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DM545/DM871 – Linear and integer programming

Sheet 2, Autumn 2025

Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1*

List all vertices of the polyhedron $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, characterized by the following matrices A and $\mathbf{b}$:

$$A = \begin{bmatrix} 2 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

You are encouraged to use Python for carrying out the calculations.

Exercise 2$^+$ Simplex method

This is part of the first exercise (Opgave 1) in the written exam of 2008.
Consider the following linear programming problem (P1)

$$\begin{aligned} \text{maximize } & 2x_1 + 4x_2 - x_3 \\ \text{subject to } & 2x_1 - x_3 \leq 6 \\ & 3x_2 - x_3 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- Rewrite the problem in equational standard form adding the slack variables x_4, x_5, x_6 to the three constraints above, respectively, and write the first simplex tableau with x_4, x_5, x_6 as basic solution.
- Argue that x_2 can be brought in the basis with advantage and perform one pivot iteration that brings x_2 into the basic solution.
- After another pivot iteration, it is x_1 that can be brought with advantage in the basis (you do not have to perform this iteration), reaching the following simplex tableau:

```
|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -5/3 |      1 |    2/3 |     -2 |      0 |      4 |
|      0 |      1 |   -1/3 |      0 |    1/3 |      0 |      0 |      3 |
|      1 |      0 |    1/3 |      0 |   -1/3 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -1/3 |      0 |   -2/3 |     -2 |      1 |    -14 |
|--------+--------+--------+--------+--------+--------+--------+--------+
```

  Argue that an optimal solution is found and give the solution together with its objective value.

Exercise 3* Simplex method

Solve the following LP problem carrying out the simplex operations:

$$\begin{aligned} \text{maximize } & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to } & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

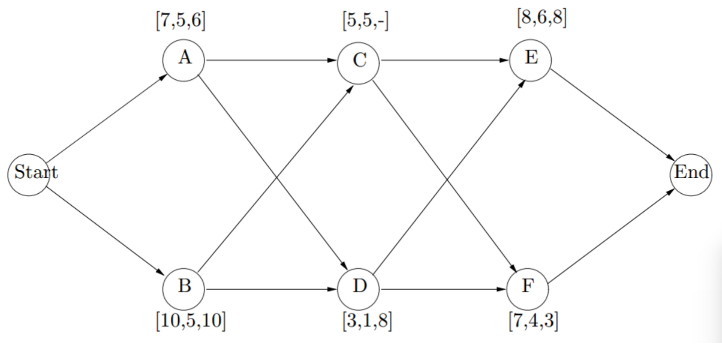


Figure 1: A network with activities on nodes for a small project with 6 activities. For each activity the following data is given in that order from left to right: normal time, minimum time in weeks, and the cost of shortening the duration of the activity by one week.

You are free to use any of the two representations, tableau or dictionary.
You can also get help from Python. You find a tutorial in the external web page.

Exercise 4

Solve the following linear programming problem applying the simplex algorithm:

$$\begin{aligned}
\text{maximize} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & x_1 - 2x_2 \leq 1 \\
& x_1 - x_2 \leq 2 \\
& 2x_1 - x_2 \leq 6 \\
& x_1 \leq 5 \\
& 2x_1 + x_2 \leq 16 \\
& x_1 + x_2 \leq 12 \\
& x_1 + 2x_2 \leq 21 \\
& x_2 \leq 10 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

[Hint: you can plot the feasibility region with one of the tools linked in the Tutorial “Preparation for the Take-Home Assignment”: Stefan Waner and Steven R. Costenoble, Simplex Method tool, and use the clairvoyant’s rule to minimize the number of operations to carry out.]

Exercise 5* Project Scheduling

This exercise is a part of one that appeared in Exam 2011.
A small project has 6 sub-activities A, B, C, D, E, F whose individual dependency (shown by the immediate predecessors) is given in Figure 1. Here we also list the normal time (in weeks), the absolute minimum time and the cost of shortening the activity by one week.
The goal is to shorten the duration of the project to 19 weeks. This means that the duration of one or more activities has to be shortened. Of course we want to select these so that the total cost of shortening the duration to 19 weeks is minimized. Formulate this problem as a linear programming problem and argue that the optimal solution to this LP will provide the correct answer. Write the model in explicit form, that is, with the actual data inserted in the formulation.

Exercise 6+ Quzzies

1. In 4D, how many hyperplanes need to intersect to give a point?
2. In 4D, can a point be described by more than 4 hyperplanes?
3. Consider the intersection of n hyperplanes in n dimensions: when does it uniquely identify a point?

Vertices of Polyhedra:
Consider the polyhedron described by $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$, that is:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\ & & \vdots & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m \end{array}$$

4. For a point $\mathbf{x}$ of a polyhedron, we define as *active* constraints those that are satisified to equality by $\mathbf{x}$. How many constraints are *active* in a *vertex* of a polyhedron $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$?

5. Does every point x that activates n constraints form a vertex of the polyhedron?

6. Can a vertex activate more than n constraints?

7. What if there are more variables than constraints? If $n < m$ then we can find a subset of constraints and then activate but what if $n > m$, can we have a vertex?

8. Combinatorial explosion of vertices: how many constraints and vertices has an n-dimensional hypercube?

9. If there are m constraints, n original variables, and n' final variables in the equational form, $m < n'$, what is an upper bound to the number of vertices?

Tableaux and Vertices

10. For each of these three statements, say if they are true or false:

    – One tableau $\Longrightarrow$ one vertex of the feasible region

    – One tableau $\Longleftarrow$ one vertex of the feasible region

    – One tableau $\Longleftrightarrow$ one vertex of the feasible region

11. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{lrcrcl} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
x_1	1	0	−1/5	1/2	0	8
	0	0	−2/5	−1	1	−64

– How many variables (original and slack) can be different from zero?

– $(x_3, x_4) = (0, 0)$ are non basic, what does this tell us about the original constraints?

Let's generalize the previous case. Consider an LP with m constraints, n original variables and m slack variables. In an optimal solution:

– if $m > n$, how many variables (original and slack) can be nonzero at most?

– if $m < n$ how many original variables must be zero at least? In other terms, in a mix planning problem with n products and m resources, $m < n$, how many products at most will be to be produced in an optimal solution?

12. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{lrcrcl} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	0	1	1/2	0	1
x_1	1	1	0	−1/2	0	1
	0	−2	0	1/2	1	−1

$(x_2, x_4) = (0, 0)$ are non basic variables, what does this tell us about the original constraints of the problem?

13. If in the original space of an LP problem we have 3 variables and there are 6 constraints, how many constraints are active in the optimal solution?

14. For the general case with n original variables:
One basic feasible solution $\Longleftrightarrow$ a matrix of active constraints has rank n. True or False?

15. Consider an LP problem with m constraints and n original variables, $m > n$. We saw that in $\mathbb{R}^n$ a point is the intersection of at least n hyperplanes. In LP this corresponds to say that in a vertex there are n active constraints. Let a tableau be associated with a solution that makes exactly $n+1$ constraints active, what can we say about the corresponding basic and non-basic variable values?

16. Given a polyhedron, what is the algebraic definition of vertex adjacency in 2, 3 and n dimensions?
How does this condition translate in terms of tableau?

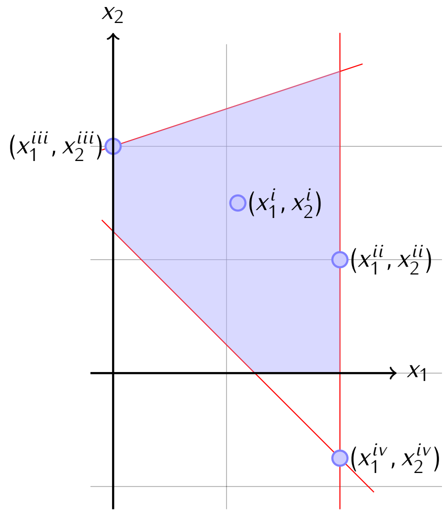
Exercise 7$^+$

What argument is used to prove that the simplex algorithm always terminates in a finite number of iterations if it does not encounter a situation in which one of the basic variables is zero? What may happen instead if the latter situation arises and which remedies are introduced?

Exercise 8*

Exercise 3 from Exam 2013.
Consider the following LP problem:

$$
(P) \qquad \begin{aligned}
\max\ & z = x_1 + 5x_2 \\
\text{s.t. } & -x_1 + 3x_2 \leq 6 \\
& 4x_1 + 4x_2 \geq 5 \\
& 0 \leq x_1 \leq 2 \\
& x_2 \geq 0
\end{aligned}
$$



(Note: the following subtasks can be carried out independently; use fractional mode for numerical calculations)

a. The polyhedron representing the feasibility region is depicted in the figure. Indicate for each of the four points represented whether they are feasible and/or basic solutions. Justify your answer.

b. Write the initial tableau or dictionary for the simplex method. Write the corresponding basic solution and its value. State whether the solution is feasible or not and whether it is optimal or not.

c. Consider the following tableau:

	x1	x2	x3	x4	x5	-z	b
I	0	4	1	-1/4	0	0	29/4
II	1	1	0	-1/4	0	0	5/4
III	0	-1	0	1/4	1	0	3/4
IV	0	4	0	1/4	0	1	-5/4

and the following three pivoting rules:

- largest coefficient
- largest increase
- steepest edge.

Which entering and leaving variables would each of them indicate? In this specific case, which rule would be convenient to follow? Report the details of the computations for the first two rules and carry out graphically the application of the third rule using the plot in the figure above (tikz code to reproduce the figure available in the online version.)

Exercise 9

The two following LP problems lead to two particular cases when solved by the simplex algorithm. Identify these cases and characterize them, that is, give indication of which conditions generate them in general.

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_2 \leq 5 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

Exercise 10*

Consider the following problem:

$$\begin{array}{rl} \max & z = 4x_2 \\ \text{s.t.} & 2x_2 \geq 0 \\ & -3x_1 + 4x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

a. Write the LP in equational standard form and say why it does not provide immediately an initial feasible basis for the simplex method.

b. To overcome the situation of infeasible basis construct the auxiliary problem for a phase I-phase II solution approach. Determine which variables are initially in basis and which are not in basis in the auxiliary problem.

c. Answer the following questions

   i) Is the initial basis in the auxiliary problem feasible in the original problem?
   ii) Is it optimal in the auxiliary problem?
   iii) Is it degenerate?
   iv) Can we say at this stage if phase I will terminate?
   v) If it will terminate, can we say at this stage that it will terminate with a basis that corresponds to a feasible solution in the original problem?
   vi) Solve the problem by carrying out Phase I and Phase II of the simplex algorithm.

Exercise 11$^+$ LP modeling — Investment plan

An investor has 10,000 Dkk to invest in four projects. The following table gives the cash flow for the four investments.
The information in the table can be interpreted as follows: For project 1, 1.00 Dkk invested at the start of year 1 will yield 0.50 Dkk at the start of year 2, 0.30 Dkk at the start of year 3, 1.80 Dkk at the start of year 4, and 1.20 Dkk at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. The investor has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of 1 year can be

Project	Year 1	Year 2	Year 3	Year 4	Year 5
1	-1.00	0.50	0.30	1.80	1.20
2	-1.00	0.60	0.20	1.50	1.30
3	0.00	-1.00	0.80	1.90	0.80
4	-1.00	0.40	0.60	1.80	0.95

reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.
[Taken from Operations Research: An Introduction, Taha]

Exercise 12* LP modeling — Budget Allocation

A company has six different opportunities to invest money. Each opportunity requires a certain investment over a period of 6 years or less. See Figure 2.
The company wants to invest in those opportunities that maximize the combined *Net Present Value* (NPV). It also has an investment budget that needs to be met for each year. (The Net Present Value is calculated with an interest rate of 5%).
How should the company invest?
We assume that it is possible to invest partially in an opportunity. For instance, if the company decides to invest 50% of the required amount in an opportunity, the return will also be 50%.

Net present value:
A debtor wants to delay the payment back of a loan for t years. Let P be the value of the original payment presently due. Let r be the market rate of return on a similar investment asset. The future value of P is

$$F = P(1 + r)^t$$

Viceversa, consider the task of finding the present value P of \$100 that will be received in five years, or equivalently, which amount of money today will grow to \$100 in five years when subject to a constant discount rate. Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1 + r)^t} = \frac{\$100}{(1 + 0.05)^5} = \$78.35.$$

Expected Investment Cash Flows and Net Present Value	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6	Budget
Year 1	-\$5.00	-\$9.00	-\$12.00	-\$7.00	-\$20.00	-\$18.00	\$45.00
Year 2	-\$6.00	-\$6.00	-\$10.00	-\$5.00	\$6.00	-\$15.00	\$30.00
Year 3	-\$16.00	\$6.10	-\$5.00	-\$20.00	\$6.00	-\$10.00	\$20.00
Year 4	\$12.00	\$4.00	-\$5.00	-\$10.00	\$6.00	-\$10.00	\$0.00
Year 5	\$14.00	\$5.00	\$25.00	-\$15.00	\$6.00	\$35.00	\$0.00
Year 6	\$15.00	\$5.00	\$15.00	\$75.00	\$6.00	\$35.00	\$0.00
NPV	\$8.01	\$2.20	\$1.85	\$7.51	\$5.69	\$5.93	

Figure 2: