Strassens algoritme

Matricer (repetition)

Matrix = firkant af tal:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix}$$

Ovenstående er en 3×3 matrix.

I dag: alle matricer er $n \times n$ kvadratiske matricer.

Matricer

Plus for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 6+2 & 4+1 \\ 2+4 & 5+3 & 7+2 \\ 9+5 & 1+4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 5 \\ 6 & 8 & 9 \\ 14 & 5 & 4 \end{bmatrix}$$

Tid? $\Theta(n^2)$.

Optimalt, da output er af størrelse n^2.

Matricer

Gange for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & ? & ? \end{bmatrix}$$

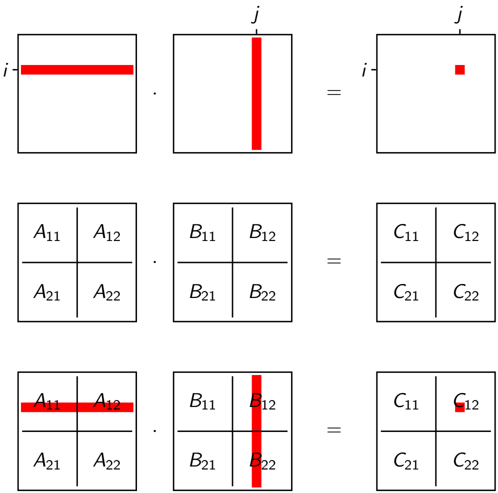
$$33 = 2 \cdot 1 + 5 \cdot 2 + 7 \cdot 3$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & 25 & ? \end{bmatrix}$$

$$25 = 9 \cdot 2 + 1 \cdot 3 + 1 \cdot 4$$

Tid? $\Theta(n^3)$. Optimalt?? Andre algoritmer??

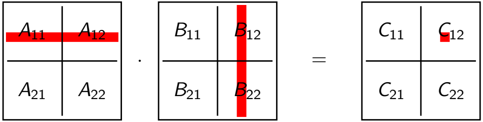
Rekursiv algoritme for multiplikation?



Bemærk:

$$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$$

Rekursiv algoritme for multiplikation



$$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = C_{11}$$

$$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$$

$$A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = C_{21}$$

$$A_{21} \cdot B_{12} + A_{22} \cdot B_{22} = C_{22}$$

Matrix addition: $O(n^2)$

Matrix multiplikation: Rekursivt kald til matrixmultiplikation på $n/2 \times n/2$ matricer. (Base case: $n = 1 \Rightarrow$ multiplikation af tal.)

$$T(n) = 8T(n/2) + n^2$$

Rekursiv algoritme for multiplikation

$$T(n) = 8T(n/2) + n^2$$

Master theorem:

- $\alpha = \log_b(a) = \log_2(8) = 3$
- $f(n) = n^2$

$n^2 = O(n^{\alpha - 0.1}) \Rightarrow$ Case 1

$$T(n) = \Theta(n^\alpha) = \Theta(n^3)$$

Det samme som den almindelige algoritme. Øv.

Strassen [1969]

Beregn:

$S_1 = B_{12} - B_{22}$

$S_2 = A_{11} + A_{12}$

$S_3 = A_{21} + A_{22}$

$\vdots$

$S_9 = A_{11} - A_{21}$

$S_{10} = B_{11} + B_{12}$

Tid: $O(n^2)$

Strassen [1969]

Beregn:

$P_1 = A_{11} \cdot S_1$

$P_2 = S_2 \cdot B_{22}$

$P_3 = S_3 \cdot B_{11}$

$P_4 = A_{22} \cdot S_4$

$P_5 = S_5 \cdot S_6$

$P_6 = S_7 \cdot S_8$

$P_7 = S_9 \cdot S_{10}$

7 rekursive kald til matrixmultiplikation på $n/2 \times n/2$ matricer.

Strassen [1969]

Check nu at der gælder:

$$
\begin{aligned}
P_5 + P_4 - P_2 + P_6 &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
P_1 + P_2 &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
P_3 + P_4 &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
P_5 + P_1 - P_3 - P_7 &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}
\end{aligned}
$$

Dvs. output kan beregnes i $O(n^2)$ tid ud fra $P_1, \ldots, P_7$, eftersom

$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = C_{11}$

$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$

$A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = C_{21}$

$A_{21} \cdot B_{12} + A_{22} \cdot B_{22} = C_{22}$

$$T(n) = 7\,T(n/2) + n^2$$

Strassen [1969]

$$T(n) = 7T(n/2) + n^2$$

Master theorem:

- $\alpha = \log_b(a) = \log_2(7) = 2.80735\ldots$
- $f(n) = n^2$

$n^2 = O(n^{\alpha - 0.1}) \Rightarrow$ Case 1

$$T(n) = \Theta(n^\alpha) = O(n^{2.81})$$

Bedre end den almindelige algoritme!