

DM545/DM871  
Linear and Integer Programming

Lecture 5  
Duality

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1. Derivation and Motivation

2. Theory

# Outline

1. Derivation and Motivation

2. Theory

Dual variables  $\mathbf{y}$  in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad & z = \mathbf{c}^T \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad & w = \mathbf{b}^T \mathbf{y} \\ & A^T \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

# Bounding approach

$$\begin{aligned} z^* &= \max 4x_1 + x_2 + 3x_3 \\ x_1 + 4x_2 &\leq 1 \\ 3x_1 + x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

a feasible solution is a **lower bound** but how good?

By tentatives:

$$(x_1, x_2, x_3) = (1, 0, 0) \rightsquigarrow z^* \geq 4$$

$$(x_1, x_2, x_3) = (0, 0, 3) \rightsquigarrow z^* \geq 9$$

What about **upper bounds**?

$$\begin{array}{rcl} 2 \cdot (x_1 + 4x_2) & \leq & 2 \cdot 1 \\ + 3 \cdot (3x_1 + x_2 + x_3) & \leq & 3 \cdot 3 \\ \hline 4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 11x_2 + 3x_3 \leq 11 \end{array}$$
$$\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T \mathbf{A} \mathbf{x} \leq \mathbf{y}^T \mathbf{b}$$

Hence  $z^* \leq 11$ . Is this the best upper bound we can find?

multipliers  $y_1, y_2 \geq 0$  that preserve sign of inequality

$$\begin{array}{rcl} y_1 \cdot (x_1 + 4x_2) & \leq & y_1(1) \\ y_2 \cdot (3x_1 + x_2 + x_3) & \leq & y_2(3) \\ \hline (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 & \leq & y_1 + 3y_2 \end{array}$$

Coefficients

$$\begin{array}{rcl} y_1 + 3y_2 & \geq & 4 \\ 4y_1 + y_2 & \geq & 1 \\ y_2 & \geq & 3 \end{array}$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 \leq y_1 + 3y_2$   
then to attain the best upper bound:

$$\begin{array}{rcl} \min & y_1 + 3y_2 \\ y_1 + 3y_2 & \geq & 4 \\ 4y_1 + y_2 & \geq & 1 \\ y_2 & \geq & 3 \\ y_1, y_2 & \geq & 0 \end{array}$$

# Multipliers Approach

$$\begin{array}{l} \pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1} \end{array} \left[ \begin{array}{cccc|cccc|c|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\ \vdots & \vdots & & \ddots & & & & & & \\ \vdots & \vdots & & \vdots & & & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 0 & b_m \\ \hline c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0 \end{array} \right]$$

Working columnwise, since at optimum  $\bar{c}_k \leq 0$  for all  $k = 1, \dots, n + m$ :

$$\left\{ \begin{array}{l} \pi_1 a_{11} + \pi_2 a_{21} + \dots + \pi_m a_{m1} + \pi_{m+1} c_1 \leq 0 \\ \vdots \\ \pi_1 a_{1n} + \pi_2 a_{2n} + \dots + \pi_m a_{mn} + \pi_{m+1} c_n \leq 0 \\ \pi_1 a_{1,n+1}, \quad \pi_2 a_{2,n+1}, \dots, \pi_m a_{m,n+1} \leq 0 \\ \vdots \\ \pi_1 a_{1,n+m}, \quad \pi_2 a_{2,n+m}, \dots, \pi_m a_{m,n+m} \leq 0 \\ \pi_{m+1} = 1 \\ \pi_1 b_1 + \pi_2 b_2 + \dots + \pi_m b_m \quad (\leq 0) \end{array} \right.$$

(from the last row we have also  $z = -\pi b$ )

[illegible]

$$y = -\pi$$

$$\begin{array}{rcl} -z & = & (-y_1 b_1) + (-y_2 b_2) \dots + (-y_m b_m) \\ & & (-y_1 a_{11}) + (-y_2 a_{21}) \dots + (-y_m a_{m1}) \leq -c_1 \\ & & \vdots \qquad \ddots \qquad \qquad \qquad \vdots \\ & & (-y_1 a_{1n}) + (-y_2 a_{2n}) \dots + (-y_m a_{mn}) \leq -c_n \\ & & -y_1, -y_2, \dots -y_m \leq 0 \end{array}$$

as we will see  $\mathbf{b}^T \mathbf{y} \geq \mathbf{c}^T \mathbf{x}$ , hence it is more interesting to minimize. It yields:

$$\begin{aligned} \min \quad & \mathbf{b}^T \mathbf{y} \\ & \mathbf{A}^T \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$



# Example

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ \text{s.t.} \quad & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{cases} 5\pi_1 + 4\pi_2 + 6\pi_3 \leq 0 \\ 10\pi_1 + 4\pi_2 + 8\pi_3 \leq 0 \\ 1\pi_1 + 0\pi_2 + 0\pi_3 \leq 0 \\ 0\pi_1 + 1\pi_2 + 0\pi_3 \leq 0 \\ 0\pi_1 + 0\pi_2 + 1\pi_3 = 1 \\ 60\pi_1 + 40\pi_2 \end{cases}$$

$$y_1 = -\pi_1 \geq 0$$

$$y_2 = -\pi_2 \geq 0$$

# Duality Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \dots, x_n$	$y_1, y_2, \dots, y_m$
Matrix	$A$	$A^T$
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
Constraints	$i$ th constraint has $\leq$ $\geq$ $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
	$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	$j$ th constraint has $\geq$ $\leq$ $=$

1. Derivation and Motivation

2. Theory

The dual of the dual is the primal:

Primal Problem:

$$\begin{aligned} \max \quad & z = c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Let's put the dual in the standard form

Dual Problem:

$$\begin{aligned} \min \quad & b^T y \equiv -\max -b^T y \\ & -A^T y \leq -c \\ & y \geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad & w = b^T y \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Dual of Dual:

$$\begin{aligned} & -\min -c^T x \\ & -Ax \geq -b \\ & x \geq 0 \end{aligned}$$

# Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

## Theorem (Weak Duality Theorem)

Given:

$$(P) \max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$$

$$(D) \min\{\mathbf{b}^T \mathbf{y} \mid A^T \mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq 0\}$$

for any feasible solution  $\mathbf{x}$  of (P) and any feasible solution  $\mathbf{y}$  of (D):

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$$

Proof:

From (D)  $c_j \leq \sum_{i=1}^m y_i a_{ij} \forall j$  and from (P)  $\sum_{j=1}^n a_{ij} x_j \leq b_i \forall i$

From (D)  $y_i \geq 0$  and from (P)  $x_j \geq 0$

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left( \sum_{i=1}^m y_i a_{ij} \right) x_j = \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j \right) y_i \leq \sum_{i=1}^m b_i y_i$$

# Strong Duality Theorem

Due to Von Neumann and Dantzig 1947 and Gale, Kuhn and Tucker 1951.

## Theorem (Strong Duality Theorem)

Given:

$$(P) \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$(D) \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

*exactly one of the following occurs:*

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution, then let an optimal be:  $x^* = [x_1^*, \dots, x_n^*]$   
(D) has feasible solution, then let an optimal be:  $y^* = [y_1^*, \dots, y_m^*]$ , then:*

$$c^T x^* = b^T y^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.
- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)
- The last row of the final tableau will give us that for every  $\mathbf{x}$ :

$$\begin{aligned} z &= z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k = z^* + \sum_{j=1}^n \bar{c}_j x_j + \sum_{i=1}^m \bar{c}_{n+i} x_{n+i} \\ &= z^* + \bar{c}_B x_B + \bar{c}_N x_N \end{aligned} \quad (*)$$

In addition,  $z^* = \sum_{j=1}^n c_j x_j^*$  ( $c_j$ , original values) because optimal value

- We define  $y_i^* = -\bar{c}_{n+i}$ ,  $i = 1, 2, \dots, m$
- We claim that  $(y_1^*, y_2^*, \dots, y_m^*)$  is a dual feasible solution satisfying  $c^T x^* = b^T y^*$ .

- Let's verify the claim:

We substitute in (\*): i)  $z = \sum_{j=1}^n c_j x_j$ ; ii)  $\bar{c}_{n+i} = -y_i^*$ ; and iii)  $x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j$  for  $i = 1, 2, \dots, m$  ( $n+i$  are the slack variables)

$$\begin{aligned} \sum_{j=1}^n c_j x_j &= z^* + \sum_{j=1}^n \bar{c}_j x_j - \sum_{i=1}^m y_i^* \left( b_i - \sum_{j=1}^n a_{ij} x_j \right) \\ &= \left( z^* - \sum_{i=1}^m y_i^* b_i \right) + \sum_{j=1}^n \left( \bar{c}_j + \sum_{i=1}^m a_{ij} y_i^* \right) x_j \end{aligned}$$

This must hold for every  $(x_1, x_2, \dots, x_n)$  hence:

$$z^* = \sum_{i=1}^m b_i y_i^* \quad \implies y^* \text{ satisfies } c^T x^* = b^T y^*$$

$$c_j = \bar{c}_j + \sum_{i=1}^m a_{ij} y_i^*, j = 1, 2, \dots, n$$



Since  $\bar{c}_k \leq 0$  for every  $k = 1, 2, \dots, n + m$ :

$$\bar{c}_j \leq 0 \rightsquigarrow c_j - \sum_{i=1}^m y_i^* a_{ij} \leq 0 \rightsquigarrow$$

$$\sum_{i=1}^m y_i^* a_{ij} \geq c_j \quad j = 1, 2, \dots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow y_i^* = -\bar{c}_{n+i} \geq 0,$$

$$i = 1, 2, \dots, m$$

$\Rightarrow y^*$  is also dual feasible solution

# Complementary Slackness Theorem

## Theorem (Complementary Slackness)

A feasible solution  $\mathbf{x}^*$  for (P)

A feasible solution  $\mathbf{y}^*$  for (D)

Necessary and sufficient conditions for optimality of both:

$$\left( c_j - \sum_{i=1}^m y_i^* a_{ij} \right) x_j^* = 0, \quad j = 1, \dots, n$$

If  $x_j^* \neq 0$  then  $\sum y_i^* a_{ij} = c_j$  (no surplus)

If  $\sum y_i^* a_{ij} > c_j$  then  $x_j^* = 0$

Proof:

$$\mathbf{z}^* = \mathbf{c}^T \mathbf{x}^* \leq \mathbf{y}^* \mathbf{A} \mathbf{x}^* \leq \mathbf{b}^T \mathbf{y}^* = w^*$$

Hence from strong duality theorem:

$$\mathbf{c} \mathbf{x}^* - \mathbf{y}^* \mathbf{A} \mathbf{x}^* = 0$$

In scalars

$$\sum_{j=1}^n \underbrace{\left( c_j - \sum_{i=1}^m y_i^* a_{ij} \right)}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be  $= 0$

Proof in scalar form:

$$c_j x_j^* \leq \left( \sum_{i=1}^m a_{ij} y_i^* \right) x_j^* \quad j = 1, 2, \dots, n \quad \text{from feasibility in D}$$

$$\left( \sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \leq b_i y_i^* \quad i = 1, 2, \dots, m \quad \text{from feasibility in P}$$

Summing in  $j$  and in  $i$ :

$$\sum_{j=1}^n c_j x_j^* \leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} y_i^* \right) x_j^* = \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \leq \sum_{i=1}^m b_i y_i^*$$

For the strong duality theorem the left hand side is equal to the right hand side and hence all inequalities become equalities.

$$\sum_{j=1}^n \underbrace{\left( c_j - \sum_{i=1}^m y_i^* a_{ij} \right)}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

# Economic Interpretation of Duality Theory

$$\begin{aligned}
 \max \quad & 5x_0 + 6x_1 + 8x_2 \\
 & 6x_0 + 5x_1 + 10x_2 \leq 60 \\
 & 8x_0 + 4x_1 + 4x_2 \leq 40 \\
 & 4x_0 + 5x_1 + 6x_2 \leq 50 \\
 & x_0, x_1, x_2 \geq 0
 \end{aligned}$$

final tableau:

$$\begin{array}{ccccccc|c}
 x_0 & x_1 & x_2 & s_1 & s_2 & s_3 & -z & b \\
 \hline
 & 0 & 1 & & 0 & & & 5/2 \\
 & 1 & 0 & & 0 & & & 7 \\
 & 0 & 0 & & 1 & & & 2 \\
 \hline
 -1/5 & 0 & 0 & -1/5 & 0 & -1 & & -62
 \end{array}$$

- Which values have the variables, the reduced costs, the shadow prices (or marginal prices), the dual variables?
- If one slack variable  $> 0$  then overcapacity:  $s_2 = 2$  then the second constraint is not tight
- How many products can be produced at most? at most  $m$
- How much more expensive a product not selected should be?  
look at reduced costs:  $c_j + \pi a_j > 0$
- What is the value of extra capacity of manpower? In +1 out +1/5

# Economic Interpretation of Duality Theory

Game: Suppose two economic operators:

- P owns the factory and produces goods
- D is in the market buying and selling raw material and resources
- D asks P to close and sell him/her all resources
- P considers if the offer is convenient
- D wants to spend least possible
- $y$  are prices that D offers for the resources
- $\sum y_i b_i$  is the amount D has to pay to have all resources of P
- $\sum y_i a_{ij} \geq c_j$  total value to make  $j >$  price per unit of product
- P either sells all resources  $\sum y_i a_{ij}$  or produces product  $j$  ( $c_j$ )
- without  $\geq$  there would not be negotiation because P would be better off producing and selling
- ▶ at optimality the situation is indifferent (strong th.)
- ▶ resource 2 that was not totally utilized in the primal has been given value 0 in the dual.  
(complementary slackness th.) Plausible, since we do not use all the resource, likely to place not so much value on it.
- ▶ for product 0  $\sum y_i a_{ij} > c_j$  hence not profitable producing it. (complementary slackness th.)

- Derivation:
  - Economic Interpretation
  - Bounding Approach
  - Multiplier Approach
  - Recipe
  - Lagrangian Multipliers Approach (next time)
- Theory:
  - Symmetry
  - Weak Duality Theorem
  - Strong Duality Theorem
  - Complementary Slackness Theorem