DM545/DM871

Linear and Integer Programming

Lecture 12

Network Flows

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Well Solved Problems
2. (Minimum Cost) Network Flows
3. Application Example

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1. Well Solved Problems

2. (Minimum Cost) Network Flows

3. Application Example

Separation problem

$\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in X\} \equiv \max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in \mathrm{conv}(X)\}$
$X \subseteq \mathbb{Z}^n$, P a polyhedron $P \subseteq \mathbb{R}^n$ and $X = P \cap \mathbb{Z}^n$

Definition (Separation problem for a COP)

Given $\boldsymbol{x}^* \in P$; is $\boldsymbol{x}^* \in \mathrm{conv}(X)$? If not find an inequality $\boldsymbol{ax} \leq \boldsymbol{b}$ satisfied by all points in X but violated by the point $\boldsymbol{x}^*$.

(Farkas' lemma states the existence of such an inequality.)

Properties of Easy Problems

Four properties that often go together:

Definition

(i) Efficient optimization property: $\exists$ a polynomial algorithm for $\max\{\boldsymbol{c}\boldsymbol{x} : \boldsymbol{x} \in X \subseteq \mathbb{R}^n\}$

(ii) Strong duality property: $\exists$ strong dual D $\min\{w(\boldsymbol{u}) : \boldsymbol{u} \in U\}$ that allows to quickly verify optimality

(iii) Efficient separation problem: $\exists$ efficient algorithm for separation problem

(iv) Efficient convex hull property: a compact description of the convex hull is available

Example:

If explicit convex hull strong duality holds
efficient separation property (just description of $\mathrm{conv}(X)$)

Theoretical analysis to prove results about

- strength of certain inequalities that are facet defining
  2 ways
- descriptions of convex hull of some discrete $X \subseteq \mathbb{Z}^*$
  several ways, we see one next

Example

Let

$$X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^{m} x_i \leq my, x_i \leq 1 \text{ for } i = 1, \ldots, m\}$$

$$P = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$$

.

Polyhedron P describes $\text{conv}(X)$

Totally Unimodular Matrices

When the LP solution to this problem

$$IP : \max\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}^n_+\}$$

with all data integer will have integer solution?

$$\left[\begin{array}{c:c:c:c} A_N & A_B & 0 & \boldsymbol{b} \\ \hdashline \boldsymbol{c}_N^T & \boldsymbol{c}_B^T & 1 & 0 \end{array}\right]$$

$A_B x_B + A_N x_N = b$

$\boldsymbol{x}_N = 0 \rightsquigarrow A_B \boldsymbol{x}_B = \boldsymbol{b}$,
A_B $m \times m$ non singular matrix
$\boldsymbol{x}_B \geq 0$

Cramer's rule for solving systems of linear equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \boldsymbol{x} = A_B^{-1}\boldsymbol{b} = \frac{A_B^{adj}\boldsymbol{b}}{\det(A_B)}$$

Definition

- A square integer matrix B is called unimodular (UM) if $\det(B) = \pm 1$
- An integer matrix A is called totally unimodular (TUM) if every square, nonsingular submatrix of A is UM

Proposition

- *If A is TUM then all vertices of $R_1(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ are integer if $\mathbf{b}$ is integer*
- *If A is TUM then all vertices of $R_2(A) = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$ are integer if $\mathbf{b}$ is integer.*

Proof: if A is TUM then $\left[A \,\vdots\, I \right]$ is TUM
Any square, nonsingular submatrix C of $\left[A \,\vdots\, I \right]$ can be written as

$$C = \left[\begin{array}{c:c} B & 0 \\ \hdashline D & I_k \end{array} \right]$$

where B is square submatrix of A. Hence $\det(C) = \det(B) = \pm 1$

Proposition

The transpose matrix A^T of a TUM matrix A is also TUM.

Theorem (Sufficient condition)

An integer matrix A is TUM if

1. $a_{ij} \in \{0, -1, +1\}$ *for all* i, j
2. *each column contains at most two non-zero coefficients* ($\sum_{i=1}^{m} |a_{ij}| \leq 2$)
3. *if the rows can be partitioned into two sets* I_1, I_2 *such that:*
   - *if a column has 2 entries of same sign, their rows are in different sets*
   - *if a column has 2 entries of different signs, their rows are in the same set*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: by induction

Basis: one matrix of one element $\{0, +1, -1\}$ is TUM

Induction: let C be of size k.
If C has column with all 0s then it is singular.
If a column with only one 1 then expand on that by induction
If 2 non-zero in each column then

$$\forall j : \sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij}$$

but then a linear combination of rows is zero and $\det(C) = 0$

Other matrices with integrality property:

- TUM
- Balanced matrices
- Perfect matrices
- Integer vertices

Defined in terms of forbidden substructures that represent fractionating possibilities.

Proposition

A is always TUM if it comes from

- *node-edge incidence matrix of undirected bipartite graphs (ie, no odd cycles)* ($I_1 = U, I_2 = V, B = (U, V, E)$)
- *node-arc incidence matrix of directed graphs* ($I_2 = \emptyset$)

Eg: Shortest path, max flow, min cost flow, bipartite weighted matching

Summary

1. Well Solved Problems

2. (Minimum Cost) Network Flows

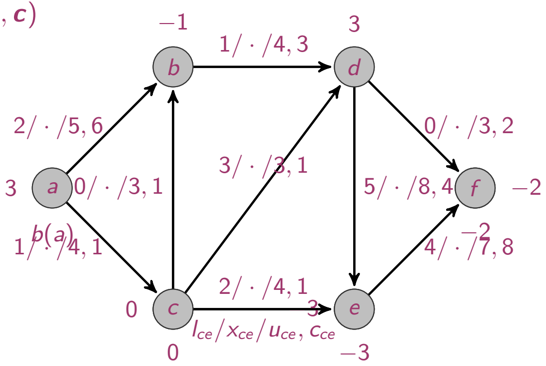
3. Application Example

Outline

Terminology

Network: • directed graph $D = (V, A)$
• arc, directed link, from tail to head
• lower bound $l_{ij} > 0, \forall ij \in A$, capacity $u_{ij} \geq l_{ij}, \forall ij \in A$
• cost c_{ij}, linear variation (if $ij \notin A$ then $l_{ij} = u_{ij} = 0, c_{ij} = 0$)
• balance vector $b(i)$, $b(i) > 0$ supply node (source), $b(i) < 0$ demand node (sink, tank), $b(i) = 0$ transhipment node (assumption $\sum_i b(i) = 0$)

$N = (V, A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{b}, \boldsymbol{c})$



Network Flows

Flow $\boldsymbol{x} : A \to \mathbb{R}$

balance vector of $\boldsymbol{x}$: $b_{\boldsymbol{x}}(v) = \sum_{vu \in A} x_{vu} - \sum_{wv \in A} x_{wv}, \ \forall v \in V$

$$b_{\boldsymbol{x}}(v) \begin{cases} > 0 & \text{source} \\ < 0 & \text{sink/target/tank} \\ = 0 & \text{balanced} \end{cases}$$

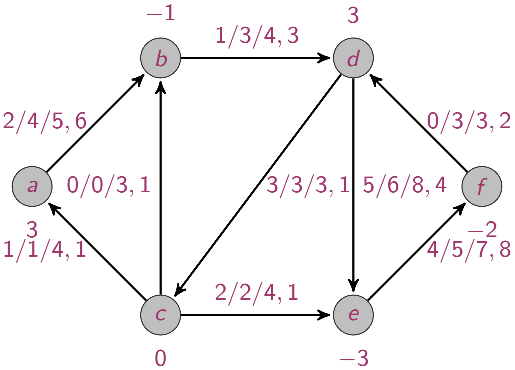
(generalizes the concept of path with $b_{\boldsymbol{x}}(v) = \{0, 1, -1\}$)

feasible $\quad l_{ij} \leq x_{ij} \leq u_{ij}$, $b_{\boldsymbol{x}}(i) = b(i)$

cost $\quad \boldsymbol{c}^T \boldsymbol{x} = \sum_{ij \in A} c_{ij} x_{ij}$ (varies linearly with $\boldsymbol{x}$)

If iji is a 2-cycle and all $l_{ij} = 0$, then at least one of x_{ij} and x_{ji} is zero.

Example



Feasible flow of cost 109

Minimum Cost Network Flows

Find cheapest flow through a network in order to satisfy demands at certain nodes from available supplier nodes.

Variables:

$$x_{ij} \in \mathbb{R}_0^+$$

Objective:

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

Constraints: mass balance + flow bounds

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b(i) \quad \forall i \in V$$

$$l_{ij} \leq x_{ij} \leq u_{ij}$$

$$\begin{aligned} \min\ & \boldsymbol{c}^T \boldsymbol{x} \\ & N\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \end{aligned}$$

N node arc incidence matrix

if flow of indivisible goods: under the assumption that all parameter values are integer (we can multiply if rational) the LP relaxation solution is integer.

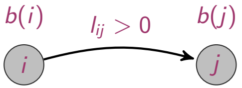
	x_{e_1}	x_{e_2}	$\dots$	x_{ij}	$\dots$	x_{e_m}		
	c_{e_1}	c_{e_2}	$\dots$	c_{ij}	$\dots$	c_{e_m}		
1	1	$.$	$\dots$	$.$	$\dots$	$.$	$=$	b_1
2	$.$	$.$	$\dots$	$.$	$\dots$	$.$	$=$	b_2
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
i	-1	$.$	$\dots$	1	$\dots$	$.$	$=$	b_i
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
j	$.$	$.$	$\dots$	-1	$\dots$	$.$	$=$	b_j
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
n	$.$	$.$	$\dots$	$.$	$\dots$	$.$	$=$	b_n
e_1	1						$\leq$	u_1
e_2		1					$\leq$	u_2
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
(i,j)				1			$\leq$	u_{ij}
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
e_m						1	$\leq$	u_m

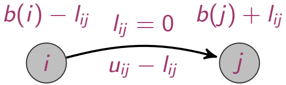
Reductions/Transformations

Lower bounds

Let $N = (V, A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{b}, \boldsymbol{c})$

$$\begin{aligned} N' &= (V, A, \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{b}', \boldsymbol{c}) \\ b'(i) &= b(i) - l_{ij} \\ b'(j) &= b(j) + l_{ij} \\ u'_{ij} &= u_{ij} - l_{ij} \\ l'_{ij} &= 0 \end{aligned}$$

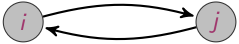




$$\boldsymbol{c}^T\boldsymbol{x}$$

$$\boldsymbol{c}^T\boldsymbol{x}' + \sum_{ij \in A} c_{ij} l_{ij}$$

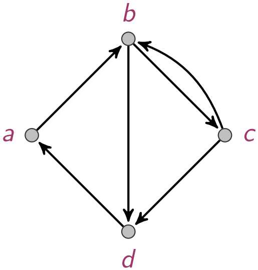
Undirected arcs

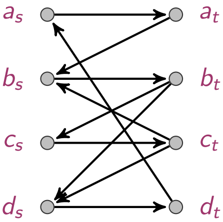


Vertex splitting

If there are bounds and costs of flow passing through vertices where $b(v) = 0$ (used to ensure that a node is visited):

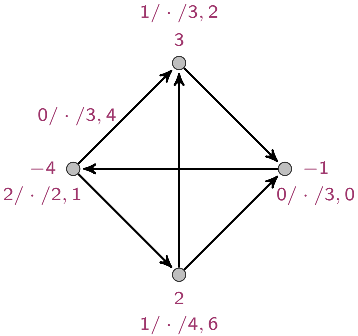
$N = (V, A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{c}, \boldsymbol{l}^*, \boldsymbol{u}^*, \boldsymbol{c}^*)$

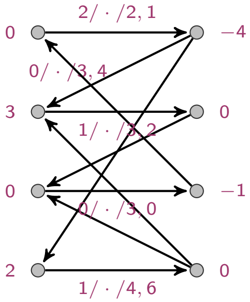




From D to D_{ST} as follows:

$$\begin{aligned} \forall v \in V \quad & \rightsquigarrow v_s, v_t \in V(D_{ST}) \text{ and } v_s v_t \in A(D_{ST}) \\ \forall xy \in A(D) & \rightsquigarrow x_t y_s \in A(D_{ST}) \end{aligned}$$



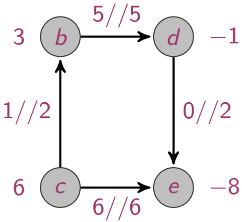


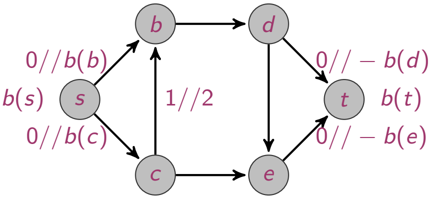
$$\forall v \in V \text{ and } v_s v_t \in A_{ST} \rightsquigarrow h'(v_s v_t) = h^*(v), \quad h^* \in \{l^*, u^*, c^*\}$$
$$\forall xy \in A \text{ and } x_t y_s \in A_{ST} \rightsquigarrow h'(x_t y_s) = h(x, y), \; h \in \{l, u, c\}$$

If $b(v) = 0$, then $b'(v_s) = b'(v_t) = 0$
If $b(v) < 0$, then $b'(v_s) = 0$ and $b'(v_t) = b(v)$
If $b(v) > 0$, then $b'(v_s) = b(v)$ and $b'(v_t) = 0$

(s,t)-flow:

$$b_{\boldsymbol{x}}(v) = \begin{cases} k & if \;\; v = s \\ -k & if \;\; v = t \\ 0 & otherwise \end{cases}, \qquad |\boldsymbol{x}| = |b_{\boldsymbol{x}}(s)|$$





$$b(s) = \sum_{v:b(v)>0} b(v) = M$$
$$b(t) = \sum_{v:b(v)<0} b(v) = -M$$

$\exists$ feasible flow in $N \iff \exists\ (s,t)$-flow in N_{st} with $|x| = M \iff$ max flow in N_{st} is M

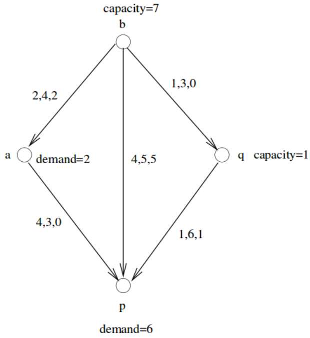
Residual Network

Residual Network $N(\boldsymbol{x})$**:** given that a flow $\boldsymbol{x}$ already exists, how much flow excess can be moved in G?

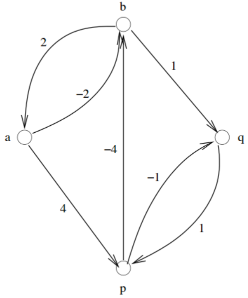
Replace arc $ij \in N$ with arcs:

	residual capacity	cost
ij :	$r_{ij} = u_{ij} - x_{ij}$	c_{ij}
ji :	$r_{ji} = x_{ij}$	$-c_{ij}$

$(N, \boldsymbol{c}, \boldsymbol{u}, \boldsymbol{x})$



$(N(\boldsymbol{x}), \boldsymbol{r}, \boldsymbol{c}')$



Special cases

Shortest path problem path of minimum cost from s to t with costs $\lesseqgtr 0$

$b(s) = 1, b(t) = -1, b(i) = 0$

if to any other node? $b(s) = (n-1), b(i) = -1, u_{ij} = n-1$

Max flow problem incur no cost but restricted by bounds

steady state flow from s to t

$b(i) = 0 \ \forall i \in V, \qquad c_{ij} = 0 \ \forall ij \in A \qquad ts \in A$

$c_{ts} = -1, \qquad u_{ts} = \infty$

Assignment problem min weighted bipartite matching,

$|V_1| = |V_2|, A \subseteq V_1 \times V_2$

c_{ij}

$b(i) = 1 \ \forall i \in V_1 \qquad b(i) = -1 \ \forall i \in V_2 \qquad u_{ij} = 1 \ \forall ij \in A$

Special cases

Transportation problem/Transhipment distribution of goods, warehouses-costumers

$|V_1| \neq |V_2|, \qquad u_{ij} = \infty$ for all $ij \in A$

$$\begin{aligned}
\min \sum c_{ij} x_{ij} & \\
\sum_i x_{ij} \geq b_j & \qquad \forall j \in V_2 \\
\sum_j x_{ij} \leq a_i & \qquad \forall i \in V_1 \\
x_{ij} \geq 0 &
\end{aligned}$$

if $\sum a_i = \sum b_i$ then $\geq / \leq$ become $=$
if $\sum a_i > \sum b_i$ then add dummy tank nodes
if $\sum a_i < \sum b_i$ then infeasible

Multi-commodity flow problem ship several commodities using the same network, different origin destination pairs separate mass balance constraints, share capacity constraints, min overall flow

$$
\begin{aligned}
\min \textstyle\sum_k \boldsymbol{c}^k \boldsymbol{x}^k & \\
N\boldsymbol{x}^k \geq \boldsymbol{b}^k & \qquad \forall k \\
\textstyle\sum_k \boldsymbol{x}_{ij}^k \leq \boldsymbol{u}_{ij} & \qquad \forall ij \in A \\
0 \leq \boldsymbol{x}_{ij}^k \leq \boldsymbol{u}_{ij}^k &
\end{aligned}
$$

What is the structure of the matrix now? Is the matrix still TUM?

Outline

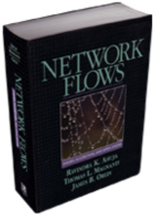
1. Well Solved Problems

2. (Minimum Cost) Network Flows

3. Application Example

Ship loading problem

Plenty of applications. See Ahuja Magnanti Orlin, Network Flows, 1993

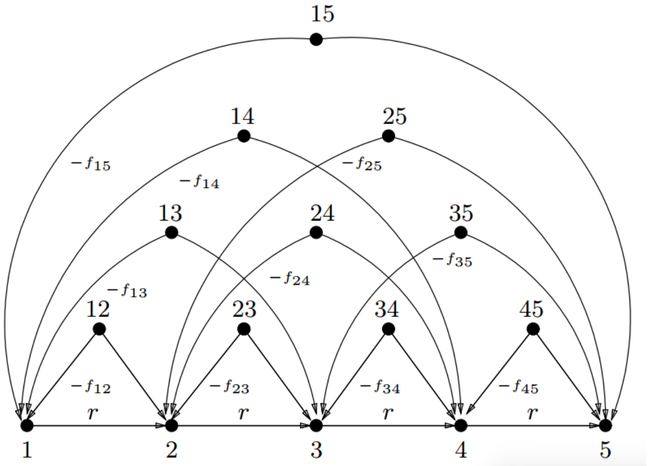


- A cargo company (eg, Maersk) uses a ship with a capacity to carry at most r units of cargo.
- The ship sails on a long route (say from Southampton to Alexandria) with several stops at ports in between.
- At these ports cargo may be unloaded and new cargo loaded.
- At each port there is an amount b_{ij} of cargo which is waiting to be shipped from port i to port $j > i$
- Let f_{ij} denote the income for the company from transporting one unit of cargo from port i to port j.
- The goal is to plan how much cargo to load at each port so as to maximize the total income while never exceeding ship's capacity.

Application Example: Modeling

- n number of stops including the starting port and the terminal port.
- $N = (V, A, \boldsymbol{l} \equiv 0, \boldsymbol{u}, \boldsymbol{c})$ be the network defined as follows:
  - $V = \{v_1, v_2, \dots, v_n\} \cup \{v_{ij} : 1 \leq i < j \leq n\}$
  - $A = \{v_1 v_2, v_2 v_3, \dots v_{n-1} v_n\} \cup \{v_{ij} v_i, v_{ij} v_j : 1 \leq i < j \leq n\}$
  - capacity: $u_{v_i v_{i+1}} = r$ for $i = 1, 2, \dots, n-1$ and all other arcs have capacity ∞.
  - cost: $c_{v_{ij} v_i} = -f_{ij}$ for $1 \leq i < j \leq n$ and all other arcs have cost zero (including those of the form $v_{ij} v_j$)
  - balance vector: $b(v_{ij}) = b_{ij}$ for $1 \leq i < j \leq n$ and the balance vector of $b(v_i) = -b_{1i} - b_{2i} - \dots - b_{i-1,i}$ for $i = 1, 2, \dots, n$

Application Example: Modeling



Application Example: Modeling

Claim: the network models the ship loading problem.

- suppose that $t_{12}, t_{13}, \ldots, t_{1n}, t_{23}, \ldots, t_{n-1,n}$ are cargo numbers, where t_{ij} ($\leq b_{ij}$) is the amount of cargo the ship will transport from port i to port j and that the ship is never loaded above capacity.
- total income is

$$I = \sum_{1 \leq i < j \leq n} t_{ij} f_{ij}$$

- Let $\mathbf{x}$ be the flow in N defined as follows:
  - flow on an arc of the form $v_{ij} v_i$ is t_{ij}
  - flow on an arc of the form $v_{ij} v_j$ is $b_{ij} - t_{ij}$
  - flow on an arc of the form $v_i v_{i+1}$, $i = 1, 2, \ldots, n-1$, is the sum of those t_{ab} for which $a \leq i$ and $b \geq i + 1$.
- since t_{ij}, $1 \leq i < j \leq n$, are legal cargo numbers then $\mathbf{x}$ is feasible with respect to the balance vector and the capacity restriction.
- the cost of $\mathbf{x}$ is $-I$.

Application Example: Modeling

- Conversely, suppose that $\boldsymbol{x}$ is a feasible flow in N of cost J.
- we construct a feasible cargo assignment $s_{ij}, 1 \leq i < j \leq n$ as follows:
  - let s_{ij} be the value of $\boldsymbol{x}$ on the arc $v_{ij}v_i$.
- income $-J$