DM545/DM871

Linear and Integer Programming

Lecture 9

IP Modeling
Formulations, Relaxations

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Outline

Outline

Uncapacited Facility Location (UFL)

Given:

- depots $N = \{1, \ldots, n\}$
- clients $M = \{1, \ldots, m\}$
- f_j fixed cost to use depot j
- transport cost for all orders c_{ij}

Task: Which depots to open and which depots serve which client

Variables: $y_j = \begin{cases} 1 & \text{if depot opened} \\ 0 & \text{otherwise} \end{cases}$, $\quad x_{ij}$ fraction of demand of i satisfied by j

Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

Constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, m$$

$$\sum_{i \in M} x_{ij} \leq m y_j \qquad \forall j \in N$$

Outline

Good and Ideal Formulations

Definition (Formulation)

A polyhedron $P \subseteq \mathbb{R}^{n+p}$ is a formulation for a set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if and only if $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region X.

There are infinitely many formulations

Definition (Convex Hull)

Given a set $X \subseteq \mathbb{Z}^n$ the convex hull of X is defined as:

$$\text{conv}(X) = \Big\{ \boldsymbol{x} : \boldsymbol{x} = \sum_{i=1}^{t} \lambda_i \boldsymbol{x}^i, \quad \sum_{i=1}^{t} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{for } i = 1, \dots, t, \\ \text{for all finite subsets } \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^t\} \text{ of } X \Big\}$$

Proposition

$\mathrm{conv}(X)$ is a polyhedron (ie, representable as $A\mathbf{x} \leq \mathbf{b}$)

Proposition

Extreme points of $\mathrm{conv}(X)$ all lie in X

Hence:

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\} \equiv \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \mathrm{conv}(X)\}$$

However it might require exponential number of inequalities to describe $\mathrm{conv}(X)$
What makes a formulation better than another?

$$X \subseteq \mathrm{conv}(X) \subseteq P_2 \subset P_1$$

$$P_2 \text{ is better than } P_1$$

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, P_2 is a better formulation than P_1 if $P_2 \subset P_1$

Example

$P_1 = \text{UFL with } \sum_{i \in M} x_{ij} \le m y_j \quad \forall j \in N$
$P_2 = \text{UFL with } x_{ij} \le y_j \quad \forall i \in M, j \in N$

$$P_2 \subset P_1$$

- $P_2 \subseteq P_1$ because summing $x_{ij} \le y_j$ over $i \in M$ we obtain $\sum_{i \in M} x_{ij} \le m y_j$
- $P_2 \subset P_1$ because there exists a point in P_1 but not in P_2: $m = 6 = 3 \cdot 2 = k \cdot n$

$$x_{10} = 1,\ x_{20} = 1,\ x_{30} = 1, \qquad \sum_i x_{i0} \le 6 y_0 \quad y_0 = 1/2$$
$$x_{41} = 1,\ x_{51} = 1,\ x_{61} = 1 \qquad \sum_i x_{i1} \le 6 y_1 \quad y_1 = 1/2$$

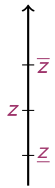
Outline

1. Formulations
    - Uncapacited Facility Location
    - Alternative Formulations

2. Relaxations

Optimality and Relaxation

$$z = \max\{c(\boldsymbol{x}) : \boldsymbol{x} \in X \subseteq \mathbb{Z}^n\}$$



How can we prove that $\boldsymbol{x}^*$ is optimal?
$\overline{z}$ is UB
$\underline{z}$ is LB
stop when $\overline{z} - \underline{z} \leq \epsilon$

- Primal bounds (here lower bounds): every feasible solution gives a primal bound, it may be easy or hard to find, heuristics
- Dual bounds (here upper bounds): Relaxations

Optimality gap (SCIP):

- If primal and dual bounds have opposite signs, the gap is "Infinity".
- If primal and dual bounds have the same sign, the gap is

$$\frac{|pb - db|}{\min(|pb|, |db|)}$$ decreases monotonously during the solving process.

Proposition

$$\begin{aligned} \textit{Given:} \quad (IP)\ z &= \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{R}^n\} \\ \textit{a relaxation of it is:} \quad (RP)\ z^R &= \max\{f(\mathbf{x}) : \mathbf{x} \in T \subseteq \mathbb{R}^n\} \qquad \textit{if:} \end{aligned}$$

(i) $X \subseteq T$ *or*

(ii) $f(\mathbf{x}) \geq c(\mathbf{x})\ \forall \mathbf{x} \in X$

In other terms:

$$\max_{\mathbf{x} \in T} f(\mathbf{x}) \geq \left\{ \begin{array}{l} \max_{\mathbf{x} \in T} c(\mathbf{x}) \\ \max_{\mathbf{x} \in X} f(\mathbf{x}) \end{array} \right\} \geq \max_{\mathbf{x} \in X} c(\mathbf{x})$$

- T: candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(\mathbf{x}) \geq c(\mathbf{x})\ \forall \mathbf{x} \in X$

Relaxations

How to construct relaxations?

1. $IP : \max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in P \cap \mathbb{Z}^n\}, \qquad P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$
   $LP : \max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in P\}$
   Better formulations give better bounds ($P_1 \subseteq P_2$)

   Proposition

   (i) *If a relaxation LP is infeasible, the original problem IP is infeasible.*

   (ii) *Let* $\boldsymbol{x}^*$ *be optimal solution for LP. If* $\boldsymbol{x}^* \in X$ *and* $f(\boldsymbol{x}^*) = c(\boldsymbol{x}^*)$ *then* $\boldsymbol{x}^*$ *is optimal for IP.*

2. Combinatorial relaxations to easy problems that can be solved rapidly
   Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$IP: \qquad z = \max\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \in X \subseteq \mathbb{Z}^n\}$$

$$LR: \qquad z(\boldsymbol{u}) = \max\{\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{b} - A\boldsymbol{x}) : \boldsymbol{x} \in X\}$$

$$z(\boldsymbol{u}) \geq z \qquad \forall \boldsymbol{u} \geq 0$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(\boldsymbol{x}) : \boldsymbol{x} \in X\} \qquad w = \min\{w(\boldsymbol{u}) : \boldsymbol{u} \in U\}$$

form a weak-dual pair if $c(\boldsymbol{x}) \leq w(\boldsymbol{u})$ for all $\boldsymbol{x} \in X$ and all $\boldsymbol{u} \in U$.
When $z = w$ they form a strong-dual pair

Proposition

$z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_+\}$ and $w^{LP} = \min\{\mathbf{u}^T\mathbf{b} : A^T\mathbf{u} \geq \mathbf{c}, \mathbf{u} \in \mathbb{R}^m_+\}$ (ie, dual of linear relaxation) form a weak-dual pair.

Proposition

Let IP and D be weak-dual pair:

(i) *If D is unbounded, then IP is infeasible*

(ii) *If $\mathbf{x}^* \in X$ and $\mathbf{u}^* \in U$ satisfy $c(\mathbf{x}^*) = w(\mathbf{u}^*)$ then $\mathbf{x}^*$ is optimal for IP and $\mathbf{u}^*$ is optimal for D.*

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Examples

Weak pairs:

Matching: $z = \max\{1^T \boldsymbol{x} : A\boldsymbol{x} \leq 1, \boldsymbol{x} \in \mathbb{Z}^m_+\}$

V. Covering: $w = \min\{1^T \boldsymbol{y} : A^T y \geq 1, \boldsymbol{y} \in \mathbb{Z}^n_+\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:

S. Packing: $z = \max\{1^T \boldsymbol{x} : A\boldsymbol{x} \leq 1, \boldsymbol{x} \in \mathbb{Z}^n_+\}$

S. Covering: $w = \min\{1^T \boldsymbol{y} : A^T \boldsymbol{y} \geq 1, \boldsymbol{y} \in \mathbb{Z}^m_+\}$