 & 2\cdot 5 & 2\cdot 7 \\ 2\cdot 39 & 2\cdot 24 & 2\cdot 13 \end{bmatrix} = \begin{bmatrix} -4 & 32 & 16 \\ -44 & 10 & 14 \\ 78 & 48 & 26 \end{bmatrix}$$

6) $(C^T B)A^T$

$$\left(\begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}^T = \left(\begin{bmatrix} 4 & -3 & 2 \\ 9 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix}$$

We cannot perform this multiplication since C^T doesn't have the same number of columns as B has rows.

7) $2tr(AB)$

$$2tr\left(\begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}\right) = 2tr\left(\begin{bmatrix} 2\cdot 1 + 0\cdot 5 & 2\cdot -7 + 0\cdot 3 & 2\cdot 2 + 0\cdot 0 \\ -4\cdot 1 + 6\cdot 5 & -4\cdot -7 + 6\cdot 3 & -4\cdot 2 + 6\cdot 0 \end{bmatrix}\right)$$

$$2tr\left(\begin{bmatrix} 2 & -14 & 4 \\ 26 & 46 & -8 \end{bmatrix}\right)$$

Trace is not defined for non-square matrices.

8) $det(E)$

We're using cofactor expansion to get the determinator.

$$det\left(\begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}\right) = 0 \cdot \begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix} - 3 \cdot \begin{bmatrix} -5 & 1 \\ 7 & 2 \end{bmatrix} - 0 \cdot \begin{bmatrix} -5 & 1 \\ 7 & 6 \end{bmatrix}$$

we can discard the zeroes and we're then left with:

$$-3 \cdot (-5 \cdot 2 - 7 \cdot 1) = -3 \cdot (-10 - 7) = -3 \cdot -17 = 51$$

Exercise 2

Consider the following system of linear equations in the variables $x, y, z \in \mathbb{R}$.

$$-2y + 3z = 3$$
$$3x + 6y - 3z = -2$$
$$-3x - 8y + 6z = 5$$

1. Write the augmented matrix of this system.
2. Reduce this matrix to row echelon form by performing a sequence of elementary row operations.
3. Solve the system and write its general solution in parametric form.

Solution:

```
import numpy as np

# The augmented matrix
AA = np.array([[ 0, -2, 3, 3],
               [ 3, 6, -3, -2],
               [-3, -8, 6, 5]])

In [30]: import sympy as sy
```

```
    ...: # np.linalg.solve(A,b)
    ...:
    ...: sy.Matrix(AA).rref()
    ...:
Out[30]:
(Matrix([
 [1, 0, 2, 7/3],
 [0, 1, -3/2, -3/2],
 [0, 0, 0, 0]]), (0, 1))
```

Hence, the solution is:

$$\mathbf{x} = \begin{bmatrix} 7/3 - 2t \\ -3/2 + 3/2t \\ t \end{bmatrix} = \begin{bmatrix} 7/3 \\ -3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 3/2 \\ 1 \end{bmatrix} t \qquad t \in \mathbb{R}$$

Exercise 3

Consider the following matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

1. Find M^{-1} by performing row operations on the matrix $[M \mid I]$.
2. Is it possible to express M as a product of elementary matrices? Explain why or why not.

Solution:

```
import numpy as np

M = np.array([[ 1, 0, 1],[-1,1,0],[2,2,2]])
MM = np.concatenate([M,np.identity(3)],axis=1)

import sympy as sy
sy.Matrix(MM).rref()
```

```
(Matrix([
 [1, 0, 0, -1.0, -1.0,  0.5],
 [0, 1, 0, -1.0,    0,  0.5],
 [0, 0, 1,  2.0,  1.0, -0.5]]), (0, 1, 2))
```

Yes, it is possible. Since the matrix is invertible we have shown above that we can go from an identity matrix to M^{-1} and consequently also to M from an identity matrix with elementary row operations. Elementary row operations can be expressed as products between elementary matrices.

Exercise 4

1. Given the point $[3, 2]$ and the vector $[-1, 0]$ find the vector and parametric (Cartesian) equation of the line containing the point and parallel to the vector.
2. Find the vector and parametric (Cartesian) equations of the plane in $\mathbb{R}^3$ that passes through the origin and is orthogonal to $\mathbf{v} = [3, -1, -6]$.

Solution:

The vector equation: Let $[3,2]^T = \mathbf{p}$ and $[-1,0]^T = \mathbf{v}$. Any point $\mathbf{x}$ on the line can be expressed as:

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \qquad \forall t \in \mathbb{R}$$

We can derive the Cartesian equation by eliminating t from the equation above:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

From the first coordinate: $x_1 = 3 - t$ and from the second: $x_2 = 2$. The Cartesian equation is $x_2 = 2$ since x_1 is free to get any value.

The plane through the origin orthogonal to $\mathbf{v} = [3, -1, -6]$ is given by:

$$\mathbf{x}^T\mathbf{v} = 0$$

That is: $3x_1 - x_2 - 6x_3 = 0$.

Exercise 5

Write a one line description of the methods you know to compute the inverse of a square matrix.

Solution:

- using cofactors for the adjoint matrix and dividing by the determinant
- by row reduction of $[A|I]$

Exercise 6

Calculate by hand the inverse of

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

and check your result with the function `numpy.linalg.inv`.

Solution:

$$\begin{bmatrix} 0. & 1. & -1.5 \\ 1. & -3. & 4. \\ 0. & 0. & 0.5 \end{bmatrix}$$

Exercise 7

Use Cramer's rule to express the solution of the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Solution:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained from A by replacing the ith column with the vector b.

$$x_1 = -4.5 \qquad x_2 = 14 \qquad x_3 = 1.5$$

Exercise 8

Given two points in the Cartesian plane $\mathbb{R}^2$, $A = (1, 2)$ and $B = (3, 4)$ write the vector parametric equation and the Cartesian equation of the line that passes through them. Express the vector equation as an affine combination of the two points.

Solution:

The vector equation is an affine combination of the two points:

$$\mathbf{x} = [1, 2]^T + t([3, 4]^T - [1, 2]^T), \forall t \in \mathbb{R}^2$$

To find a, b, c such that $ax + by + c = 0$ describes the line we can rewrite the equation above as

$$[x, y]^T = [1, 2]^T + t([1, 2]^T), \forall t \in \mathbb{R}^2$$

we then eliminate t by substituting in the two equations.

Exercise 9

Express the segment in $\mathbb{R}^2$ between the points $A = (1, 2)$ and $B = (3, 4)$ as a convex combination of its extremes.

Solution:

$$\{\alpha[1, 2]^T + \beta[3, 4]^T \mid \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$$

Exercise 10

Write a generic vector parametric equation and a generic Cartesian equation of a plane in $\mathbb{R}^3$.

Solution:

$$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad x, t \in \mathbb{R}$$

$$ax + by + cz + d = 0$$

Exercise 11

Write a generic Cartesian equation of an hyperplane in $\mathbb{R}^n$ that does not pass through the origin.

Solution:

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$

Exercise 12

Prove that the following vectors in $\mathbb{R}^3$ linearly independent?
- $[6, 9, 5]^T$
- $[5, 5, 7]^T$
- $[2, 0, 7]^T$

Solution:

We need to solve homogeneous system $A\mathbf{x} = \mathbf{0}$, where the matrix A has the three vectors forming its columns. The matrix A has $\det(A) \neq 0$ and rank 3. Therefore the only solution to the homogeneous system is the trivial solution $\mathbf{0}$. The three column vectors are therefore linearly independent.

```
A=np.array([[6, 9, 5],[5, 5, 7],[2, 0, 7]]).T
np.linalg.det(A)
np.linalg.matrix_rank(A)
```