Obligatory Problems MM513

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Problem 1

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_n)$ be a filtration of $\mathcal{F}$. Further let $(X_n)_{n\geq 0}$ be a sequence of s.v.'s so that X_n is $\mathcal{F}_n$–measurable for all $n \geq 0$ and let $(C_n)_{n\geq 1}$ be a sequence of sv.'s so that C_n is $\mathcal{F}_{n-1}$ for all $n \geq 1$ and so that there is a constant K so that $0 \leq C_n \leq K$ a.s. for all $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we put

$$Y_n = \sum_{k=1}^{n} C_k(X_k - X_{k-1}), \quad Y_0 = 0.$$

(i) Prove that (Y_n) is a martingale, if (X_n) is a martingale, a submartingale, if (X_n) is a submartingale, and a supermartingale, if (X_n) is a supermartingale.

Let now T be a stopping time so that $P(T < \infty) = 1)$.

(ii) Prove that the set $(T < n) \in \mathcal{F}_{n-1}$ for all $n \geq 1$ and conclude that $1_{(T\geq n)}$ is $\mathcal{F}_{n-1}$–measurable for all $n \geq 1$.

(iii) Prove that for all $n \geq 1$ we have

$$X_{T\wedge n} - X_0 = \sum_{k=1}^{n} 1_{(T\geq k)}(X_k - X_{k-1}).$$

Conclude from (i) that $(X_{T\wedge n})$ is a martingale (relative to $(\mathcal{F}_n)$), if (X_n) is a martingale, a submartingale, if (X_n) is a submartingale, and a supermartingale, if (X_n) is a supermartingale.

iv) Prove that for almost all $\omega \in \Omega$ we have that $T(\omega) \wedge n = T(\omega)$ for n sufficiently large and hence also $X_{T\wedge n}(\omega) = X_T(\omega)$ for n sufficienly large.

In particular $X_T(\omega) = \lim_n X_{T\wedge n}(\omega)$ for almost all $\omega \in \Omega$.

(v) Assume now that (X_n) is a supermartingale with $X_n \geq 0$ a.s for all $n \geq 0$. Prove that

$$E(X_T) \leq E(X_0).$$

Problem 2, the mother of all martingales

We consider the probality space $([0,1],\mathcal{B},P)$ where P denotes the Lebesgue measure on $[0,1]$ and $\mathcal{B}$ denotes the σ–algebra of all Borel subsets of $[0,1]$. We define a sequence (h_n) of measurable functions by:

$$h_1(t) = 1 \quad \text{for all } t \in [0,1].$$

For all $k \in \mathbb{N} \cup \{0\}$ og $1 \le \ell \le 2^k$ we put:

$$h_{2^k+\ell}(t) = \left\{ \begin{array}{rll} 1 & \text{if} & t \in [(2\ell-2)2^{-k-1}, (2\ell-1)2^{-k-1}[\\ -1 & \text{if} & t \in [(2\ell-1)2^{-k-1}, 2\ell 2^{-k-1}[\\ 0 & \text{otherwise.} & \end{array} \right\}$$

(h_n) is called the Haar system on $[0,1]$.

1. Draw the graphs of the first 5 Haar functions.
2. Let for all $n \in \mathbb{N}$ $\mathcal{F}_n = \sigma\{h_m \mid 1 \le m \le n\}$. Recall that the atoms in $\mathcal{F}_n$ are precisely the sets in $\mathcal{F}_n$ on which all the h_m's are constant for $1 \le m \le n$. Let now $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ with $2^k < n \le 2^{k+1}$ and $B \in \mathcal{F}_n$. Prove that B is an atom in $\mathcal{F}_n$ if and only if: Either there is an m with $2^k < m \le n$ so that

   $$B = \{t \in [0,1] \mid h_m(t) = 1\}$$

   or

   $$B = \{t \in [0,1] \mid h_m(t) = -1\}$$

   or in the case where $n < 2^{k+1}$: B is an atom in $\mathcal{F}_{2^k}$ and $B \subseteq \cap_{m=2^k+1}^{n} h_m^{-1}(0)$.

   This desciption is of course modulo zero–sets.
3. Show that if $m < n$ and $B \in \mathcal{F}_m$ is an atom, then

   $$\int_B h_n(t)dt = 0$$

4. Let $(t_n) \subseteq \mathbb{R}$ and define $X_n : [0,1] \to \mathbb{R}$ by

   $$X_n = \sum_{m=1}^{n} t_m h_m \quad \text{for all } n \in \mathbb{N}.$$

   Show that (X_n) is a martingale. Here it is a good idea to consult exercise 2 in Exercises for MM513.

For every $n \in \mathbb{N}$ we put

$$A_n = \{t \in [0,1] \mid h_n(t) = 1\} \cup \{t \in [0,1] \mid h_n(t) = -1\}.$$

5. Let $n \in \mathbb{N}$, $f \in L_1([0,1])$. Show that if either $B = \{t \in [0,1] \mid h_{n+1}(t) = 1\}$ or $B = \{t \in [0,1] \mid h_{n+1}(t) = -1\}$, then

$$\int_B E(f \mid \mathcal{F}_n)(t)dt = \frac{1}{2}\int_{A_{n+1}} f(t)dt.$$

Hint: Use that an $\mathcal{F}_n$–measurable stochastic variable is constant on an atom in $\mathcal{F}_n$.

Let now $f \in L_1([0,1])$ be fixed in the rest of this problem. For every $n \in \mathbb{N}$ we put

$$t_n = P(A_n)^{-1}\int_{A_n} f(t)h_n(t)dt.$$

By induction we want to prove that:

$$E(f \mid \mathcal{F}_n) = \sum_{m=1}^{n} t_m h_m \quad \text{for all } n \in \mathbb{N}, \tag{0.1}$$

but we do it stepwise.

6. To prove (0.1) it is enough to prove that for all $n \in \mathbb{N}$ and every atom $B \in \mathcal{F}_n$ we have:

$$\int_B f(t)dt = \int_B \sum_{m=1}^{n} t_n h_m(t)dt. \tag{0.2}$$

Why?

7. Prove (0.1) by induction. Hint: If (0.1) is proved for n, we can write:

$$\sum_{m=1}^{n+1} t_m h_m = E(f \mid \mathcal{F}_n) + t_{n+1}h_{n+1}.$$

Now use 5. and (0.2) in a suitable manner.

8. Prove that

$$f = \sum_{m=1}^{\infty} t_m h_m,$$

where the convergence is in $L_1[0,1]$.

Hint: Let $\mathcal{B}$ be the Borel algebra on $[0,1]$ and $\mathcal{F}_\infty$ as in the notes. Without proof you may use that $\mathcal{B} = \mathcal{F}_\infty$.

You have now proved that (h_n) is a basis for $L_1([0,1])$ in the sense of Banach spaces..

Problem 3

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let (Y_n) be a sequence of independent stochastic variables so that $E(Y_n) = 1$ og $Y_n > 0$ n.s. for alle $n \in \mathbb{N}$. Put $\mathcal{F}_n = \sigma\{Y_j \mid 1 \leq j \leq n\}$ and $X_n = \prod_{j=1}^n Y_j$.

(i) Prove that (X_n) is a martingale and show that there is a stochastic variable X so that $X_n \to X$ n.s.

(ii) Show that $E(X) \leq \lim_n \prod_{j=1}^n E(Y_j) = \prod_{j=1}^\infty E(Y_j) = 1$.

(iii) Assume further that we have:

$$P(Y_n = \frac{1}{2}) = P(Y_n = \frac{3}{2}) = \frac{1}{2}$$

for alle $n \in \mathbb{N}$ so that the Y_n's are identically distributed. Show that $X = 0$ n.s. Hint: Compute $E(\log Y_n)$ and realize that $E(\log Y_n) < 0$. Now apply the Strong Law of Large numbers (either Theorem 5.3.1 or Theorem 5.4.4 in the book; the latter will be proved in week 19) on the the sequence $(\log Y_n)$.

Hence it can happen that the inequality in (ii) is sharp.

(iv) Show that under the conditions in (iii) the sequence (X_n) **does not** converge to X in $L_1(P)$.

Problem 4

Let$(\Omega, \mathcal{F}, P)$ be a probability space and let $(X_n)_{n\geq 0} \subseteq L_2(P)$ be a martingale (relative to some filtration $(\mathcal{F}_n)$ of $\mathcal{F}$) with $X_0 = 0$.

(i) Prove that (X_n^2) is a submartingale and conclude that there exists a martingale (M_n) and a non–decreasing process (A_n) so that $M_0 = A_0 = 0$, A_n is $\mathcal{F}_{n-1}$–measurable for all $n \geq 1$, and

$$X_n^2 = M_n + A_n \quad \text{for all } n \geq 0.$$

Put $A_\infty = \lim_n A_n$ and prove that $E(X_n^2) = E(A_n)$ and that $E(X_n^2) \to E(A_\infty)$ for $n \to \infty$.

Conclude that (X_n) is bounded in $L_2(P)$ if and only if $E(A_\infty) < \infty$.

From now on we assume that $E(A_\infty) < \infty$.

(ii) Prove that (X_n) is uniformly integrable and that there is an $X \in L_1(P)$ so that $X = \lim_n X_n$ a.s. and in $L_1(P)$.

(iii) Prove that for all $m \leq n$ we have that

$$E((X_n - X_m)^2 \mid \mathcal{F}_m) = E(X_n^2 \mid \mathcal{F}_m) - X_m^2$$

and conclude that

$$E((X_n - X_m)^2) = E(X_n^2) - E(X_m^2).$$

(iv) Let now m be fixed for a moment. Use (i) and (ii) to prove that

$$E((X - X_m)^2) \leq E(A_\infty) - E(X_m^2) \tag{0.3}$$

and conclude that $X \in L_2(P)$.

Finally use (0.3) to prove that $X_m \to X$ in $L_2(P)$