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THE POSITIVE APPROXIMATION PROPERTY OF BANACH LATTICES

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ABSTRACT

In this paper we study the positive approximation property (p.a.p.) of Banach lattices. The main results give some characterizations of the p.a.p. and the bounded p.a.p. Some perturbation results on positive operators, which are of interest in other contexts, too, are proved.

Introduction

A Banach lattice X is said to have the positive approximation property (p.a.p.) if the identity can be approximated uniformly on compact sets by positive, bounded finite rank operators. If these operators in addition can be chosen to be uniformly bounded in norm, we say that X has the bounded positive approximation property (b.p.a.p.).

It is an open problem whether every Banach lattice with the usual approximation property of Grothendieck also has the p.a.p., and we have not been able to solve this problem here, except for a very special case (Theorem 2.1 due to L. Tzafriri and the author) which actually covers all known examples of Banach lattices with the approximation property. There are some indications that the problem has an affirmative answer for super-reflexive spaces, but so far Theorem 3.6 of [2] is the only result which in general links the a.p. of a Banach lattice to the order (it is formulated for the uniform a.p., but a similar result holds for b.a.p.).

In section 1 of this paper we give a few characterizations of the p.a.p. in terms of traces of positive nuclear operators.

Section 2 contains some of the main results of the paper. Here we give

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several conditions on a Banach lattice X, equivalent to X having the b.p.a.p. Some of these conditions are similar to the ones which are known for the b.a.p. for Banach spaces, the proofs, however, normally do not carry over, the obstacle being that perturbations of operators often spoil positivity. We solve this by proving some perturbation theorems which keep positivity. As a corollary, we get that every reflexive Banach lattice with the p.a.p. has the metric p.a.p.

Part of this work was done during my visit to the Department of Mathematics, Texas A and M University, in August 1986. I wish to thank W. B. Johnson and J. Zinn for many fruitful discussions during that period.

0. Notation and preliminaries

Throughout the paper, we shall use the notation and terminology of the theory of Banach spaces and Banach lattices, as it appears in [4] and [5].

If X and Y are Banach spaces, we let $B(X, Y)$ ($B(X)$ if $X = Y$) denote the space of all bounded operators from X to Y and $K(X, Y)$ ($K(X)$) if $X = Y$) the space of all compact operators from X to Y. If X and Y are Banach lattices, and $\mathscr{A}(X, Y)$ is a subspace of $B(X, Y)_+$, we let $\mathscr{A}(X, Y)_+$ denote the cone of all positive $T \in \mathscr{A}(X, Y)$.

We shall identify the tensor product $X^* \otimes Y$ with the space of all bounded finite rank operators from X to Y. $X \otimes_\pi Y$ denotes the completed π-tensor product of the Banach spaces X and Y, and if $\mathscr{U} = \sum_{n=1}^{\infty} x_n^* \otimes x_n \in X^* \otimes_\pi X$ we put $\mathrm{Tr}\, \mathscr{U} = \sum_{n=1}^{\infty} x_n^*(x_n)$ (that is, Tr is taken in the sense of tensors and not in the sense of operators). If X is a Banach lattice, then $(X^* \otimes_\pi X)_+$ consists of those $\mathscr{U} \in X^* \otimes_\pi X$ for which $\mathscr{U}x \geqq 0$ for all $x \in X$, $x \geqq 0$ (that is $\mathscr{U}$ is positive considered as an operator).

If T is a nuclear operator, $n_1(T)$ denotes the nuclear norm, and if T is integral, we let $i_1(T)$ denote the integral norm. Finally, if E is a Banach space and X a Banach lattice, an operator $T \in B(E, X)$ is called order bounded if T maps the unit ball of E into an order bounded subset of X. The order bounded norm

$$\| T \|_m = \inf\{ \| z \| \, z \in X, z \geqq 0, \, | Tx | \leqq \| x \| z \text{ for all } x \in E \}.$$

$X^* \otimes_m X$ denotes the completion of $X^* \otimes X$ in the norm $\| \cdot \|_m$, and it is known to be a Banach lattice. For further information on order bounded operators we refer to [2], [7], and [8].

1. The positive approximation property of Banach lattices

We start with the following definition:

1.1. DEFINITION. A Banach lattice X is said to have the positive approximation property (p.a.p.) if for every compact set $K \subseteq X$ and every $\varepsilon > 0$ there is a bounded positive, finite rank operator T on X with $\| x - Tx \| \leqq \varepsilon$ for all $x \in K$.

All known examples of Banach lattices with the usual approximation property (a.p.) also have the p.a.p.

The next theorem gives some equivalent formulation of the p.a.p. Some of these are very similar to the corresponding ones for the a.p. (see e.g. [4]). Before we prove it, we need the following lemma.

1.2. LEMMA. *Let* $\mathcal{U} \in (X^* \otimes_\pi X)_+$. *Then* $\operatorname{Tr}(S\mathcal{U}) \geqq 0$ *for all* $S \in (X^* \otimes X)_+$.

PROOF. By considering X^{**} instead of X, if necessary, we can without loss of generality assume that X is order complete.

Let $\mathcal{U} = \Sigma_{m=1}^{\infty} x_m^* \otimes x_m \in (X^* \otimes_\pi X)_+$. If $S = \Sigma_{j=1}^{k} y_j^* \otimes y_j \in X^* \otimes X$ with the y_j's mutually disjoint and positive, then $y_j^* \geqq 0$ for all $1 \leqq j \leqq k$, and we get

$$\operatorname{Tr}(S\mathcal{U}) = \sum_{j=1}^{k} y_j^*(\mathcal{U} y_j) \geqq 0. \tag{1}$$

If $S \in (X^* \otimes X)_+$ is arbitrary, then by e.g. [6], theorem 2.9, there is a sequence $(S_n) \subseteq (X^* \otimes X)_+$ of the form considered in (1) so that $\| S - S_n \| \rightarrow 0$. From this we obtain

$$|\operatorname{Tr}((S - S_n)\mathcal{U})| \leqq \| S - S_n \| \sum_{m=1}^{\infty} \| x_m^* \| \; \| x_m \| \rightarrow 0 \tag{2}$$

for $n \rightarrow \infty$.

(1) and (2) now give that $\operatorname{Tr}(S\mathcal{U}) \geqq 0$.

1.3. THEOREM. *Let X be a Banach lattice. The following statements are equivalent*:

(i) *X has the p.a.p.*

(ii) *For every Banach lattice Y, $(Y^* \otimes X)_+$ is dense in $B(Y, X)_+$ for the topology τ of uniform convergence on compact sets.*

(iii) *For every Banach lattice Y, $(X^* \otimes Y)_+$ is dense in $B(X, Y)_+$ for the topology τ of uniform convergence on compact sets.*

(iv) *For every* $\mathscr{U} \in (X^* \otimes_\pi X)_+$ $\mathrm{Tr}\,\mathscr{U} \geqq 0$.

PROOF. (i)$\Leftrightarrow$(iv): Since every τ-continuous linear functional on $B(X)$ is given by a $\mathscr{U} \in X^* \otimes_\pi X$, Lemma 1.2 shows that (iv) holds if and only if every τ-continuous functional on $B(X)$, which is non-positive on $(X^* \otimes X)_+$, is also non-positive on the identity. By the bipolar theorem this is equivalent to (i).

The other equivalences are obvious. ∎

It seems to be an open problem whether the statements in the above theorem are equivalent to that for all Banach lattices Y, $(Y^* \otimes X)_+$ is dense in $K(Y, X)_+$ in the operator norm. We have not been able to settle this.

2. The bounded positive approximation property

In this section we turn our attention to the investigation of the bounded positive approximation property (b.p.a.p.), which is defined as follows:

2.1. DEFINITION. Let $\lambda \geqq 1$. A Banach lattice X is said to have the λ-b.p.a.p., if it has the p.a.p. and the operator T in Definition 1.1 can be chosen to have norm less than or equal to λ.

The next theorem which gives a method to perturb positive finite rank operators on a Banach lattice, preserving positivity, shall be very useful for us in the sequel.

2.2. THEOREM. *Let X be an order complete Banach lattice, $\{z_j \mid 1 \leqq j \leqq k\}$ and $\{x_i \mid 1 \leqq i \leqq n\}$ be finite sets of mutually disjoint, positive elements in X, so that there are $p_i \in \mathbf{N}$, $p_i < p_{i+1}$ for $1 \leqq i \leqq n-1$, and $(t_j) \subseteq \mathbf{R}$, so that*

$$\text{(i)} \qquad x_i = \sum_{j=p_i+1}^{p_{i+1}} t_j z_j \qquad \text{for } 1 \leqq i \leqq n.$$

If $\varepsilon > 0$ and $T \in X^* \otimes [z_j]$ with $\| Tx_i - x_i \| \leqq \varepsilon$ for all $1 \leqq i \leqq n$, then there is an operator $S \in X^* \otimes [x_j]$, so that

$$\| Sx_i - x_i \| \leqq \varepsilon, \quad | Sx_i | \leqq x_i$$

$$\text{for all } 1 \leqq i \leqq n \quad \text{and} \quad \| \, |S| \, \| \leqq \| \, |T| \, \|.$$

If T is positive, then S can be chosen positive as well.

PROOF. Without loss of generality, we may assume that $\| x_i \| = \| z_j \| = 1$ for all $1 \leqq i \leqq n$ and all $1 \leqq j \leqq k$. Let, for all $1 \leqq i \leqq n$, P_i denote the band projection of X onto the band generated by x_i and define

$$T_1 = \sum_{i=1}^{n} P_i T P_i. \tag{1}$$

For every $x \in X$, $x \geqq 0$ and every $y \in X$, $|y| \leqq x$, we obtain

$$\begin{aligned} |T_1 y| &= \left| \sum_{i=1}^{n} P_i T P_i y \right| \\ &= \bigvee_{i=1}^{n} |P_i T P_i y| \leqq \bigvee_{i=1}^{n} |T P_i y| \leqq |T|(|y|) \leqq |T|(x), \end{aligned} \tag{2}$$

which gives that $\| \, |T_1| \, \| \leqq \| \, |T| \, \|$ and for $i \leqq n$

$$\| T_1 x_i - x_i \| = \left\| \sum_{j=1}^{n} P_j T P_j x_i - x_i \right\| = \| P_i T x_i - x_i \| \leqq \| T x_i - x_i \| \leqq \varepsilon. \tag{3}$$

If $(z_j^*) \subseteq X^*$ so that $T = \sum_{j=1}^{k} z_j^* \otimes z_j$, we get the following formula for T_1 in terms of the x_i's:

$$T_1 x = \sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} z_j^*(P_i x) z_j \qquad \text{for all } x \in X. \tag{4}$$

For every $1 \leqq i \leqq n$ and every $p_i + 1 \leqq j \leqq p_{i+1}$ we define

$$\alpha_{ij} = \begin{cases} \dfrac{t_j}{z_j^*(x_i)} & \text{when } |z_j^*(x_i)| > t_j \\ 1 & \text{else} \end{cases} \tag{5}$$

and put

$$S = \sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} \alpha_{ij} P_i^* z_j^* \otimes z_j. \tag{6}$$

Since $|\alpha_{ij}| \leqq 1$, we get for all $x \in X$, $x \geqq 0$, that $|S| x \leqq |T_1| x$ and hence

$$\| \, |S| \, \| \leqq \| \, |T_1| \, \| \leqq \| \, |T| \, \| . \tag{7}$$

Further, for $1 \leqq i \leqq n$,

$$|S x_i| = \sum_{j=p_i+1}^{p_{i+1}} |\alpha_{ij}| \; |z_j^*(x_i)| z_j \leqq \sum_{j=p_i+1}^{p_{i+1}} t_j z_j = x_i \tag{8}$$

and

$$\| Sx_i - x_i \| = \left\| \sum_{j=p_i+1}^{p_{i+1}} (\alpha_{ij} z_j^*(x_i) - t_j) z_j \right\|$$

$$(9) \qquad \leqq \left\| \sum_{j=p_i+1}^{p_{i+1}} (z_j^*(x_i) - t_j) z_j \right\|$$

$$= \| T_1 x_i - x_i \| \leqq \varepsilon.$$

We note that if T is positive, then both T_1 and S are positive, too. ■

We shall also need the following two lemmas, which reduce a more general situation to that of Theorem 2.2.

2.3. LEMMA. *Let X be an order continuous Banach lattice, $\{x_1, x_2, \ldots, x_n\} \subseteq X$ a set of mutually disjoint positive vectors, and $E \subseteq X$ a finite dimensional subspace. For every $\varepsilon > 0$ there are mutually disjoint positive elements $z_1, z_2, \ldots, z_k \in X$ and an operator $T: E \rightarrow [z_j]$ so that $x_i \in [z_j]$ for $1 \leqq i \leqq n$ and $\| Tx - x \| \leqq \varepsilon \| x \|$ for all $x \in E$.*

PROOF. Let $\varepsilon > 0$ and put $x_0 = \sum_{i=1}^n x_i$. x_0 is a weak order unit in the band Y generated by x_0. Put

$$(1) \qquad \mathscr{A} = \{ e \in Y \mid 0 \leqq e \leqq x_0, e_0 \wedge (x_0 - e) = 0 \}.$$

Since Y is order continuous, $\mathscr{A}$ is a complete Boolean algebra, and therefore every $x \in Y$ can be approximated by linear combinations of mutually disjoint elements of $\mathscr{A}$, see [5], 1.a.13. Hence letting P denote the band projection of X onto Y, we can find mutually disjoint elements $z_1, z_2, \ldots, z_r \in \mathscr{A}$ and a bounded operator $T_1 : P(E) \rightarrow [z_j]_{j=1}^r$, so that $\| T_1 x - x \| \leqq \varepsilon \| x \|$ for all $x \in P(E)$. Using the Boolean algebra structure of $\mathscr{A}$ and that $x_i \in \mathscr{A}$ for $1 \leqq i \leqq n$, we can without loss of generality assume that $x_i \in [z_j]_{j=1}^r$ for all $1 \leqq i \leqq n$.

It follows from [5] that we can find positive mutually disjoint elements $z_{r+1}, z_{r+2}, \ldots, z_k \in X$ and a bounded operator $T_2 : (I - P)(E) \rightarrow [z_j]_{j=r+1}^k$ so that $\| T_2 x - x \| \leqq \varepsilon \| x \|$ for all $x \in (I - P)(E)$. If we let $T = T_1 P_{|E} + T_2 (I - P)_{|E}$ then for all $x \in E$,

$$\| Tx - x \| \leqq \| T_1 Px - Px \| + \| T_2 (I - P)x - (I - P)x \| \leqq 2\varepsilon \| x \|. \quad ■$$

2.4. LEMMA. *Let X be an order continuous Banach lattice. Let $T \in X^* \otimes X$ and $\{x_1, x_2, \ldots, x_n\}$ mutually disjoint, positive elements. If $\varepsilon > 0$, then there is* *an $S \in X^* \otimes X$ of the form $S = \Sigma_{j=1}^k z_j^* \otimes z_j$, where the z_j's are mutually disjoint and positive, so that*

(i) $$\| T - S \|_m \leqq \varepsilon,$$

(ii) $$x_i \in [z_j] \qquad \textit{for all } 1 \leqq i \leqq n.$$

If T is positive, S can be chosen positive.

Proof. Using Lemma 2.3 and the ideas of Lemma 2.15 in [2], we can find a sequence (S_n) of operators of the required form with (ii) satisfied and so that $n_1(T - S_n) \to 0$ for $n \to \infty$. Hence also $\| T - S_n \|_m \to 0$ for $n \to \infty$ so that (i) can be achieved. If $T \geqq 0$, then since $X^* \otimes_m X$ is a Banach lattice, $\| T - S_n^+ \|_m \to 0$ for $n \to \infty$. ■

Combining the two last lemmas with Theorem 2.2, we can prove the following perturbation theorem, where the ideas of the proof go back to Johnson, Rosenthal and Zippin [3].

2.5. Theorem. *Let X be an order continuous Banach lattice, $\{x_1, x_2, \ldots, x_n\} \subseteq X$ a set consisting of mutually disjoint, positive norm one vectors. If $\varepsilon > 0$ and $T \in X^* \otimes X$ with $\| Tx_i - x_i \| \leqq \varepsilon$ for all $1 \leqq i \leqq n$, then there is a $U \in X^* \otimes X$ so that $Ux_i = x_i$ for $1 \leqq i \leqq n$ and $\| \, |U| \, \| \leqq \| \, |T| \, \| + (2n^2 + 1)\varepsilon$.*

If T is positive, then U can be chosen positive, as well.

Proof. By Lemma 2.4 we can find an operator $T_1 = \Sigma_{j=1}^k z_j^* \otimes z_j \in X^* \otimes X$, where the z_j's are positive and mutually disjoint, so that $\| T_1 - T \|_m \leqq \varepsilon$ and $(x_i) \subseteq [z_j]$. Hence $\| T_1 x_i - x_i \| \leqq 2\varepsilon$ for all $i \leqq n$ and

(1) $$\begin{aligned} \| \, |T_1| \, \| &\leqq \| \, |T| \, \| + \| \, |T_1 - T| \, \| \leqq \| \, |T| \, \| + \| T_1 - T \|_m \\ &\leqq \| \, |T| \, \| + \varepsilon. \end{aligned}$$

Using Theorem 2.2, we can now find an $S \in X^* \otimes X$ so that $S(X) \subseteq [z_j]_{j=1}^k$, $\| \, |S| \, \| \leqq \| \, |T_1| \, \| \leqq \| \, |T| \, \| + \varepsilon$, $|Sx_i| \leqq x_i$ and $\| Sx_i - x_i \| \leqq 2\varepsilon$ for $i \leqq n$.

Let now Q be a positive projection of X onto $[x_i]$ with $\| Q \| \leqq n$, and define $U \in X^* \otimes X$ by

(2) $$U = Q + S - SQ.$$

If $x \in X$, $x \geqq 0$, then $SQx \leqq Qx$ by the above, so that $Q - SQ \geqq 0$. For every $i \leqq n$ we get

$$Ux_i = Qx_i + Sx_i - SQx_i = x_i. \tag{3}$$

Finally (2) gives

$$|U| \leqq |S| + (Q - SQ) \tag{4}$$

and hence

$$\begin{aligned} \| \, |U| \, \| &\leqq \| \, |S| \, \| + \| Q - SQ \| \leqq \| \, |T| \, \| + \varepsilon + 2\varepsilon n \| Q \| \\ &= \| \, |T| \, \| + (2n^2 + 1)\varepsilon. \end{aligned} \tag{5}$$

It follows immediately that if T is positive, then S and hence U are positive as well.

We are now able to prove one of our main results on the b.p.a.p.

2.6. THEOREM. *Let X be an order continuous Banach lattice and $\lambda \geqq 1$. The following statements are equivalent*:

(i) X has the λ-*b.p.a.p.*

(ii) *For every $\varepsilon > 0$ and every compact set $K \subseteq X$, there is a $T \in X^* \otimes X$ so that $\| x - Tx \| \leqq \varepsilon$ for all $x \in K$ and $\| \, |T| \, \| \leqq \lambda$.*

PROOF. (i)$\Rightarrow$(ii) is trivial, so let us prove (ii)$\Rightarrow$(i). Let $\varepsilon > 0$ and $\{x_1, x_2, \dots, x_n\} \subseteq X$ a finite set consisting of positive, mutually disjoint elements. By assumption there is a $T \in X^* \otimes X$, $\| \, |T| \, \| \leqq \lambda$ and $\| Tx_i - x_i \| \leqq \varepsilon$ for $1 \leqq i \leqq n$. By Lemma 2.4, we may assume that there are mutually disjoint, positive elements $z_1, z_2, \dots, z_k \in X$ so that $T(X) \subseteq [z_j]_{j=1}^k$ and that there are $p_i \in N$, $p_i < p_{i+1}$ and $t_j \geqq 0$ so that

$$x_i = \sum_{j=p_i+1}^{p_{i+1}} t_j z_j. \tag{1}$$

By the proof of Theorem 2.2 (the construction of the operator T_1), we may further assume that there are $z_j^* \in X^*$, $1 \leqq j \leqq k$ so that

$$T = \sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} P_i^* z_j^* \otimes z_j. \tag{2}$$

For every $1 \leqq i \leqq n$ and every j, $p_i + 1 \leqq j \leqq p_{i+1}$ we put

$$\alpha_{ij} = \begin{cases} \dfrac{|z_j^*(x_i)|}{|z_j^*|(x_i)} & \text{if } |z_j^*|(x_i) \neq 0 \\ 1 & \text{else} \end{cases} \tag{3}$$

and define

$$S = \sum_{i=1}^{n} \sum_{j=p_i+1}^{p_{i+1}} \alpha_{ij} P_i^* |z_j^*| \otimes z_j. \tag{4}$$

Clearly $S \geqq 0$, and since $\alpha_{ij} \leqq 1$ we get that $S \leqq \Sigma_{i=1}^{n} \Sigma_{j=p_i+1}^{p_{i+1}} P_i^* |z_j^*| \otimes z_j = |T|$, so that $\| S \| \leqq \lambda$. Further

$$Sx_i = \sum_{j=p_i+1}^{p_{i+1}} |z_j^*(x_i)| z_j = |Tx_i| \tag{5}$$

and therefore

$$\| Sx_i - x_i \| = \| \, |Tx_i| - x_i \| \leqq \| Tx_i - x_i \| \leqq \varepsilon \tag{6}$$

for all $1 \leqq i \leqq n$.

By Proposition 2.9 of [6], which is a weaker version of Lemma 2.3, it now follows that X has the λ-b.p.a.p. ■

Using the perturbation Theorem 2.5, we are now able to show the following theorem which gives some equivalent formulations of the b.p.a.p.

2.7. Theorem. *Let X be an order continuous Banach lattice and $\lambda \geqq 1$. The following statements are equivalent*:

(i) *X has the $(\lambda + \varepsilon)$-b.p.a.p. for all $\varepsilon > 0$.*

(ii) *For every $\varepsilon > 0$ and every finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$ consisting of positive, mutually disjoint elements, there is a positive $T \in X^* \otimes X$, so that $Tx_i = x_i$ for all $1 \leqq i \leqq n$ and $\| T \| \leqq \lambda + \varepsilon$.*

(iii) *For every $\varepsilon > 0$ and every finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$ consisting of positive, mutually disjoint elements, there is a $T \in X^* \otimes X$, so that $Tx_i = x_i$ for all $1 \leqq i \leqq n$ and $\| \, |T| \, \| \leqq \lambda + \varepsilon$.*

Proof. (i)$\Rightarrow$(ii) follows from Theorem 2.5, (ii)$\Rightarrow$(iii) is trivial and (iii)$\Rightarrow$(i) follows from Theorem 2.6. ■

As a corollary to Theorem 2.6 we get the following

2.8. Corollary. *Let X be an order continuous Banach lattice with the Radon–Nikodym property (RNP), so that there is a positive contractive projection of X^{**} onto X (e.g. let X be reflexive). If X has the p.a.p., then X has the 1-b.p.a.p.*

PROOF. Assume that X has the p.a.p., let τ denote the topology on $B(X)$ of uniform convergence on compact sets. Since X has the a.p., $(B(X),\tau)^* = N_1(X)$, the space of nuclear operators (see e.g. [4]), so by the bipolar theorem and Theorem 2.6 we have to show that if $T \in N_1(X)$ with $|\mathrm{Tr}(ST)| \leqq \| \, |S| \, \|$ for all $S \in X^* \otimes X$ then $|\mathrm{Tr}(T)| \leqq 1$.

If $T \in N_1(X)$ satisfies the left-hand side of this implication and $S = \sum_{j=1}^k \varepsilon_j z_j^* \otimes z_j \in X^* \otimes X$ with $\varepsilon_j = \pm 1$ for $1 \leqq j \leqq k$, then

$$\left| \sum_{j=1}^k \varepsilon_j z_j^* T z_j \right| = |\mathrm{Tr}(ST)| \leqq \| \, |S| \, \|$$

$$\leqq \left\| \sum_{j=1}^k |z_j^*| \otimes |z_j| \right\| \tag{1}$$

$$= \sup\left\{ \sum_{j=1}^k x^*(|z_j|) x^{**}(|z_j^*|) \,\middle|\, x^* \in K^*, x^{**} \in K \right\}$$

where

$$\begin{aligned} K &= \{x^{**} \in X^{**} \mid x^{**} \geqq 0, \ \| x^{**} \| \leqq 1\}, \\ K^* &= \{x^* \in X^* \mid x^* \geqq 0, \ \| x^* \| \leqq 1\}. \end{aligned} \tag{2}$$

By choosing the signs, we obtain for all finite sets $(z_j)_{j=1}^k \subseteq X$, $(z_j^*)_{j=1}^k \subseteq X^*$:

$$\sum_{j=1}^k |z_j^* T z_j| \leqq \sup\left\{ \sum_{j=1}^k x^*(|z_j|) x^{**}(|z_j^*|) \,\middle|\, x^{**} \in K, x^* \in K^* \right\}. \tag{3}$$

Equipped with the respective ω^*-topologies, K and K^* are compact sets. We now consider the following two sets:

$$F_1 = \{ f \in C(K \times K^*) \mid \sup\{ f(x^{**}, x^*) \mid x^{**} \in K, x^* \in K^*\} < 1\}, \tag{4}$$

and

$$\begin{aligned} F_2 = \mathrm{conv}\{ f \in C(K \times K^*) \mid \ & \exists\, z^* \in X^*, z^* \geqq 0, \ \exists\, z \in X, z \geqq 0 \\ & \text{with } |z^* T z| = 1 \text{ and } f(x^{**}, x^*) = x^{**}(z^*) x^*(z) \\ & \text{for all } x^{**} \in K, x \in K^*\}. \end{aligned} \tag{5}$$

It follows from (3) that $F_1 \cap F_2 = \varnothing$.

By the first separation theorem and the Riesz representation theorem, there is a $\lambda \in \mathbf{R}$ and a $\mu \in M(K \times K^*)$, so that

$$\int f d\mu \leqq \lambda \qquad \text{for all } f \in F_1 \tag{6}$$

and

$$\int f d\mu \geqq \lambda \qquad \text{for all } f \in F_2. \tag{7}$$

Since the negative cone of $C(K \times K^*)$ is a subset of F_1, we get that $\lambda \geqq 0$ and $\mu \geqq 0$, and we can then assume that $\mu(K \times K^*) = 1$. Since the open unit ball of $C(K \times K^*)$ is contained in F_1, we get with this normalization that $\lambda \geqq 1$, and the definition of F_1 and F_2 now gives: for all $z^* \in X^*$, $z^* \geqq 0$ and all $z \in X$, $z \geqq 0$

$$|z^*Tz| \leqq \int x^{**}(z^*)x^*(z)d\mu(x^{**}, x^*). \tag{8}$$

It is easy to see that there is a positive operator $U \in B(X, X^{**})$, so that for all $z \geqq 0$ and $z^* \geqq 0$, $(Uz)(z^*)$ is equal to the right-hand side of (8). If $S \in B(X, L_\infty(\mu))$ is defined by $(Sz)(x^{**}, x^*) = x^*(z)$ for $x^{**} \in K$, $x^* \in K^*$ and $z \in X$, and $R \in B(X^*, L_\infty(\mu))$ is defined by $(Rz^*)(x^{**}, x^*) = x^{**}(z^*)$ for $x^{**} \in K$, $x^* \in K$ and $z^* \in X^*$, we get that $U = R^*IS$, where I denotes the formal identity map from $L_\infty(\mu)$ to $L_1(\mu)$. Hence U is integral with $i_1(U) \leqq 1$ (note also that both S and R are positive), and (8) gives

$$|T| \leqq U. \tag{9}$$

If P denotes a positive, contractive projection of X^{**} onto X and we put $V = PU$, then by (9), $|T| \leqq V$. V is integral from X to X, and since X has the RNP, V is nuclear as well with $n_1(V) = i_1(V) \leqq 1$.

Since X has the p.a.p. and $V \pm T \geqq 0$, Theorem 1.3 gives that

$$|\mathrm{Tr}(T)| \leqq \mathrm{Tr}(V) \leqq 1. \tag{10}$$

■

Remark. It follows from Lemma 0.1 in [7] that if X satisfies the assumptions above, then X is a band in X^{**}.

We end this section by proving a theorem and a corollary due to L. Tzafriri and the author, which show that in a special case the a.p. implies the p.a.p. See also C. Schütt [9].

2.9. Theorem. *Let X be an order continuous Banach lattice and $\{x_1, x_2, \ldots, x_n\} \subseteq X$ a finite set of positive, mutually disjoint elements. If P is a* *projection of X onto $[x_i]$, then there is a positive projection Q of X onto $[x_i]$ with* $\| Q \| \leqq \| P \|$.

PROOF. Let $F = \{y_1, y_2, \dots, y_k\}$ be a set of mutually disjoint positive elements, so that there are finite subsets $\sigma_i \subseteq \mathbf{N}$, $1 \leqq i \leqq n$, $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$ and a sequence $(t_j) \subseteq \mathbf{R}$, $t_j \geqq 0$, so that

$$x_i = \sum_{j \in \sigma_i} t_j y_j. \tag{1}$$

Further, let $y_j^* \in X^*$ be mutually disjoint, positive biorthogonals to (y_j) and let $\{\alpha_{ij} \mid 1 \leqq i \leqq n, 1 \leqq j \leqq k\}$ be the matrix for P with respect to (y_j) and (x_i).

Since $Px_i = x_i$, we get

$$\sum_{j \in \sigma_i} \alpha_{ij} t_j = 1 \qquad \text{for all } 1 \leqq i \leqq n. \tag{2}$$

If $Q_1 : [y_j] \to [x_i]$ is defined by

$$Q_1 = \sum_{i=1}^{n} \sum_{j \in \sigma_i} \alpha_{ij} y_j^* \otimes x_i \tag{3}$$

then it follows from [4], proposition 1e 8, that Q_1 is bounded with $\| Q_1 \| \leqq \| P \|$.

By (2), $Q_1 x_i = x_i$ for all $1 \leqq i \leqq n$. For every choice (ε_j) of signs we get

$$\begin{aligned} \left\| \sum_{i=1}^{n} \sum_{j \in \sigma_i} \varepsilon_j \alpha_{ij} y_j^*(y) x_i \right\| &\leqq \| Q_1 \| \left\| \sum_{j=1}^{k} \varepsilon_j y_j^*(y) y_j \right\| \\ &= \| Q_1 \| \, \| y \| \qquad \text{for all } y \in [y_j]. \end{aligned} \tag{4}$$

(4) shows that the operator $Q_2 : [y_j] \to [x_i]$ defined by

$$Q_2 = \sum_{i=1}^{n} \sum_{j \in \sigma_i} |\alpha_{ij}| y_j^* \otimes x_i \tag{5}$$

is bounded with $\| Q_2 \| \leqq \| Q_1 \| \leqq \| P \|$.

For $1 \leqq i \leqq n$ we let

$$s_i = \left(\sum_{j \in \sigma_i} |\alpha_{ij}| t_j \right)^{-1} \tag{6}$$

and define $Q_F : [y_j] \to [x_i]$ by

$$Q_F = \sum_{i=1}^{n} s_i \sum_{j \in \sigma_i} |\alpha_{ij}| y_j^* \otimes x_i. \tag{7}$$

Since $|s_i| \leqq 1$, Q_F is bounded with $\| Q_F \| \leqq \| Q_2 \| \leqq \| P \|$.

It is readily seen that $Q_F x_i = x_i$ for all $1 \leqq i \leqq n$, so that Q_F is a positive projection of $[y_j]$ onto $[x_i]$.

Let now

$$\mathscr{A} = \{F \subseteq X \mid [x_i] \subseteq \operatorname{span} F, \quad F \text{ finite set of mutually disjoint, positive elements}\}. \tag{8}$$

By Lemma 2.3 we get that $\overline{\operatorname{span}}\, \mathscr{A} = X$, and we can order $\mathscr{A}$ partially by

$$F_1 \leqq F_2 \qquad \text{if } \operatorname{span} F_1 \subseteq \operatorname{span} F_2. \tag{9}$$

For each $F \in \mathscr{A}$ we define

$$R_F \in \Pi = \prod_{x \in X} \{y \in [x_i] \mid \| y \| \leqq \| P \| \ \| x \| \}$$

by

$$R_F(x) = \begin{cases} Q_F x & \text{if } \in \operatorname{span} F, \\ 0 & \text{else.} \end{cases}$$

Since Π is compact, (R_F) has a convergent subnet with limit $R \in \Pi$. It is readily verified that $R_{|\operatorname{span} \mathscr{A}}$ is a bounded linear map with norm less than or equal to $\| P \|$, and it can therefore be extended to a $Q \in B(X)$ by continuity. Clearly $Qx = x$ for all $x \in [x_i]$, so that Q is a projection, and from the construction it follows that $Q \geqq 0$. ■

As a corollary we obtain

2.10. Theorem. *Let X be an order continuous Banach lattice. If there is a $\lambda \geqq 1$ and a net $(P_t)_{t \in I}$ of projections converging pointwise to the identity, so that $\| P_t \| \leqq \lambda$ and $P_t(X)$ is the span of finitely many mutually disjoint elements for all $t \in I$, then X has the λ-b.p.a.p. with projections.*

3. Some concluding remarks

Theorem 2.10 explains in a way why it is difficult to find an example (if there is one) of a Banach with the a.p. and without the p.a.p. Indeed, in all Banach lattices known to have the a.p., this is proved by constructing a net of projections as in Theorem 2.10.

It is also clear that an example has to be extremely non-symmetric, since e.g.

every r.i. space with non-trivial Boyd indices has the p.a.p. due to the fact that conditional expectations are bounded in such a space.

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