

DM545/DM871  
Linear and Integer Programming

Lecture 3  
**The Simplex Method**

Marco Chiarandini

Department of Mathematics & Computer Science  
University of Southern Denmark

## 1. Simplex Method

- Standard Form

- Basic Feasible Solutions

- Algorithm

- Tableaux and Dictionaries

## 1. Simplex Method

- Standard Form

- Basic Feasible Solutions

- Algorithm

- Tableaux and Dictionaries

# A Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

## 1. Simplex Method

- Standard Form

- Basic Feasible Solutions

- Algorithm

- Tableaux and Dictionaries

Every LP problem can be converted in the **standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

$$\mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

- if equations, then put two constraints,  $\mathbf{a}\mathbf{x} \leq \mathbf{b}$  and  $\mathbf{a}\mathbf{x} \geq \mathbf{b}$
- if  $\mathbf{a}\mathbf{x} \geq \mathbf{b}$  then  $-\mathbf{a}\mathbf{x} \leq -\mathbf{b}$
- if  $\min \mathbf{c}^T \mathbf{x}$  then  $-\max(-\mathbf{c}^T \mathbf{x})$

and then be put in **equational standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

1. “=” constraints
2.  $\mathbf{x} \geq 0$  nonnegativity constraints
3. ( $\mathbf{b} \geq 0$ )
4. max

# Transformation to Std Form

Every LP problem can be transformed in eq. std. form

1. introduce slack variables (or surplus)

$$5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 + x_4 = 40$$

2. if  $x_1 \geq 0$  then  $x_1 = x'_1 - x''_1$   
 $x'_1 \geq 0$   
 $x''_1 \geq 0$

3. ( $b \geq 0$ )

4.  $\min c^T x \equiv -\max(-c^T x)$

LP in  $m \times n$  converted into LP with at most  $(m + 2n)$  variables and  $m$  equations ( $n \neq$  original variables,  $m \neq$  constraints)

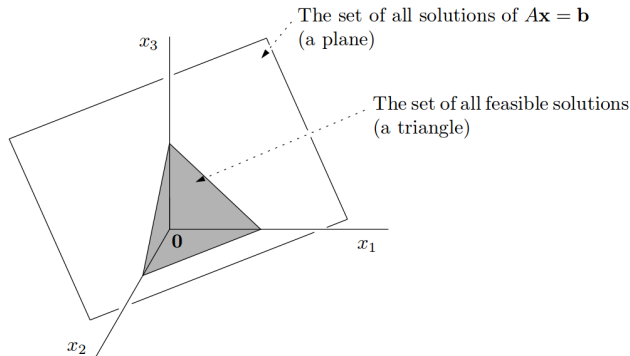
# Geometry of LP in Eq. Std. Form

$$\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$$

In  $\mathbb{R}^3$ :

From linear algebra:

- the set of solutions of  $A\mathbf{x} = \mathbf{b}$  is an affine space (hyperplane not passing through the origin).
- $\mathbf{x} \geq 0$  nonnegative orthant (octant in  $\mathbb{R}^3$ )





- $Ax = b$  is a system of equations that we can solve by Gaussian elimination
- Elementary row operations of  $[A \mid b]$  do not affect set of feasible solutions
  - multiplying all entries in some row of  $[A \mid b]$  by a nonzero real number  $\lambda$
  - replacing the  $i$ th row of  $[A \mid b]$  by the sum of the  $i$ th row and  $j$ th row for some  $i \neq j$
- Let  $n'$  be the number of vars in eq. std. form.

we assume  $n' \geq m$  and  $\text{rank}([A \mid b]) = \text{rank}(A) = m$

ie, rows of  $A$  are linearly independent  
otherwise, remove linear dependent rows

## 1. Simplex Method

Standard Form

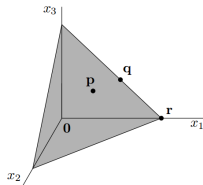
Basic Feasible Solutions

Algorithm

Tableaux and Dictionaries

# Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:

Let  $B = \{1 \dots m\}$ ,  $N = \{m+1 \dots n+m = n'\}$  be subsets partitioning the columns of  $A$ :  $A_B$  be made of columns of  $A$  indexed by  $B$ :

## Definition

$\mathbf{x} \in \mathbb{R}^n$  is a **basic feasible solution** of the linear program  $\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$  for an index set  $B$  if:

- $x_j = 0 \ \forall j \notin B$
- the square matrix  $A_B$  is nonsingular, ie, all columns indexed by  $B$  are lin. indep.
- $\mathbf{x}_B = A_B^{-1} \mathbf{b}$  is nonnegative, ie,  $\mathbf{x}_B \geq 0$  (feasibility)

We call  $x_j$  for  $j \in B$  **basic variables** and remaining variables **nonbasic variables**.

### Theorem

A **basic feasible solution** is uniquely determined by the set  $B$ .

Proof:

$$Ax = A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$x_B = A_B^{-1} b$$

$A_B$  is nonsingular hence one solution

Note: we call  $B$  a **(feasible) basis**

### Definition

A **basic feasible solution** of a linear program with  $n$  variables is a feasible solution for which some  $n$  linearly independent constraints hold with equality.

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

### Theorem

Let  $P$  be a (convex) polyhedron made of all feasible solutions of an LP in eq. std. form. For a point  $v \in P$  the following are equivalent:

- (i)  $v$  is an extreme point (vertex) of  $P$
- (ii)  $v$  is a basic feasible solution of LP

Proof: see text book [MG] sec. 4.4.

## Theorem

Let  $LP = \max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$  be feasible and bounded,  
then there is an optimal solution that is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming

A similar theorem is valid for arbitrary linear programs (not in eq. form)

However, an optimal solution does not need to be basic:

$$\max x_1 + x_2 \text{ subject to } x_1 + x_2 \leq 1$$

(unbounded case with infinite solutions)

- Idea for solution method:
- examine all basic solutions.
- There are finitely many:  $\binom{m+n}{m}$ .
- However, if  $n = m$  then  $\binom{2m}{m} \approx 4^m$ .

## 1. Simplex Method

Standard Form

Basic Feasible Solutions

**Algorithm**

Tableaux and Dictionaries



# Simplex Method

$$\max \quad z = [6 \ 8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical eq. std. form: one decision variable is isolated in each constraint with coefficient 1 and does not appear in the other constraints nor in the obj. func. and  $b$  terms are positive

It gives immediately a basic feasible solution:

$$x_1 = 0, x_2 = 0, x_3 = 60, x_4 = 40$$

Is it optimal? Look at signs in  $z \rightsquigarrow$  if positive then an increase would improve.

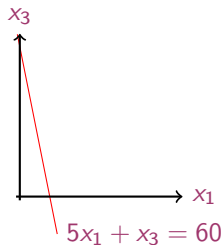
Let's try to increase a promising variable, ie,  $x_1$ , one with positive coefficient in  $z$

$$5x_1 + x_3 = 60$$

$$x_1 = \frac{60}{5} - \frac{x_3}{5}$$

$$x_3 = 60 - 5x_1 \geq 0$$

If  $x_1 > 12$  then  $x_3 < 0$

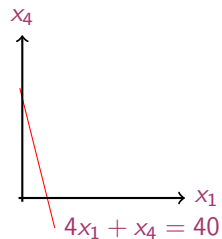


$$4x_1 + x_4 = 40$$

$$x_1 = \frac{40}{4} - \frac{x_4}{4}$$

$$x_4 = 40 - 4x_1 \geq 0$$

If  $x_1 > 10$  then  $x_4 < 0$



we can take the minimum of the two  $\rightsquigarrow x_1$  increased to 10  
 $x_1$  enters the basis and  $x_4$  leaves it.

Hence, we increase  $x_1$  to 10 and consequently set  $x_4 = 0$   
That is,  $x_1$  enters the basis and  $x_4$  leaves it.

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 + x_3 = 60 \\ & 4x_1 + 4x_2 + x_4 = 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{array}{rcl} x_3 & = & 60 - 5x_1 - 10x_2 \\ x_4 & = & 40 - 4x_1 - 4x_2 \\ \hline z & = & \phantom{00} + 6x_1 + 8x_2 \end{array}$$

$$\begin{array}{rcl} x_3 & = & 10 - 5x_2 + 5/4x_4 \\ x_1 & = & 10 - x_2 - 1/4x_4 \\ \hline z & = & 60 + 2x_2 - 6/4x_4 \end{array}$$

# Simplex Tableau

First simplex tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
$x_3$	5	10	1	0	0	60
$x_4$	4	4	0	1	0	40
	6	8	0	0	1	0

we want to reach this new tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
$x_3$	0	?	1	?	0	?
$x_1$	1	?	0	?	0	?
	0	?	0	?	1	?

Pivot operation:

1. Choose pivot:

column: one  $s$  with positive coefficient in obj. func.

row: ratio between coefficient  $b$  and pivot column: choose the one with smallest ratio:

$$\theta = \min_i \left\{ \frac{b_i}{a_{is}} : a_{is} > 0 \right\}, \quad \theta \text{ increase value of entering var.}$$

2. elementary row operations to update the tableau

- $x_4$  leaves the basis,  $x_1$  enters the basis
  - Divide pivot row by pivot
  - Send to zero the coefficient in the pivot column of the first row
  - Send to zero the coefficient of the pivot column in the third (cost) row

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
I' = I - 5II'	0	5	1	-5/4	0	10
II' = II/4	1	1	0	1/4	0	10
III' = III - 6II'	0	2	0	-6/4	1	-60

From the last row we read:  $2x_2 - 3/2x_4 - z = -60$ , that is:  $z = 60 + 2x_2 - 3/2x_4$ .  
 Since  $x_2$  and  $x_4$  are nonbasic we have  $z = 60$  and  $x_1 = 10, x_2 = 0, x_3 = 10, x_4 = 0$ .

- Done? No! Let  $x_2$  enter the basis

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
I' = I/5	0	1	1/5	-1/4	0	2
II' = II - I'	1	0	-1/5	1/2	0	8
III' = III - 2I'	0	0	-2/5	-1	1	-64

### Definition (Reduced costs)

We call **reduced costs** the coefficients in the objective function of the nonbasic variables,  $\bar{c}_N$

### Proposition (Optimality Condition)

The basic feasible solution is **optimal** when the **reduced costs** in the corresponding simplex tableau are **nonpositive**, ie, such that:

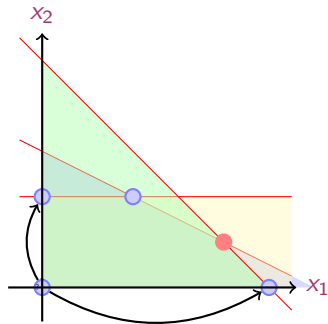
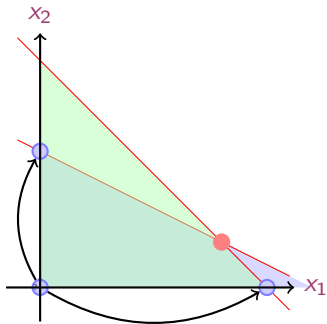
$$\bar{c}_N \leq 0$$

Proof: Let  $z_0$  be the obj value when  $\bar{c}_N \leq 0$ .

For any other feasible solution  $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_B, \tilde{\mathbf{x}}_N]$ ,  $\tilde{\mathbf{x}}_N \geq 0$  reachable from the tableau we have:

$$z = \mathbf{c}^T \tilde{\mathbf{x}} \quad \text{and from the last line of the tableau} \quad \mathbf{c}^T \tilde{\mathbf{x}} = z = z_0 + \bar{\mathbf{c}}_B^T \tilde{\mathbf{x}}_B + \bar{\mathbf{c}}_N^T \tilde{\mathbf{x}}_N \leq z_0$$

# Graphical Representation



## 1. Simplex Method

Standard Form

Basic Feasible Solutions

Algorithm

Tableaux and Dictionaries



# Tableaux and Dictionaries

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \sum_{j=1}^n a_{ij} x_j & \leq b_i, \quad i = 1, \dots, m \\ x_j & \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j, \quad i = 1, \dots, m$$

$$z = \sum_{j=1}^n c_j x_j$$

Tableau

$$\left[ \begin{array}{c|c|c|c} I & \bar{A}_N & 0 & \bar{b} \\ \hline 0 & \bar{c}_N & 1 & -\bar{d} \end{array} \right]$$

Dictionary

$$\begin{aligned} x_r &= \bar{b}_r - \sum_{s \notin B} \bar{a}_{rs} x_s, \quad r \in B \\ z &= \bar{d} + \sum_{s \notin B} \bar{c}_s x_s \end{aligned}$$

pivot operations in dictionary form:

choose col  $s$  with r.c.  $> 0$

choose row with  $\min\{-\bar{b}_i/\bar{a}_{is} \mid \bar{a}_{is} < 0, i = 1, \dots, m\}$

update: express entering variable and substitute in other rows

# Example

$$\begin{aligned}
 \max \quad & 6x_1 + 8x_2 \\
 \text{s.t.} \quad & 5x_1 + 10x_2 \leq 60 \\
 & 4x_1 + 4x_2 \leq 40 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
$x_3$	5	10	1	0	0	60
$x_4$	4	4	0	1	0	40
	6	8	0	0	1	0

$$x_3 = 60 - 5x_1 - 10x_2$$

$$x_4 = 40 - 4x_1 - 4x_2$$

$$z = 64 + 6x_1 + 8x_2$$

After 2 iterations:

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
$x_2$	0	1	$1/5$	$-1/4$	0	2
$x_1$	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

$$x_2 = 2 - 1/5x_3 + 1/4x_4$$

$$x_1 = 8 + 1/5x_3 - 1/2x_4$$

$$z = 64 - 2/5x_3 - 1x_4$$

## 1. Simplex Method

- Standard Form

- Basic Feasible Solutions

- Algorithm

- Tableaux and Dictionaries