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DM545/DM871 – Linear and integer programming

Sheet 2, Autumn 2024

Solution:

Included.

Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1* List all vertices of the polyhedron $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, characterized by the following matrices A and $\mathbf{b}$:

$$A = \begin{bmatrix} 2 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

You are encouraged to use Python for carrying out the calculations.

Solution:

- Let denote the polyhedron in focus by $P(A', \mathbf{b}') = \{\mathbf{x} \in \mathbb{R}^4 \mid A'\mathbf{x} = \mathbf{b}'\} = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where:

  $$A' := \begin{bmatrix} A \\ I \end{bmatrix} \qquad \mathbf{b}' := \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

  There are 3 + 4 constraints (the 3 of $A\mathbf{x} \leq \mathbf{b}$ and the 4 of $\mathbf{x} \geq \mathbf{0}$) and 4 variables.

- An equivalent way to write the polyhedron $P(A', \mathbf{b}')$ is in its equational standard form $P^=(\bar{A}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^p \mid \bar{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, setting $p = n + m$. We showed in class how go from one representation to the other by introducing slack variables.

  In our specific case:

  $$\bar{A} = \begin{bmatrix} 2 & 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  Note that we do not include the $\mathbf{x} \geq \mathbf{0}$ constraints in $\bar{A}$, since we include them in the definition of feasible basic solutions and hence they are handled implicitly there.

- We saw that in the context of $P^=(\bar{A}, \mathbf{b})$ the concept of geometrical vertices of the polyhedron corresponds to the concept of basic feasible solutions.

- Hence, to determine the vertices of the polyhedron $P(A', \mathbf{b}')$ we need to enumerate all basic solutions of $P^=(\bar{A}, \mathbf{b})$.

- Recall that a basic feasible solution is given by a subset B of size m of the indices of columns of the matrix $\bar{A}$ and is such that A_B, the so-called basis matrix, is non-singular and $\bar{x}_B = A_B^{-1} b \geq 0$ and $\bar{x}_N = 0$. To determine all bases we need to generate all combinations of size m of the p columns. In our case, $m = 3$ and $p = 7$. We write a python script to do the calculations for us:

```
import numpy as np
import numpy.linalg as la
import itertools as it

A = np.array([[2,0,1,-4],[0,1,0,2],[0,0,1,0]])
I = np.identity(3)
```

```
A = np.concatenate([A,I],axis=1)
print(A)
b = np.array([3,1,1])

for e in it.combinations(range(7),3):
    if la.det(A[:,e]) != 0:
        x = np.dot(la.inv(A[:,e]),b)
        if (x>=0).all:
            print(f"basis: {str(e)} --> values: {str(x)}")
        else:
            print(f"basis: {str(e)} --> infeasible")
    else:
        print(f"basis: {str(e)} --> not invertible, not a point")
```

The output of the code snippet is omitted here for reasons of space but in the take-home assginments it must be reported.

We can reason on the geometrical representation of $P^=(\bar{A}, \mathbf{b})$. Let m be the number of constraints, n the number of original variables and $p > n$ the number of variables in the system of linear equations $\bar{A}\bar{x} = b$. Hence, $\bar{A} \in \mathbb{R}^{m \times p}$, $\bar{x} \in \mathbb{R}^p$, $b \in \mathbb{R}^m$.
Recall from linear algebra that:

- the system admits solutions if and only if: $\text{rank}(\bar{A}) = \text{rank}(\bar{A} \mid b)$.
- If $p < m$ then $\text{rank}(\bar{A}) < m$ the system is overdetermined and likely infeasible, unless $\text{rank}(\bar{A}) = \text{rank}(\bar{A} \mid b)$, in which case some constraint(s) are linearly dependent.
- Linearly dependent constraints can be removed as they are redundant.
- We can assume $p > m$ since we introduced a slack variable for each constraint (hence, $p = n+m > m$).
- Since we assume to have removed linearly dependent constraints, it is $\text{rank}(\bar{A}) = m$. Consequently it must be $\text{rank}(\bar{A}) = \text{rank}(\bar{A} \mid b)$.
- Under the assumption that $p > m$ the system is underdetermined. The solutions have $p - m$ variables set free and the solution space has dimension $p - m$. The *simplex* is the intersection between this space of $\bar{A}\mathbf{x} = \mathbf{b}$ and the positive orthant $\mathbf{x} \geq \mathbf{0}$.
- Algebraically, we can find some solutions of the solution space of $\bar{A}\bar{x} = \mathbf{b}$ by setting $p-m$ variables to zero and solving the linear system in the remaining variables. This corresponds to finding the basic feasible solutions of the linear system $\bar{A}\bar{x} = \mathbf{b}$. It can be proved that these basic solutions are the vertices of $P^=$.
- The simplex $P^=$ is the set of points that can be described by a convex combination of these basic feasible solutions.
- The vertices of this simplex in $\mathbb{R}^p$ are the vertices of the polyhedron defined by A' and $\mathbf{b}'$, $P(A', \mathbf{b}')$.
- There can be $\binom{p}{m}$ vertices in $P^=$ but not all of them are linearly independent with respect to the other vertices. Some of these basic feasible solutions may indicate the same geometrical vertex. When is that the case?

Exercise 2$^+$ Simplex method

This is part of the first exercise (Opgave 1) in the written exam of 2008.
Consider the following linear programming problem (P1)

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 4x_2 - x_3 \\
\text{subject to} \quad & 2x_1 - x_3 \leq 6 \\
& 3x_2 - x_3 \leq 9 \\
& x_1 + x_2 \leq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

- Rewrite the problem in equational standard form adding the slack variables x_4, x_5, x_6 to the three constraints above, respectively, and write the first simplex tableau with x_4, x_5, x_6 as basic solution.

Solution:

We write the initial tableaux:

$$\begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 0 & 0 & 6 \\ 0 & 3 & -1 & 0 & 1 & 0 & 0 & 9 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 4 \\ 2 & 4 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Or also:

x1	x2	x3	x4	x5	x6	-z	b
2	0	-1	1	0	0	0	6
0	3	-1	0	1	0	0	9
1	1	0	0	0	1	0	4
2	4	-1	0	0	0	1	0

- Argue that x_2 can be brought in the basis with advantage and perform one pivot iteration that brings x_2 into the basic solution.

Solution:

x_2 has positive reduced cost, hence worth bringing up.

pivot column: 2 pivot row: 2 pivot: 3

$$\begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 & 0 & 3 \\ 1 & 0 & 1/3 & 0 & -1/3 & 1 & 0 & 1 \\ 2 & 0 & 1/3 & 0 & -4/3 & 0 & 1 & -12 \end{bmatrix}$$

- After another pivot iteration, it is x_1 that can be brought with advantage in the basis (you do not have to perform this iteration), reaching the following simplex tableau:

x1	x2	x3	x4	x5	x6	-z	b
0	0	-5/3	1	2/3	-2	0	4
0	1	-1/3	0	1/3	0	0	3
1	0	1/3	0	-1/3	1	0	1
0	0	-1/3	0	-2/3	-2	1	-14

Argue that an optimal solution is found and give the solution together with its objective value.

Solution:

The optimal solution is found because all reduced costs are non-positive. The objective function value is 14.

We show here for completeness all iterations of the simplex from the first tableau above:

```
pivot column: 2
pivot row: 2
 pivot: 3

|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      2 |      0 |     -1 |      1 |      0 |      0 |      0 |      6 |
|      0 |      1 |   -1/3 |      0 |    1/3 |      0 |      0 |      3 |
|      1 |      0 |    1/3 |      0 |   -1/3 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      2 |      0 |    1/3 |      0 |   -4/3 |      0 |      1 |    -12 |
|--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 1
pivot row: 3
 pivot: 1

|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -5/3 |      1 |    2/3 |     -2 |      0 |      4 |
|      0 |      1 |   -1/3 |      0 |    1/3 |      0 |      0 |      3 |
|      1 |      0 |    1/3 |      0 |   -1/3 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -1/3 |      0 |   -2/3 |     -2 |      1 |    -14 |
|--------+--------+--------+--------+--------+--------+--------+--------+
```

Exercise 3* Simplex method

Solve the following LP problem carrying out the simplex operations:

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

You are free to use any of the two representations, tableau or dictionary.
You can also get help from Python. You find a tutorial in the external web page.

Solution:

```
%run utils
A=array([[2,f(3,1),1,1,0,0,0,5],[4,1,2,0,1,0,0,11],[3,4,2,0,0,1,0,8],[5,4,3,0,0,0,1,0]])
tableau(A)
# enough that one is a fraction to make all matrix of type fraction
# First simplex iteration
A[0,:] = f(1,2)*A[0,:]
A[1,:] = A[1,:]-f(4,1)*A[0,:]
A[2,:] = A[2,:]-f(3,1)*A[0,:]
A[3,:] = A[3,:]-f(5,1)*A[0,:]
tableau(A)
```

```
|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      2 |      3 |      1 |      1 |      0 |      0 |      0 |      5 |
|      4 |      1 |      2 |      0 |      1 |      0 |      0 |     11 |
```

```
|       3 |       4 |       2 |       0 |       0 |       1 |       0 |       8 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|       5 |       4 |       3 |       0 |       0 |       0 |       1 |       0 |
|--------+--------+--------+--------+--------+--------+--------+--------+
pivot column: 1
pivot row: 1
pivot: 2


|--------+--------+--------+--------+--------+--------+--------+--------+
|      x1 |      x2 |      x3 |      x4 |      x5 |      x6 |      -z |       b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|       1 |     3/2 |     1/2 |     1/2 |       0 |       0 |       0 |     5/2 |
|       0 |      -5 |       0 |      -2 |       1 |       0 |       0 |       1 |
|       0 |    -1/2 |     1/2 |    -3/2 |       0 |       1 |       0 |     1/2 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|       0 |    -7/2 |     1/2 |    -5/2 |       0 |       0 |       1 |   -25/2 |
|--------+--------+--------+--------+--------+--------+--------+--------+


pivot column: 3
pivot row: 3
pivot: 1/2
|--------+--------+--------+--------+--------+--------+--------+--------+
|      x1 |      x2 |      x3 |      x4 |      x5 |      x6 |      -z |       b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|       1 |       2 |       0 |       2 |       0 |      -1 |       0 |       2 |
|       0 |      -5 |       0 |      -2 |       1 |       0 |       0 |       1 |
|       0 |      -1 |       1 |      -3 |       0 |       2 |       0 |       1 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|       0 |      -3 |       0 |      -1 |       0 |      -1 |       1 |     -13 |
|--------+--------+--------+--------+--------+--------+--------+--------+
```

Exercise 4

Solve the following linear programming problem applying the simplex algorithm:

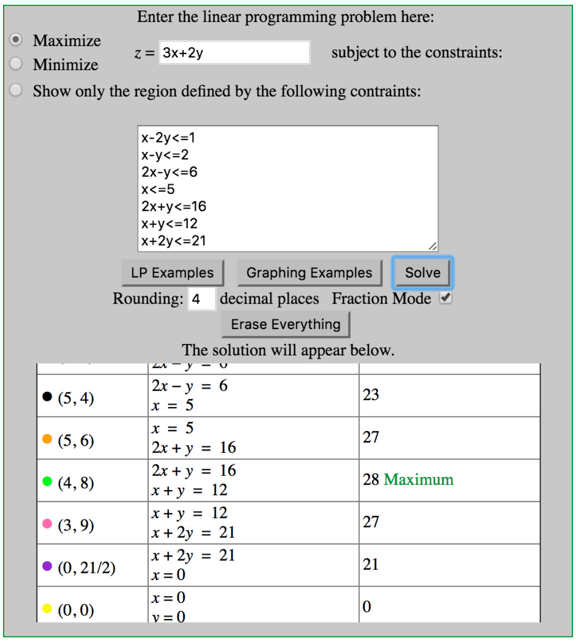
$$\begin{aligned}
\text{maximize} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & x_1 - 2x_2 \leq 1 \\
& x_1 - x_2 \leq 2 \\
& 2x_1 - x_2 \leq 6 \\
& x_1 \leq 5 \\
& 2x_1 + x_2 \leq 16 \\
& x_1 + x_2 \leq 12 \\
& x_1 + 2x_2 \leq 21 \\
& x_2 \leq 10 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

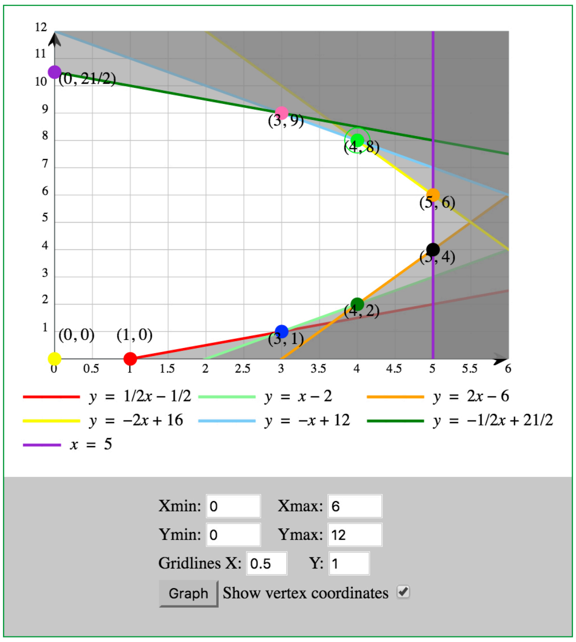
[Hint: you can plot the feasibility region with one of the tools linked at the course web page: "Tools" -> "Web applications on the simplex" -> "LP Simplex" and use the clairvoyant's rule to minimize the number of operations to carry out.]

Solution:

The initial solution of the simplex is at [0,0]. Then we can follow one of the two paths. The clairvoyant rule says that we should choose the direction that minimizes the path. This can be achieved by taking x_2. This yields a path of 3 arcs. Taking instead x_1 in the basis at the beginning would lead to a path of length 6.

Exercise 5* Project Scheduling





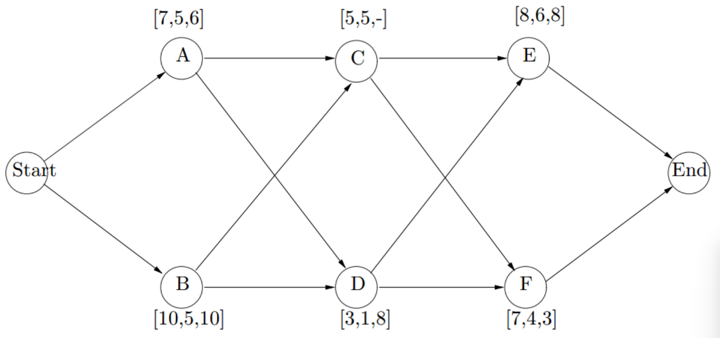


Figure 1: A network with activities on nodes for a small project with 6 activities. For each activity the following data is given in that order from left to right: normal time, minimum time in weeks, and the cost of shortening the duration of the activity by one week.

This exercise is a part of one that appeared in Exam 2011.
A small project has 6 sub-activities A, B, C, D, E, F whose individual dependency (shown by the immediate predecessors) is given in Figure 1. Here we also list the normal time (in weeks), the absolute minimum time and the cost of shortening the activity by one week.
The goal is to shorten the duration of the project to 19 weeks. This means that the duration of one or more activities has to be shortened. Of course we want to select these so that the total cost of shortening the duration to 19 weeks is minimized. Formulate this problem as a linear programming problem and argue that the optimal solution to this LP will provide the correct answer. Write the model in explicit form, that is, with the actual data inserted in the formulation.

Solution:

We will use a variable x_i to indicate how much we will shorten activity i and another set of variables y_i which will indicate the earliest starting time of activity i. For each arc $i \rightarrow j$ in the project network we will add the constraint $y_j \leq y_i + (d_i - x_i)$. We also use a variable y_{end} to express that the dummy activity "end" cannot start before all its immediate predecessors have finished. Finally we add the constraint $y_{end} \leq 19$ to force the total project time to be less or equal than 19.

$$\begin{aligned}
&\min\ 6x_A + 10x_B + 8x_D + 8x_E + 3x_F \\
\text{subject to }\ & y_C \geq y_A + (7 - x_A) \\
& y_C \geq y_B + (10 - x_B) \\
& y_D \geq y_A + (7 - x_B) \\
& y_D \geq y_B + (10 - x_B) \\
& y_E \geq y_C + (5 - x_c) \\
& y_E \geq y_D + (3 - x_D) \\
& y_F \geq y_C + (5 - x_C) \\
& y_F \geq y_D + (3 - x_D) \\
& y_{end} \geq y_E + (8 - x_E) \\
& y_{end} \geq y_F + (7 - x_F) \\
& y_{end} \leq 19 \\
& x_A \leq 2 \\
& x_B \leq 5 \\
& x_C \leq 0 \\
& x_D \leq 2 \\
& x_E \leq 3 \\
& x_F \leq 2 \\
& x_A, x_B, x_C, x_D, x_E, x_F \geq 0 \\
& y_A, y_B, y_C, y_D, y_E, y_F, y_{end} \geq 0
\end{aligned}$$

Note that we could have left x_C out of the model since its value is preset to zero. The optimal solution to this LP will tell us to shorten activity i by $x_i \geq 0$ units and since the cost we apply to each x_i is the per unit shortening cost of that activity, the cost of the solution will be that of shortening the project in the way suggested by the x_i's. Conversely, any feasible shortening of projects corresponds to a solution to this LP whose cost (in the LP) is the actual cost of shortening the activities in the way suggested.

Exercise 6$^+$ Quzzies

1. In 4D, how many hyperplanes need to intersect to give a point?

   Solution:

   4

2. In 4D, can a point be described by more than 4 hyperplanes?

   Solution:

   Yes, just think of a pyramid in 3D

3. Consider the intersection of n hyperplanes in n dimensions: when does it uniquely identify a point?

   Solution:

   when the rank of the matrix A of the linear system is n (or A is nonsingular)

Vertices of Polyhedra:
Consider the polyhedron described by $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$, that is:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
 & & \vdots & & & & & \vdots & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m
\end{array}$$

4. For a point $\mathbf{x}$ of a polyhedron, we define as *active* constraints those that are satisified to equality by $\mathbf{x}$. How many constraints are *active* in a *vertex* of a polyhedron $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$?

   Solution:

   at least n, rank of matrix of active constraints is n

5. Does every point x that activates n constraints form a vertex of the polyhedron?

   Solution:

   no, some may be not feasible, ie, intersection in a point outside of the polyhedron

6. Can a vertex activate more than n constraints?

   Solution:

   Yes, just look at the pyramid in 3 dim. However, the rank of the matrix of active constraints is still n

7. What if there are more variables than constraints? If $n < m$ then we can find a subset of constraints and then activate but what if $n > m$, can we have a vertex?

   Solution:

   No. In LP we deal with this issue by adding slack variables, they make us choose arbitrarily a vertex

8. Combinatorial explosion of vertices: how many constraints and vertices has an n-dimensional hypercube?

   Solution:

   To define a cube we need 6 constraints and there are 2^3 vertices. For an n-hypercube we need $2n$ constraints and there are 2^n vertices

9. If there are m constraints and n variables, $m > n$, what is an upper bound to the number of vertices?

   Solution:

   the number of possible active constraints is $\binom{m}{n}$ it is an upper bound because:

   – some combinations of constraints will not define a vertex, ie, if rows of matrix not independent
   – some vertices are outside the polyhedron
   – some vertices may activate more than n constraints and hence the same vertex can be given by more than n constraints

Tableaux and Vertices

10. For each of these three statements, say if they are true or false:

   – One tableau $\Longrightarrow$ one vertex of the feasible region

   – One tableau $\Longleftarrow$ one vertex of the feasible region

   – One tableau $\Longleftrightarrow$ one vertex of the feasible region

Solution:

One tableau $\Longleftarrow$ one vertex of the feasible region degenerate vertices have several tableau associated

11. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
x_1	1	0	−1/5	1/2	0	8
	0	0	−2/5	−1	1	−64

– How many variables (original and slack) can be different from zero?

Solution:

at most 2

– $(x_3, x_4) = (0, 0)$ are non basic, what does this tell us about the original constraints?

Solution:

The two original constraints are both active (that is, satisfied at equality) because their corresponding slack is zero.

Let's generalize the previous case. Consider an LP with m constraints, n original variables and m slack variables. In an optimal solution:

– if $m > n$, how many variables (original and slack) can be nonzero at most?

Solution:

at most m

– if $m < n$ how many original variables must be zero at least? In other terms, in a mix planning problem with n products and m, $m < n$ resources, how many products at most will be to be produced in an optimal solution?

Solution:

$n - m$, and hence at most $m < n$ products

Solution:

at most m

12. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	0	1	1/2	0	1
x_1	1	1	0	−1/2	0	1
	0	−2	0	1/2	1	−1

$(x_2, x_4) = (0, 0)$ are non basic variables, what does this tell us about the original constraints of the problem?

Solution:

The second constraint is active because its slack x_4 is zero.

13. If in the original space of the problem we had 3 variables, and there are 6 constraints, how many constraints would be active?

Solution:

3 constraints. With slack variables we would have 6 variables in all, if any of them is positive the constraint $x_i \geq 0$ of the original variables would be active, otherwise the corresponding constraints of the original problem are active.

14. For the general case with n original variables:
One basic feasible solution $\iff$ a matrix of active constraints has rank n. True or False?

Solution:

True

15. Consider an LP problem with m constraints and n original variables, $m > n$. We saw that in $\mathbb{R}^n$ a point is the intersection of at least n hyperplanes. In LP this corresponds to say that in a vertex there are n active constraints. Let a tableau be associated with a solution that makes exactly $n + 1$ constraints active, what can we say about the corresponding basic and non-basic variable values?

Solution:

one basic variable is zero. Indeed, in the simplex we will have $m + n$ variables and m variables in basis. We saw that the n non basic variables are set to zero and that there is an active constraint for each of them (initially they are those where the slack variables are). Hence, if there are $n + 1$ active constraints, there must be another variable that is set to zero. It must be a basic variable.

16. Given a polyhedron, what is the algebraic definition of vertex adjacency in 2, 3 and n dimensions?

Solution:

two vertices are adjacent iff:

- they have at least $n - 1$ active constraints in common
- rank of common active constraints is $n - 1$

How does this condition translate in terms of tableau?

Solution:

For what seen above this translates in $n - 1$ basis variables in common in the tableau

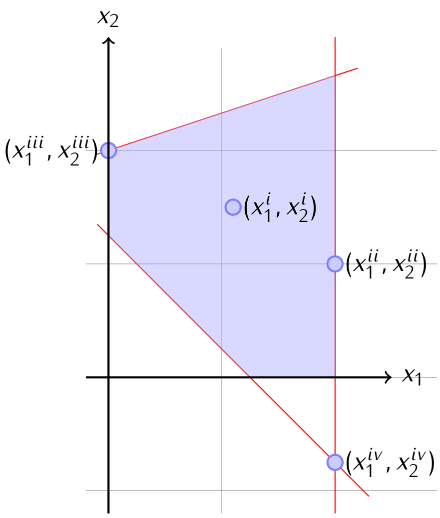
Exercise 7$^+$

What argument is used to prove that the simplex algorithm always terminates in a finite number of iterations if it does not encounter a situation in which one of the basic variables is zero? What may happen instead if the latter situation arises and which remedies are introduced?

Exercise 8

Exercise 3 from Exam 2013.
Consider the following LP problem:

$$
(P) \qquad \begin{array}{lrcrl}
\max & z = x_1 & + & 5x_2 & \\
\text{s.t.} & -x_1 & + & 3x_2 & \leq 6 \\
 & 4x_1 & + & 4x_2 & \geq 5 \\
 & 0 & \leq & x_1 & \leq 2 \\
 & & & x_2 & \geq 0
\end{array}
$$



(Note: the following subtasks can be carried out independently; use fractional mode for numerical calculations)

a. The polyhedron representing the feasibility region is depicted in the figure. Indicate for each of the four points represented whether they are feasible and/or basic solutions. Justify your answer.

Solution:

- Point 1 is a feasible solution but not basic (no constraint is active in that point).
- Point 2 is a feasible solution but not basic (only one constraint is active while two are needed)
- Point 3 is a basic feasible solution
- Point 4 is a basic solution (combination of two active constraints) but non feasible.

b. Write the initial tableau or dictionary for the simplex method. Write the corresponding basic solution and its value. State whether the solution is feasible or not and whether it is optimal or not.

Solution:

$$\begin{array}{rrcrl} z = \max & x_1 & + & 5x_2 & \\ \text{s.t.} & -x_1 & + & 3x_2 & \leq 6 \\ & -4x_1 & - & 4x_2 & \leq -5 \\ & & & x_1 & \leq 2 \\ & & & x_1 & \geq 0 \\ & & & x_2 & \geq 0 \end{array}$$

	x1	x2	x3	x4	x5	-z	b
I	-1	3	1	0	0	0	6
II	-4	-4	0	1	0	0	-5
III	1	0	0	0	1	0	2
IV	1	5	0	0	0	1	0

The basic solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $x_4 = -5$ and $x_5 = 2$. Its value is 0. The solution is not feasible.

c. Consider the following tableau:

	x1	x2	x3	x4	x5	-z	b
I	0	4	1	-1/4	0	0	29/4
II	1	1	0	-1/4	0	0	5/4
III	0	-1	0	1/4	1	0	3/4
IV	0	4	0	1/4	0	1	-5/4

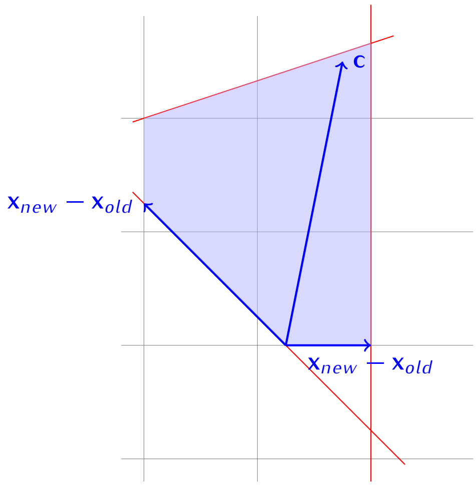
and the following three pivoting rules:

- largest coefficient
- largest increase
- steepest edge.

Which entering and leaving variables would each of them indicate? In this specific case, which rule would be convenient to follow? Report the details of the computations for the first two rules and carry out graphically the application of the third rule using the plot in the figure above (tikz code to reproduce the figure available in the online version.)

Solution:

- The two candidate entering variables are $x2$ and $x4$. The reduced cost of $x2$ is larger hence that is the entering variable. The leaving variable is consequently given by the ratio test and is $x1$ since $5/4 < 29/16$.
- The two candidate entering variables are $x2$ and $x4$. The increase possible with $x2$ is $\min\{29/4 \cdot 1/4, 5/4 \cdot 1\} \cdot 4 = 5$ while the increase with $x4$ is $\min\{3/4 \cdot 4\} \cdot 1/4 = 3/4$. Hence $x2$ is the entering variable and the leaving variable is $x1$.
- In the figure we plot the vector **c** which is the perpendicular to the objective function and the two vectors corresponding to the movement we would take by the iteration of the simplex. The angle between **c** and $\mathbf{x}_{new} - \mathbf{x}_{old}$ is smaller for the decision x_2 entering x_1 leaving.



None of the three rules is convenient, the best would be to let $x4$ enter and $x5$ leave, we would reach the optimal solution in less iterations.

Exercise 9*

The two following LP problems lead to two particular cases when solved by the simplex algorithm. Identify these cases and characterize them, that is, give indication of which conditions generate them in general.

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_2 \leq 5 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution:
In the initial tableau of the first problem there is a column, the first one, with positive reduced cost and no positive a_{ij} term. This means that the corresponding variable x_1 can be brought into the basis but the increase of its value is unlimited. This indicates that we have an unbounded problem.
The second LP problem is developed in the slides for the lecture on exception handling. After some iterations we reach a tableau in which a non basic variable has reduced cost zero. This indicates that it can be brought in the basis without a change in the objective function. Since the solution changes when we bring the variable in basis then the problem has more than one solution and it has therefore infinite solutions. They can be expressed as the convex combination of all optimal basic solutions.

Exercise 10

Consider the following problem:

$$\begin{array}{rl} \max & z = 4x_2 \\ \text{s.t.} & 2x_2 \geq 0 \\ & -3x_1 + 4x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

a. Write the LP in equational standard form and say why it does not provide immediately an initial feasible basis for the simplex method.

Solution:

In the equational standard form we have a negative b term. The implication of this is that the initial solution of the simplex is infesible because $x_B \ngeq 0$. If we try to make the term positive we end up not having an identity matrix in the tableau.

b. To overcome the situation of infeasible basis construct the auxiliary problem for a phase I-phase II solution approach. Determine which variables are initially in basis and which are not in basis in the auxiliary problem.

c. Answer the following questions

   i) Is the initial basis in the auxiliary problem feasible in the original problem?
   ii) Is it optimal in the auxiliary problem?
   iii) Is it degenerate?
   iv) Can we say at this stage if phase I will terminate?
   v) If it will terminate, can we say at this stage that it will terminate with a basis that corresponds to a feasible solution in the original problem?
   vi) Solve the problem by carrying out Phase I and Phase II of the simplex algorithm.