Continuous functions on $\mathbb{R}$

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We tacitly assume that the reader is familar with the continuity properties of the classical functions defined on the $\mathbb{R}$ or intervals of $\mathbb{R}$. However, let us recall that if $U \subseteq \mathbb{R}$ and $f : U \to \mathbb{R}$ is a function, then f is said to be continuous in a point $x_0 \in U$, if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in U : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{1.1}$$

Intuitively speaking, this means that when x gets close to x_0, then $f(x)$ gets close to $f(x_0)$. f is said to be continuous if it is continuous in all points of U. If we write this with quantifiers, we get:

$$\forall \varepsilon > 0 \quad \forall x \in U \quad \exists \delta > 0 \quad \forall y \in U : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \tag{1.2}$$

It is always a bit dangerous to interchange quantifiers in a logical statement because the statement changes radically. Let us anyway look on the following statement:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in U \quad \forall y \in U : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \tag{1.3}$$

If we do a little text analysis of the two statements we see that in (1.2) the δ depends on ε and x while in (1.3) the δ only depends on ε and thus works for all $x, y \in X$. The statement (1.3) makes perfectly sence and gives rise to the following definition:

Definition 1.1 *Let $U \subseteq \mathbb{R}$. A function $f : U \to \mathbb{R}$ is called uniformly continuous if it satisfies* (1.3)

The word "uniformly" is used because given $\varepsilon > 0$, one can use the same δ for all $x, y \in X$. The next statement is really an example, but we formulate it as a proposition.

Proposition 1.2 *Let $f : [1, \infty[\to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$ for all $1 \leq x < \infty$. Then f is uniformly continuous.*

Proof: Let $x, y \geq 1$ be arbitrary. Since f is differentiable, we can by the mean value theorem find a ξ between x and y so that

$$f(x) - f(y) = f'(\xi)(x - y).$$

Since $\xi \geq 1$ and $f'(\xi) = \frac{1}{2\sqrt{\xi}}$, we get that $|f'(\xi)| \leq \frac{1}{2}$ and hence

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

which holds for all $x, y \geq 1$. If now $\varepsilon > 0$ is arbitrary, we can choose a $0 < \delta < 2\varepsilon$ and if $|x - y| < \delta$, then by the above:

$$|f(x) - f(y)| \leq \varepsilon.$$

This shows that f is uniformly continuous. □

We shall later prove that any continuous function defined on a closed and bounded interval of $\mathbb{R}$ is uniformly continuous. Combining this with Proposition 1.2 we get that the square root function is in fact uniformly continuous on $[0, \infty[$. The next example shows that even very nice continuous functions need not be uniformly continuous.

Example 1.3 *let $g : \mathbb{R} \to \mathbb{R}$ be defind by $g(x) = x^2$ for all $x \in \mathbb{R}$. We claim that g is not uniformly continuous. It is clearly enough to prove that g is not uniformly continuous on $[0, \infty]$. To see this we put $\varepsilon = 1$ and let $0 < \delta \leq 1$ be arbitrary. If $x \geq 0$, we get*

$$0 \leq g(x + \delta) - g(x) = (x + \delta)^2 - x^2 = (2x + \delta)\delta.$$

For all $x > \frac{1}{2}(\delta^{-1} - \delta)$ we get that

$$g(x + \delta) - g(x) > 1$$

which shows that g is not uniformly continuous.