DM545/DM871

Linear and Integer Programming

Introduction to Linear Programming
Notation and Modeling

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Outline

1. Course Organization

2. Preliminaries

3. Introduction: Operations Research
    - Resource Allocation
    - Duality

Who is here?

54 in total registered in ItsLearning

DM545 (5 ECTS)
27, who??

- Math-economics (Bachelor, 3th semester)
- Others?

Prerequisites

- Programming
- Linear Algebra

DM871 (5 ECTS)
27, who??

- Computer Science (Master)
- Mathematics (Bachelor/Master)
- Applied Mathematics (Bachelor/Master)
- Data Science (Master)
- Others?

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Aims of the course

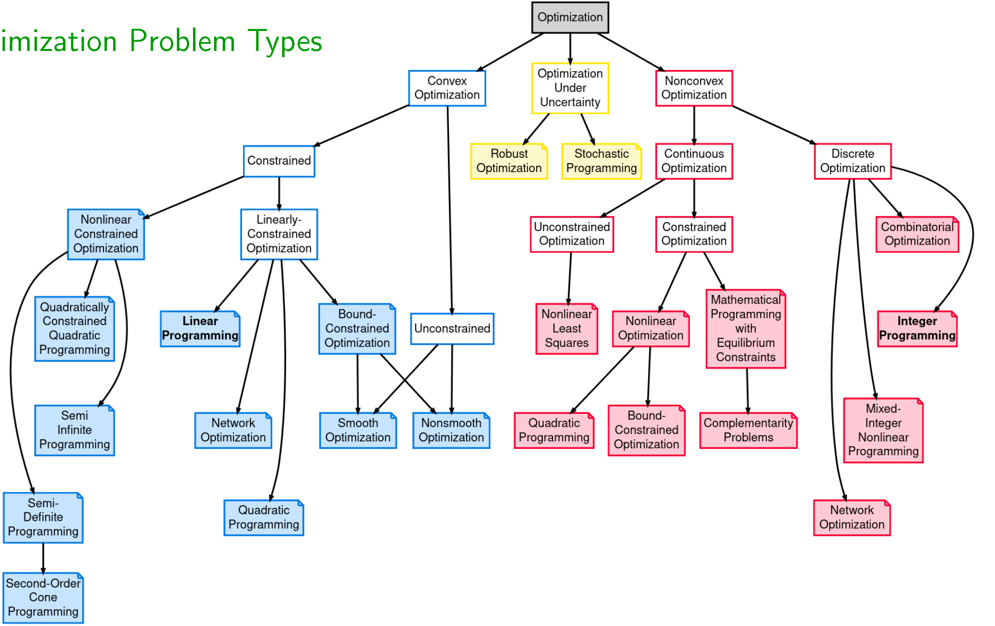
Learn about **mathematical optimization**:

- linear programming (continuous linear optimization)
- integer linear programming (discrete linear optimization)

Optimization is an important tool in decision making and in analyzing physical systems.
In mathematical terms, an optimization problem is the problem of finding the best solution from the set of all feasible solutions.

The first step in the optimization process is constructing an appropriate mathematical model

Optimization Problem Types



(NEOS Server, University of Wisconsin)

Contents of the Course (aka Syllabus)

Linear Programming

1 Introduction - Linear Programming, Notation & Modeling
2 Linear Programming, Simplex Method
3 Exception Handling
4 Duality Theory
5 Sensitivity
6 Revised Simplex Method

Integer Linear Programming

7 Modeling Examples, Good Formulations, Relaxations
8 Well Solved Problems
9 Network Optimization Models (Max Flow, Min cost flow, Matching)
10 Cutting Planes & Branch and Bound
11 More on Modeling

Practical Information

Teacher: Marco Chiarandini (imada.sdu.dk/u/marco/)
Instructors: ... (M1)
... (H1)

Schedule, alternative views:

- mitsdu.sdu.dk, SDU Mobile
- Official course description (læserplanen)
- ItsLearning
- https://dm871.github.io/

Schedule (14 weeks):

- Introductory classes: 28 hours (14 classes)
- Training classes: 16 hours (8 classes)

Communication Means

- ItsLearning $\Leftrightarrow$ External Web Page (link `https://dm871.github.io`)
- Announcements in ItsLearning
- Write to Marco (`marco@imada.sdu.dk`) or to instructor
- Ask peers

$\rightsquigarrow$ It is good to ask questions!!

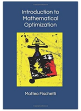
$\rightsquigarrow$ Let me know if you think we should do things differently!

Sources — Reading Material

Main references:

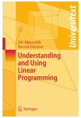
[LN] Lecture Notes (continously updated)

[F] M. Fischetti, Introduction to Mathematical Optimization, Independently published, 2019



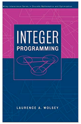
Linear Programming:

[VA] R. Vanderbei. Linear Programming: Foundations and Extensions. Springer US, 2008



Integer Programming:

[Wo] L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998



Others

[HL] Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

Course Material

External Web Page is the main reference for list of contents (ie[1], syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- tutorials for programming tasks

[1] ie = id est = that is, eg = exempli gratia = for example, wrt = with respect to, et al. = et alii = and others

Assessment

- Two obligatory 24 h Take-Home Assignments, evaluation by internal censor
  - individual !!
  - exercises similar to previous 4 hour written exams
  - style: short answers about calculations and modeling. Differences between DM871 and DM545
  - (language: Danish and/or English)

- Final grade: overall evaluation but as starting point the average grade rounded up

- Tentative plan:
  - Test 1 (about weeks 36, 38, 40, 43) in week 45 or 46
  - Test 2 (about weeks ..., 45, 47, 49) in week 50 or 51

  Which day? Which time range? (Pool coming in ItsLearning)

Training Sessions

- Do the plus exercises in advance
- Prepare the starred exercises in advance to get out the most from the session
- Participate actively in class
- Try the others, unsolved in class, after the session
- Best if carried out in small groups
- (Starred) Exercises are examples of exam questions (but not only!)

Training Session Organization

Exercise classes are about material presented the week before.

Concepts from Linear Algebra

Linear Algebra:
manipulation of matrices and vectors with some theoretical background

Linear Algebra

- Matrices and vectors - Matrix algebra
- Dot (scalar, Euclidean inner) product
- Geometric insights
- Systems of Linear Equations - Row echelon form, Gaussian elimination
- Matrix inversion and determinants
- Rank and linear dependency

Coding

- gives you the ability to create new and useful artifacts
- allows you to have more control of your world as more and more of it becomes digital
- is just fun.

It can also help you to understand math.

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand
- learn by doing, interacting with Python.

- Python 3.11+
  Commercial software: Gurobi or Cplex or Xpress $\approx$ 100 000 Dkk Open source: Pyomo, Python-MIP, PySCIPOpt: Python interfaces to SCIP Optimization Suite
- ipython, jupyter, jupyterLab (= interactive python)? Or Visual Code or Spyder3.

Outline

Sets

- A set is a collection of objects. eg.: $A = \{1, 2, 3\}$
- $A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}$
  ('$|$' reads 'such that')
- $B = \{x \mid x \text{ is a student of this course}\}$
- $x \in A$
  x belongs to A
- set of no members: empty set, denoted $\emptyset$
- if a set S is a (proper) subset of a set T, we write ($S \subset T$) $T \subseteq S$
  $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 30\}$
- for two sets A and B, the union $A \cup B$ is $\{x \mid x \in A \text{ or } x \in B\}$
- for two sets A and B, the intersection $A \cap B$ is $\{x \mid x \in A \text{ and } x \in B\}$
  $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cap B = \{2, 5\}$

Numbers

- set of real numbers: $\mathbb{R}$
- set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, ...\}$ (positive integers); $\mathbb{N}_0$ to include zero
- set of all integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$; $\mathbb{Z}_0^+$ only positives and zero
- set of rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
- set of complex numbers: $\mathbb{C}$
- absolute value (non-negative):

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

- the set $\mathbb{R}^2$ is the set of ordered pairs (x, y) of real numbers (eg, coordinates of a point wrt a pair of axes, the Cartesian plane)

Matrices and Vectors

- A matrix is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- An $n \times 1$ matrix is a column vector, or simply a vector:

$$\mathrm{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- the set $\mathbb{R}^n$ is the set of vectors $[x_1, x_2, \ldots, x_n]^T$ of real numbers (eg, coordinates of a point wrt an n-dimensional space, the Euclidean Space)

Graphs

- An undirected graph G=(V,E) is a structure made of a pair of sets: V the set of vertices and $E \subseteq \{e = \{u, v\} \mid u, v \in V\}$ the set of edges (undirected links)
- A directed graph (digraph) $D = (V, A)$ is a pair of sets: V the set of vertices and $A \subseteq \{uv = (u, v) \mid u, v \in V\}$ the set of arcs (directed links)
- Graph drawing (2D embeddings)
- Labelled graphs and weighted graphs
- Walks, Paths, Cycles, Connectedness
- Problems on graphs: eg, determining intrinsic properties such as shortest st-path
- Matrix representations: adjacency matrix, incidence matrix

- adjacency set (neighborhood set) for $v \in V$: $N(v) = \{u \mid u \in V, uv \in E\}$; vertex degree (number of incident edges) $\delta(v)$; outdegree (number of outgoing edges), $\delta^+(v)$; indegree (number of incoming edges), $\delta^-(v)$;

- subgraph and induced subgraph $G' = G[V] \subseteq G$

- cut: set of edges that separates a connected graphs into two connected components $G[X]$, $G[Y]$: $C = \{xy \mid x \in X, y \in Y\}$

- bipartite graphs $G = (U, V, E)$, trees ($=$ connected acycle graphs)

- multi-graphs, hypergraphs, mixed graphs

- Graphs are the basic subject studied by graph theory. The word "graph" was first used in this sense by J.J. Sylvester in 1878 due to a direct relation between mathematics and chemical structure.

Basic Algebra

Elementary Algebra: the study of symbols and the rules for manipulating symbols. It differs from arithmetic in the use of abstractions, such as using letters to stand for numbers that are either unknown or allowed to take on many values

- collecting up terms: eg. $2a + 3b - a + 5b = a + 8b$
- multiplication of variables: eg:

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms: eg:

$$\begin{aligned} -(a-2b) &= -a + 2b, \\ (2x-3y)(x+4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2 \end{aligned}$$

- $a^r a^s = a^{r+s}, \quad (a^r)^s = a^{rs}, \quad a^{-n} = 1/a^n,$
  $a^{m/n} = (a^{1/n})^m, \quad a^{1/n} = x \iff x^n = a, \quad a^x = n \iff x = \log_a n$

Variables

- In Mathematics and Statistics, a variable is an alphabetic character representing a value, which is unknown. They are used in symbolic calculations.
  Commonly given one-character names.

- in contrast, a parameter or constant or given is a known real number

- in contrast, in Computer Science, a variable is a storage location paired with an associated identifier, which contains a value that may be known or unknown.
  Commonly given long, explanatory names.

Functions

- a function f on a set $\mathcal{X}$ into a set $\mathcal{Y}$ is a rule that assigns a unique element $f(x)$ in $\mathcal{Y}$ to each element x in $\mathcal{X}$.

$$y = f(x)$$

y dependent variable

x independent variable

- scalar: element of a field (eg, $\mathbb{R}$) (vector of a single component)

- a linear function is a polynomial of degree one or less.
If only one variable:

$$f(x) := ax + b$$

where a and b are constants, often real numbers. The graph is a nonvertical line (a is the slope and b the intercept).
For any finite number of variables: $f(x_1, \ldots, x_k) = b + a_1x_1 + \ldots + a_kx_k$ and the graph is a hyperplane of dimension k. (actually the ones above are affine functions)

Outline

What is Operations Research?

Operations Research (aka, Management Science, Analytics):
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system,
usually under conditions requiring the allocation of scarce resources,
by means of **mathematics** and **computer science**.

Quantitative methods for planning and analysis.

It encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency:

- simulation,
- mathematical optimization,
- queueing theory and other stochastic-process models,
- Markov decision processes
- econometric methods,
- data envelopment analysis,
- machine learning,
- expert systems

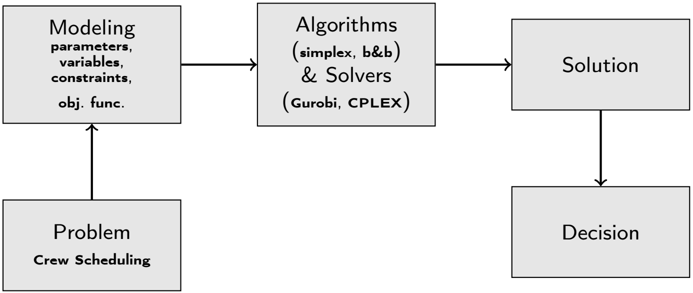
Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
  - Cheapest
  - Shortest route
  - Fewest number of people
- Not all plans are feasible - there are constraining rules
  - Limited amount of available resources
- It can be extremely difficult to figure out what to do

OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Central Idea

Build a mathematical model describing exactly what one wants, and what the "rules of the game" are. However, **what is a mathematical model and how?**

Mathematical Modeling

- Find out exactly what the decision maker needs to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Parameters, that is the given, known elements of the problem.
- Define Decision Variables of suitable type (continuous, integer, binary) corresponding to the needs
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

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   - Duality

Resource Allocation

In manufacturing industry, factory planning: find the best product mix.

Example

A factory makes two products standard and deluxe. Eg, yougurt, sleeping beds, etc.

A unit of standard gives a profit of 6(k) Dkk.
A unit of deluxe gives a profit of 8(k) Dkk.

The warming|grinding and cooling|polishing times in terms of hours per week for a unit of each type of product are given below:

	Standard	Deluxe
(Machine 1) Warming|Grinding	5	10
(Machine 2) Cooling|Polishing	4	4

Warming|Grinding capacity: 60 hours per week
Cooling|Polishing capacity: 40 hours per week

Q: How much of each product, standard and deluxe, should we produce to maximize the profit?

Mathematical Model

Decision Variables

$x_1 \geq 0$ units of product standard
$x_2 \geq 0$ units of product deluxe

Object Function

$\max\ 6x_1 + 8x_2$ maximize profit

Constraints

$$\begin{array}{rcrcll} 5x_1 & + & 10x_2 & \leq & 60 & \text{machine 1 capacity} \\ 4x_1 & + & 4x_2 & \leq & 40 & \text{machine 2 capacity} \end{array}$$

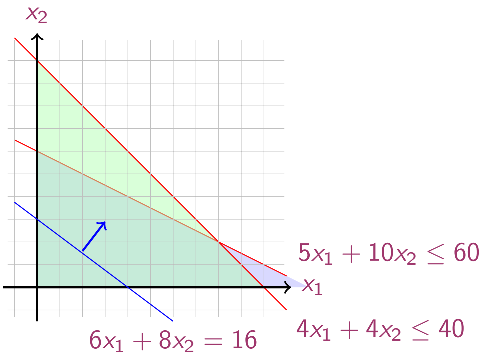
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1 & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$j = 1, 2, \ldots, n$	products
$i = 1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}$	profit per unit of product j
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1 x_1 & + & c_2 x_2 & + & c_3 x_3 & + & \ldots & + & c_n x_n & = & z \\
\text{subject to} & a_{11} x_1 & + & a_{12} x_2 & + & a_{13} x_3 & + & \ldots & + & a_{1n} x_n & \leq & b_1 \\
& a_{21} x_1 & + & a_{22} x_2 & + & a_{23} x_3 & + & \ldots & + & a_{2n} x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1} x_1 & + & a_{m2} x_2 & + & a_{m3} x_3 & + & \ldots & + & a_{mn} x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1 x_1 & + & c_2 x_2 & + & c_3 x_3 & + & \ldots & + & c_n x_n & = & z \\
\text{s.t.} & a_{11} x_1 & + & a_{12} x_2 & + & a_{13} x_3 & + & \ldots & + & a_{1n} x_n & \leq & b_1 \\
& a_{21} x_1 & + & a_{22} x_2 & + & a_{23} x_3 & + & \ldots & + & a_{2n} x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1} x_1 & + & a_{m2} x_2 & + & a_{m3} x_3 & + & \ldots & + & a_{mn} x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{rl}
\max & \displaystyle\sum_{j=1}^{n} c_j x_j \\
& \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{array}
$$

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & \mathbf{c}^T\mathbf{x} \\
& A\mathbf{x} & \leq & \mathbf{b} \\
& \mathbf{x} & \geq & 0
\end{array}
$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max \quad 6x_1 &+ 8x_2 \\ 5x_1 &+ 10x_2 \leq 60 \\ 4x_1 &+ 4x_2 \leq 40 \\ x_1&, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ A\boldsymbol{x} & \leq \boldsymbol{b} \\ \boldsymbol{x} & \geq 0 \end{aligned}$$

$$\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

Duality

Resource Valuation problem: Determine the value of the raw materials in hand such that:
(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

z_i	value of a unit of raw material i
$\sum_{i=1}^{m} b_i z_i$	total expenses for buying (or opportunity cost, cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal: for the outside company to minimize the total expenses;
(for the owing company the minimum amount of opportunity cost to accept for selling)

$$\min \sum_{i=1}^{m} b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} z_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by selling the raw material instead of producing.

$$\min \sum_{i=1}^{m} y_i b_i + \cancel{\sum_i \rho_i b_i}$$
$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1 \ldots n$$
$$y_i \geq 0, \quad i = 1 \ldots m$$

Dual Problem

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m$$
$$x_j \geq 0, \quad j = 1, \ldots, n$$

Primal Problem