Lévy's characterization of Brownian Motion

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1 Notation

In these notes we shall in general use standard notation. For every $n \in \mathbb{N}$ $\mathcal{B}_n$ denotes the Borel algebra on $\mathbb{R}^n$ and if $(\Omega, \mathcal{F}, P)$ is a probability space, $X : \Omega \to \mathbb{R}^n$ a random variable, then we let $X(P)$ denote the distribution measure (the image measure) on $\mathbb{R}^n$ of X, e.g.

$$X(P)(A) = P(X^{-1}(A)) \quad \text{for all} A \in \mathcal{B}_n. \tag{1.1}$$

If $n \in \mathbb{N}$, we let $\langle \cdot, \cdot \rangle$ denote the canonical inner product on $\mathbb{R}^n$. Hence for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ og alle $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ we have

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j. \tag{1.2}$$

All vector spaces which occur in these notes are assumed to be real unless otherwise stated.

2 The main results

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_t)_{t \geq 0}$ be an increasing family of sub-σ-algebras so that $\mathcal{F}_t$ contains all sets of measure 0 for all $t \geq 0$. We start with the following easy result.

Theorem 2.1 *Let (B_t) be a one–dimensional normalized Beownian motion, adapted to $(\mathcal{F})$ and so that $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$ (this ensures that (B_t) is a martingale with respect to $(\mathcal{F}_t)$). Then $(B_t^2 - t)$ is a martingale with respect to $(\mathcal{F}_t)$.*

Proof: If $0 \leq s < t$, then $B_t^2 = (B_t - B_s)^2 + B_s^2 + 2B_s(B_t - B_s)$ and hence

$$E(B_t^2 \mid \mathcal{F}_s) = E((B_t - B_s)^2 \mid \mathcal{F}_s) + B_s^2 + 2B_s E((B_t - B_s) \mid \mathcal{F}_s) = (t - s) + B_s^2$$

where we have used that $B_t - B_s$ and hence also $(B_t - B_s)^2$ are independent of $\mathcal{F}_s$. □

The main result of this note is to prove that the converse is also true for continuous processes, namely:

Theorem 2.2 *Let (X_t) be a continous process adapted to $(\mathcal{F}_t)$ so that $X_0 = 0$ and*

(i) (X_t) is a martingale with respect to $(\mathcal{F}_t)$.

(ii) $(X_t^2 - t)$ is a martingale with respect to $(\mathcal{F}_t)$.

Then (X_t) is a (normalized) Brownian motion.

Before we can prove it, we need yet another theorem which is a bit like Ito's formula and a lemma.

Theorem 2.3 *Let (X_t) be as in Theorem 2.2 and let $f \in C(\mathbb{R}^2)$ so that f, f' and f'' are bounded. For all $0 \leq s < t$ we have*

$$E(f(X_t) \mid \mathcal{F}_s) = X_s + \frac{1}{2}\int_s^t E(f''(X_u) \mid \mathcal{F}_s). \tag{2.3}$$

Proof: Let $\Pi = (t_k)_{k=0}^n$ be a partition of the interval $[s,t]$ so that $s = t_0, t_1 < t_2 < \cdots, < t_n = t$. By Taylor's formula we get

$$\begin{aligned} f(X_t) &= f(X_s) + \sum_{k=1}^n (f(X_{t_k}) - f(X_{t_{k-1}})) \qquad (2.4)\\ &= f(X_s) + \sum_{k=1}^n f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2}\sum_{k=1}^n f''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 + R_\Pi \end{aligned}$$

Taking conditional expectations on each side we obtain:

$$E(f(X_t) \mid \mathcal{F}_s) = f(X_s) + \sum_{k=1}^n E(E(f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) \mid \mathcal{F}_{k-1}) \mid \mathcal{F}_s) \quad +$$

$$\frac{1}{2}\sum_{k=1}^n E(E(f''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}) \mid \mathcal{F}_s) + E(R_\Pi \mid \mathcal{F}_s) = f(X_s) \quad +$$

$$\frac{1}{2}\sum_{k=1}^n E(f''(X_{t_{k-1}}) \mid \mathcal{F}_s)(t_k - t_{k-1}) + E(R_\Pi \mid \mathcal{F}_s). \tag{2.5}$$

Using the continuity of the (X_t) it can be shown that $R_\Pi \to 0$ in $L_2(P)$, when the length $|\Pi|$ of Π tends to 0. Hence also $E(R_\Pi \mid \mathcal{F}_s) \to 0$ in $L_2(P)$as $|\Pi| \to 0$. Since the function $u \to E(f''(X_u) \mid \mathcal{F}_s))$ is continuous a.s., we get that

$$\sum_{k=1}^n E(f''(X_{t_{k-1}}) \mid \mathcal{F}_s)(t_k - t_{k-1}) \to \int_s^t E(f''(X_u) \mid \mathcal{F}_s)du \quad \text{a.s.} \tag{2.6}$$

when $|\Pi| \to 0$ and since f'' is bounded, the bounded convergence theorem gives that the convergence in (2.6) is also in $L_2(P)$. Combining the above we get formula (2.3). □

We also need

Lemma 2.4 *Let $n \in \mathbb{N}$, let $Y_j : \Omega \to \mathbb{R}$, $1 \leq j \leq n$ be stochastic variables, and put $Y = (Y_1, Y_2, \cdots, Y_n) : \Omega \to \mathbb{R}^n$. Further, let ϕ_{Y_j} denote the characteristic function of Y_j for $1 \leq j \leq n$ and ϕ_Y the characteristic function of Y. Then $Y_1, Y_2, \ldots, Y_n$ are independent if and only if*

$$\phi_Y(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} \phi_{Y_j}(x_j) \tag{2.7}$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Proof: It follows from the definition of independence that $Y_1, Y_2, \ldots, Y_n$ are independent if and only if $Y(P) = \otimes_{j=1}^{n} Y_j(P)$ Noting that the right hand side of (2.7) is the characteristic function of $\otimes_{k=1}^{n} Y_j(P)$, the statement of the lemma follows from the above and the uniqueness theorem for characteristic functions [2, Theorem 2.3]. □

Proof of Theorem 2.2: The main part of the proof will be to prove that for all $0 \leq s \leq t$ we have the formula

$$E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = \exp(-\frac{1}{2}u^2(t-s)) \quad \text{for all } u \in \mathbb{R}. \tag{2.8}$$

To prove (2.8) fix an s with $0 \leq s < \infty$, a $u \in \mathbb{R}$ and apply Theorem 2.3 to the function $f(x) = \exp(iux)$ for all $x \in \mathbb{R}$. For all $s \leq t$ we then obtain:

$$E(\exp(iuX_t) \mid \mathcal{F}_s) = \exp(iuX_s) - \frac{1}{2}u^2 \int_s^t E(\exp(iuX_v) \mid \mathcal{F}_s)dv$$

or

$$E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = 1 - \frac{1}{2}u^2 \int_s^t E(\exp(iu(X_v - X_s)) \mid \mathcal{F}_s)dv. \tag{2.9}$$

Since the integrand on the right side of (2.9) is continuous, the left hand side is differentiable with respect to t and

$$\frac{d}{dt}E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = -\frac{1}{2}u^2 E(\exp(iu(X_t - X_s) \mid \mathcal{F}_s).$$

This shows that on $[s, \infty[$ $E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s)$ is the solution to the differential equation

$$g'(t) = -\frac{1}{2}u^2 g(t)$$

with the initial condition $g(s) = 1$. Hence

$$E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s) = \exp(-\frac{1}{2}u^2(t-s)) \quad \text{for all } 0 \leq s \leq t$$

and equation (2.8) is established.

Let now $0 \leq s < t$. By (2.8) the characteristic function of $X_t - X_s$ is given by:

$$E(\exp(iu(X_t - X_s)) = E(E(\exp(iu(X_t - X_s)) \mid \mathcal{F}_s)) = \exp(-\frac{1}{2}u^2(t-s))$$

and hence by [2, Theorem 2.11] $X_t - X_s$ is normally distributed with mean 0 and variance $t - s$.

Let now $0 = t_0 < t_1 < t_2 < \cdots < t_n < \infty$ and put $Y = (X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$. If ϕ_Y denotes the characteristic function of Z, then we get for all $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}$:

$$\begin{aligned} \phi_Y(u) = \exp(i < u, Y >) &= E(\prod_{k=1}^{n} \exp(iu_k(X_{t_k} - X_{t_{k-1}})) = \qquad (2.10) \\ E(E(\prod_{k=1}^{n} \exp(iu_k(X_{t_k} - X_{tk-1})) \mid \mathcal{F}_s) &= \exp(-\frac{1}{2}u_n^2(t_n - t_{n-1}))E(\prod_{k=1}^{n-1} \exp(iu_k(X_{t_k} - X_{t_{k-1}})) \end{aligned}$$

Continuing in this way we obtain:

$$\phi_Y(u) = \prod_{k=1}^{n} \exp(-\frac{1}{2}u_k^2(t_k - t_{k-1}) = \prod_{k=1}^{n} E(\exp(iu_k(X_{t_k} - X_{t_{k-1}}))$$

which together with Lemma 2.4 shows that $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

Thus we have proved that (X_t) is a normalized Brownian motion. □

In many cases where Theorem 2.2 is used $\mathcal{F}_t$ is for each t the σ–algebra generated by $\{X_s \mid 0 \leq s \leq t\}$ and the sets of measure 0. However, the theorem is often applied to cases where the $\mathcal{F}_t$'s are bigger.

We end this note by showing that the continuity assumption in Theorem 2.2 can not be omitted. Let us give the following definition:

Definition 2.5 *An $(\mathcal{F}_t)$–adapted process $(N_t)$is called a Poisson process with intensity* 1 *if $N_0 = 0$ a.s. and for $0 \leq s < t$, $N_t - N_s$ is independent of $\mathcal{F}_s$ and Poisson distributed with parameter $t - s$.*

Hence if (N_t) is a Poisson process with intensity 1, then $N_t - N_s$ takes values in $\mathbb{N} \cup \{0\}$ for all $0 \leq s < t$ and

$$P(N_t - N_s = k) = \frac{(t-s)^k}{k!}\exp(-(t-s)) \quad \text{for all } k \in \mathbb{N} \cup \{0\}$$

It follows e.g. from [1, Problem 3.2, page 11 ff] that such a process (N_t) exists.

Easy calculations show that $E(N_t - N_s) = t - s = V(N_t - N_s)$. The process (M_t), where $M_t = N_t - t$ for all $t \in [0, \infty[$, is called *the compensated Poisson process* with intensity 1. Note that (M_t) is not continuous. We have:

Theorem 2.6 *If (M_t) is a compensated Poisson process with intensity 1, then it satisfies the conditions (i) and (ii) in Theorem 2.2.*

Proof: Let $0 \leq s < t$. Since $M_t - M_s$ is independent of $\mathcal{F}_s$, we get

$$E(M_t \mid \mathcal{F}_s) = M_s + E(M_t - M_s) = M_s.$$

Since $M_t^2 = M_s^2 + (M_t - M - s)^2 + 2M_s(M_t - M_s)$, we also get

$$E(M_t^2 \mid \mathcal{F}_s) = M_s^2 + E((M_t - M_s) \mid \mathcal{F}_s) + 2M_s E(M_t - M_s \mid \mathcal{F}_s) = (t - s) + M_s^2.$$

□

References

[1] I. Karatsas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Second Edition, Springer Verlag.

[2] N.J. Nielsen, *Brownian Motion*, Lecture Notes.