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DM545/DM871 – Linear and integer programming

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Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1$^+$

Show that the dual of $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is $\min\{\mathbf{y}^T\mathbf{b} \mid \mathbf{y}^TA \geq \mathbf{c}\}$ using the bounding method.
Show that the dual of $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is $\min\{\mathbf{y}^T\mathbf{b} \mid \mathbf{y}^TA \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$ using the Lagrangian multipliers method.

Exercise 2*

Consider the following LP problem:

$$\begin{array}{ll}
\max & 2x_1 + 3x_2 \\
 & 2x_1 + 3x_2 \leq 30 \\
 & x_1 + 2x_2 \geq 10 \\
 & x_1 - x_2 \leq 1 \\
 & x_2 - x_1 \leq 1 \\
 & x_1 \geq 0
\end{array}$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Exercise 3

Consider the problem

$$\begin{array}{ll}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\
 & 4x_1 + x_2 + 2x_3 \leq 11 \\
 & 3x_1 + 4x_2 + 2x_3 \leq 8 \\
 & x_1, x_2, x_3 \geq 0
\end{array}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]

Exercise 4

Consider the following LP:

$$\begin{array}{ll}
\min & 3x_1 + 2x_2 - 4x_3 \\
 & 2x_1 + x_2 + x_3 \geq 3 \\
 & x_1 + x_2 + 2x_3 \leq 5 \\
 & x_1, x_2, x_3 \geq 0
\end{array}$$

Find the optimal solution knowing that the solution of the dual problem is $(u_1, u_2) = (10/3, 11/3)$.

Exercise 5*

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- Solving an an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. The solution of the auxiliary LP problem gives an initial feasible basis or a proof of infeasibility.
- Applying the dual simplex on a possibly modified problem to find a feasible solution. If the initial infeasible tableau of the original problem is not optimal then the objective function can be temporarily modified for this phase in order to make the initial tableau optimal although not feasible. Opposite to the primal simplex method, the dual simplex method iterates through infeasible basic solutions, while maintaining them optimal, and stops when a feasible solution is reached.

Dual Simplex: The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:

1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| \,:\, a_{i^*j} < 0\}$
3. Update the tableau around the pivot in the same way as with the primal simplex.
4. Stop if no right-hand side term is negative.

Duality can help us with the issue of initial feasible basic solutions. In the problem above, the initial tableau is infeasible and not optimal, hence we cannot apply the primal simplex nor the dual simplex. However, if the objective function was $w = -x_1 - x_2$, then we would have the conditions of infeasibility and optimality needed by the dual simplex. You can understand this also looking at the dual problem. With the new objective function the initial basic solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. In contrast, with original objective function z the primal simplex has infeasible initial basis in both problems. So we can change temporarily the objective function z with w and apply the dual simplex method to the primal problem. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and continue iterating with the primal simplex. The phase I–phase II simplex method that uses the dual simplex is also called the *dual-primal simplex method*.
Apply the two versions of the phase I–phase II simplex method (that is, phase I is carried out with the auxiliary problem or with the dual simplex) to the problem above and verify that they lead to the same solutions.

Exercise 6*

Write the dual of the following problem

$$
\begin{aligned}
(P) \qquad & \max \sum_{j \in J} \sum_{i \in I} r_j x_{ij} & \\
& \sum_{j \in J} x_{ij} \leq b_i & \forall i \in I \\
& \sum_{i \in I} x_{ij} \leq d_j & \forall j \in J \\
& \sum_{i \in I} p_i x_{ij} = p_j \sum_{i \in I} x_{ij} & \forall j \in J \\
& x_{ij} \geq 0 & \forall i \in I, j \in J
\end{aligned}
$$