# Answers to the exam problems, Januar 2013

February 7, 2013

Please note that in most of the questions in this exam set one can argue in more than one way to get the solution.

My answer should not really be considered as a standard answer to the exam questions because this document is written after the correction of your answers and therefore includes some comments on the mistakes I have seen doing this.

## Problem 1

1. Let $h$ be as in the problem. An easy computation shows that $h(\frac{\pi}{6}) = \sqrt{\frac{\pi}{6}} - \frac{1}{\sqrt{3}} > 0$ (since $\pi > 2$!). If $x \to \frac{\pi}{2}$, then $\sqrt{x} \to \sqrt{\frac{\pi}{2}}$, while $\tan x \to \infty$ and therefore $h(x) \to -\infty$. Since $[0, \frac{\pi}{2}[$ is connected and $h$ is continuous, $h([0, \frac{\pi}{2}[)$ is connected and therefore an interval. Since by the above $h$ attains both negative and positive values, there is an $a \in ]0, \frac{\pi}{2}[$ with $h(a) = 0$.

Differentiating $h$ we get that

$$h'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{\cos^2(x)} \text{for all } 0 < x < \tfrac{\pi}{2}.$$

Since cos strictly decreasing on the interval $]0, \frac{\pi}{2}[$, it follows that $h'$ is strictly decreasing on that interval, and furthermore the formula for $h'$ gives that $h'(x) \to \infty$ for $x \to 0$ and that $h'(x) \to -\infty$. Again a connectivity and continuity argument tells us that $h'$ attains the value 0, and since it is strictly decreasing, it attains the value 0 in exactly one point, say b. It follows that $h$ is strictly increasing on the interval $[0, b]$ and strictly decreasing on the interval $[b, \frac{\pi}{2}[$. Therefore $h(x) > 0$ for all $0 < x < a$ and $h(x) < 0$ for all $a < x < \frac{\pi}{2}$

This can help us make the right drawing of $S$.

2. Looking on the inequalities in the definition of the sets $A$ and $B$ it should be clear that figure 2 is the right one.

3. Since e.g. $A$ is contained in a disc with center $(0, 0)$ radius $\frac{\pi}{2}$, it is a bounded set.

Since tan and the square root are continuous functions and we have "$\leq$" in the definition of $A$, it follows that $A$ is closed.

A more stringent way to see that $A$ is closed is the following:

Let $v, w : [0, \frac{\pi}{2}[\times\mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} v(x,y) &= y - \tan x \quad \text{for } (x,y) \in [0, \tfrac{\pi}{2}[\times\mathbb{R} \\ g(x,y) &= \sqrt{x} - y \quad \text{for all } (x,y) \in [0, \tfrac{\pi}{2}[\times\mathbb{R} \end{aligned}$$

It is easy to see that $A = v^{-1}([0,\infty[) \cap w^{-1}([0,\infty[)$. Since both $v$ and $w$ are continuous, the two sets appearing on the right side of this equation are closed and therefore $A$ is closed.

Since $A$ is both closed and bounded, it follows from the Heine–Borel Theorem that $A$ is compact.

4. A point of the form $(x, \tan x)$ with $x > a$ belongs to $B$ and is also a boundary point for $B$. Therefore $B$ is not open.

**Comment:** Several of you have just proved that $B$ is not closed (which is correct) but this does **not** answer the question. Some of you have also used this to conclude falsely that $B$ is open.

Let us consider a point $z = (\frac{\pi}{2}, y)$ where $y \geq \sqrt{\frac{\pi}{2}}$. Any disc with center z will intersect both $B$ and the complement of $B$ and $z$ is therefore a boundary point of $B$. Clearly also all points $(x,y)$ with $a \leq x < \frac{\pi}{2}$ and $\sqrt{x} \leq y \leq \tan x$ belongs to $\overline{B}$. Hence clearly

$$\overline{B} = \{(x,y) \in \mathbb{R}^2 \mid a \leq x < \frac{\pi}{2} \quad \text{and } \sqrt{x} \leq y \leq \tan x\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = \frac{\pi}{2}, \quad y \geq \sqrt{\frac{\pi}{2}}\}$$

**Comment:** Many of you have forgotten the half–line above as a part of $\overline{B}$.

4 again. It is easy to see that $A$ is convex and therefore path connected. If $(x,y) \in B$, then you can go along a vertical line up to the point $(x, \tan x)$ and then go along the tangens curve until you reach the point $(a, \tan a)$ which belongs to $S$. Summing up any point from $S$ can be connected to the point $(a, \tan a)$ by a curve contained in $S$. Therefore $S$ is path connected and hence connected.

**Comment:** Some of you have claimed that also $B$ is convex but this is clearly not so.

5. Clearly $int(S) = int(A) \cup int(B)$ and note that these two open sets are disjoint and therefore by definition $S$ is disconnected. One can also argue as follows: Since $(a, \tan x) \notin S$, $S$ is not path connected and since $int(S)$ is open, it follows from Theorem 7.13 that $S$ is disconnected.

Clearly $S \setminus \{(a, \sqrt{a})\} \subseteq \{(x,y) \mid x < a\} \cup \{(x, y \mid x > a\}$. Since both sets on the right hand side are open and disjoint and non–empty, it follows from Theorem 7.3 that the set on the left hand side is disconnected.

## Problem 2

1. Since $f$ is continuous and $S$ is connected, $f(S)$ is connected too and therefore an interval. Since $1 \notin S$, this interval must be either strictly to the right or strictly to the left of 1. By (ii) $f(S)$ contains numbers arbitrarily close to 0, so we must have $f(S) \subseteq ]-\infty, 1[$.

2. Since $A$ is compact and $f$ is continuous, $f(A)$ is compact too.

3. Assume that the set $\{n \in \mathbb{N} \mid u_n \in A\}$ is infinite. Then we can construct a subsequence $(u_{n_k}) \subseteq A$ and hence $f(u_{n_k}) \in f(A)$ for all $k \in \mathbb{N}$. But since $f(u_n) \to 0$ for $n \to \infty$, also $f(u_{n_k}) \to 0$ for $k \to \infty$ and hence $0 \in \overline{f(A)}$. However, since $f(A)$ is compact, it is closed and therefore $0 \in f(A)$ which is a contradiction. Hence the set in question is either finite or empty.

One can also argue as follows: Since $A$ is compact, the sequence $(u_{n_k})$ has a convergent subsequence with a limit in $A$. This together with the continuity of $f$ now again gives that $0 \in f(A)$.

## Problem 3

1. Since $f$ is a fraction between products, sums and compositions of continuous functions and the denominator $1 + |x - y| > 0$ for all $(x, y) \in [0, 1] \times [0, 1]$, $f$ is continuous by earlier exercises.

Since $[0, 1] \times [0, 1]$ is closed and bounded, it is compact by the Heine–Borel Theorem, and hence it follows from the continuity of $f$ that $f([0, 1] \times [0, 1])$ is compact too and therefore also closed and bounded.

Since $[0, 1] \times [0, 1]$ is clearly convex and therefore connected and $f$ is continuous, $f([0, 1] \times [0, 1])$ is connected as well and therefore an interval.

Summing up we have obtained that $f([0, 1] \times [0, 1])$ is a closed and bounded interval.

2. Since $f$ is continuous and $[0, 1] \times [0, 1]$ is compact and metric, $f$ is uniformly continuous on that set by Theorem 6.32. Hence to $\varepsilon = 14$ there exists a $\delta_1 > 0$ so that for all $(x_1, y_1), (x_2, y_2) \in [0, 1] \times [0, 1]$ we have

$$\mathbf{d_2}((x_1, y_1), (x_2, y_2)) < \delta_1 \Rightarrow |f(x_1, y_1) - f(x_2, y_2)| < 14$$

If we put $\delta = \delta_1^2$, we get the desired.

**Comment:** Convexity has only meaning in a vector space and $\mathbb{R}^2$ is of course a normed space. However, several of you have said that $[0, 1] \times [0, 1]$ is a normed space which is certainly false!! (A normed space different from $\{0\}$ can of course not be bounded. The set is a **subset** of a normed space and this is enough for the argument.

## Problem 4

1. Since $X$ is connected and $f$ is continuous, $f(X) \subseteq \mathbb{R}$ is connected as well and therefore an interval.

Since $X$ is compact and $f$ is continuous, $f(X)$ is compact and therefore closed and bounded. Summing up we obtain that $f(X)$ is a closed and bounded interval.

2. We now consider $f$ as a continuous bijection of $X$ onto $[a, b]$. Since $X$ is compact, it follows from Theorem 8.3 that $f$ is a homeomorphism, which means that $f^{-1} : [a, b] \to X$ is continuous.

3. Since the function $t \to (1-t)a + tb$ and $f^{-1}$ are continuous, $h : [0, 1] \to X$ is continuous and therefore a path. Since $t \to (1-t)a + tb$ maps $[0, 1]$ onto $[a, b]$ and $f^{-1}$ maps $[a, b]$ onto $X$, $h$ maps $[0, 1]$ onto $X$ and hence $X$ is a curve and in particular path connected.

4. Assume that $g$ is continuous. Since $Y$ is compact and connected we can let $Y$ be the $X$ and let $g$ be the $f$ above. The argument above will then show that $Y$ is path connected which is a contradiction.

## Problem 5

1. Since $X$ is compact and metric, $X$ is sequentially compact and therefore $(x_n)$ has a convergent subsequence $(x_{n_k})$.

2. We wish to show that $x_n \to x$ so let $\varepsilon > 0$ be given arbitrarily and let $n_0$ be chosen so that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.

Since $n_k \to \infty$ and $x_{n_k} \to x$ for $k \to \infty$, we can find a $k_0$ so that both $n_k \geq n_0$ and $d(x, x_{n_k}) < \varepsilon$ for all $k \geq k_0$.

Let now $n \geq n_0$. We wish to estimate $d(x, x_n)$. If we put $k = k_0$, then we get by the choice of $k$

$$d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) < 2\varepsilon \quad \text{for all } n \geq n_0$$

Since this holds for all $\varepsilon > 0$, $d(x, x_n) \to 0$ for $n \to \infty$ which means that $x_n \to x$.

If you do not like the $2\varepsilon$, you could have started with $\frac{\varepsilon}{2}$ instead.

**Comment 1:** You have all tried to use the triangle inequality as suggested in the problem formulation, but some of you have inserted points so that you get terms which you cannot control. Think about it the following way: We wish to show that $d(x, x_n) \to 0$ which means that we have to estimate this number from above and hence it should be put on the left hand side of the triangle inequality. We now have to insert points so that we know something about each term that appears on the right hand side of the triangle inequality. Since we know something about $d(x, x_{n_k})$ and $d(x_n, x_{n_k})$ when $n$ and $k$ are big it is a good idea to inser the point $x_{n_k}$.

Some of you have used the triangle inequality in the right way as above and then let $n$ and $k$ tend to $\infty$ at the same time. It is a little difficult maybe to keep in one's head what happens in this situation, but it is posssible to argue succesfully.

**Comment 2:** A sequence which satisfies 2. in the formulation of the problem is called a Cauchy sequence and a metric space is called complete if every Cauchy sequence is convergent. Hence we have proved above that **a compact metric space is complete.**

Completeness is a very important property because for a given sequence it is in most cases easier to prove that it is a Cauchy sequence than to prove that it is convergent. In the latter case you have namely to guess the limit first.

Complete metric spaces will be studied in Topology 2 and I am sure that the result in Problem 5 will be a theorem there!