Strassens algoritme

Matricer (repetition)

Matrix = firkant af tal:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix}$$

Ovenstående er en 3×3 matrix.

I dag: alle matricer er $n \times n$ kvadratiske matricer. (Dvs. n angiver sidelængden af matricerne.)

Matricer

Plus for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Matricer

Plus for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 6+2 & 4+1 \\ 2+4 & 5+3 & 7+2 \\ 9+5 & 1+4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 5 \\ 6 & 8 & 9 \\ 14 & 5 & 4 \end{bmatrix}$$

Matricer

Plus for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 6+2 & 4+1 \\ 2+4 & 5+3 & 7+2 \\ 9+5 & 1+4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 5 \\ 6 & 8 & 9 \\ 14 & 5 & 4 \end{bmatrix}$$

Tid?

Matricer

Plus for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

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Tid? $\Theta(n^2)$.

Matricer

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$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

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Tid? $\Theta(n^2)$.

Optimalt, da output er af størrelse n^2.

Matricer

Gange for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Matricer

Gange for matricer:

$$\begin{bmatrix}1 & 6 & 4\\ 2 & 5 & 7\\ 9 & 1 & 1\end{bmatrix} \cdot \begin{bmatrix}3 & 2 & 1\\ 4 & 3 & 2\\ 5 & 4 & 3\end{bmatrix} = \begin{bmatrix}? & ? & ?\\ ? & ? & ?\\ ? & ? & ?\end{bmatrix}$$

$$\begin{bmatrix}1 & 6 & 4\\ 2 & 5 & 7\\ 9 & 1 & 1\end{bmatrix} \cdot \begin{bmatrix}3 & 2 & 1\\ 4 & 3 & 2\\ 5 & 4 & 3\end{bmatrix} = \begin{bmatrix}? & ? & ?\\ ? & ? & 33\\ ? & ? & ?\end{bmatrix}$$

$$33 = 2 \cdot 1 + 5 \cdot 2 + 7 \cdot 3$$

Matricer

Gange for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & ? & ? \end{bmatrix}$$

$$33 = 2 \cdot 1 + 5 \cdot 2 + 7 \cdot 3$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & 25 & ? \end{bmatrix}$$

$$25 = 9 \cdot 2 + 1 \cdot 3 + 1 \cdot 4$$

Matricer

Gange for matricer:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

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$$25 = 9 \cdot 2 + 1 \cdot 3 + 1 \cdot 4$$

Tid?

Matricer

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$$\begin{bmatrix} 1 & 6 & 4 \\ \cancel{2} & \cancel{5} & \cancel{7} \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & \cancel{1} \\ 4 & 3 & \cancel{2} \\ 5 & 4 & \cancel{3} \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & ? & ? \end{bmatrix}$$

$$33 = 2 \cdot 1 + 5 \cdot 2 + 7 \cdot 3$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ \cancel{9} & \cancel{1} & \cancel{1} \end{bmatrix} \cdot \begin{bmatrix} 3 & \cancel{2} & 1 \\ 4 & \cancel{3} & 2 \\ 5 & \cancel{4} & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & 25 & ? \end{bmatrix}$$

$$25 = 9 \cdot 2 + 1 \cdot 3 + 1 \cdot 4$$

Tid? $\Theta(n^3)$.

Matricer

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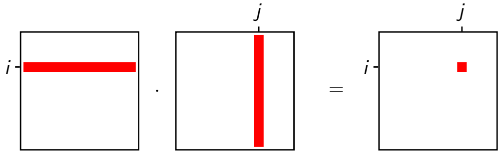
$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & 33 \\ ? & 25 & ? \end{bmatrix}$$

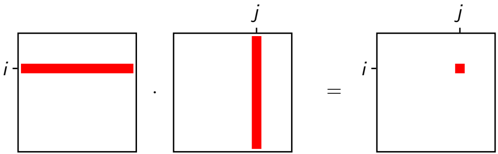
$$25 = 9 \cdot 2 + 1 \cdot 3 + 1 \cdot 4$$

Tid? $\Theta(n^3)$. Optimalt?? Andre algoritmer??

Rekursiv algoritme for multiplikation?

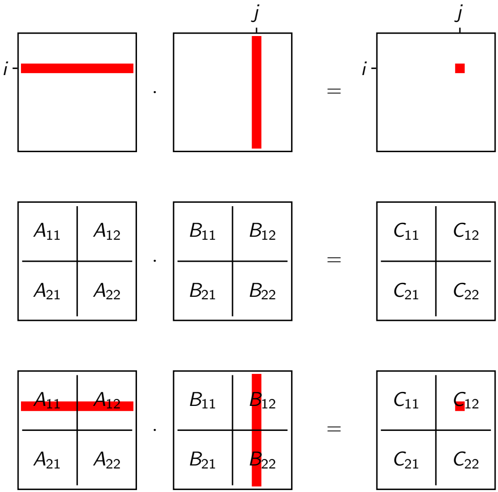


Rekursiv algoritme for multiplikation?

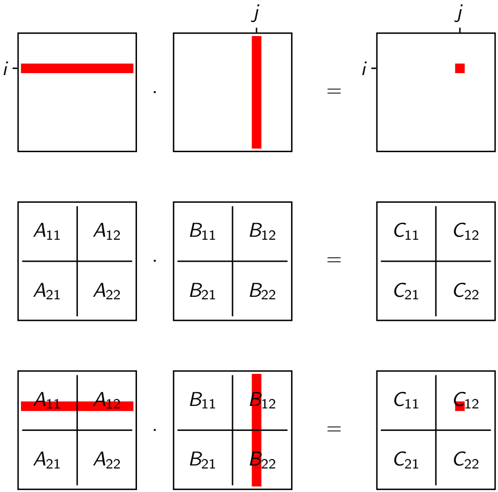


$$\begin{array}{|c|c|} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \cdot \begin{array}{|c|c|} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array} = \begin{array}{|c|c|} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}$$

Rekursiv algoritme for multiplikation?



Rekursiv algoritme for multiplikation?



Bemærk:

$$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$$

Rekursiv algoritme for multiplikation

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = C_{11}$$

$$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$$

$$A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = C_{21}$$

$$A_{21} \cdot B_{12} + A_{22} \cdot B_{22} = C_{22}$$

Rekursiv algoritme for multiplikation

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

$$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = C_{11}$$

$$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$$

$$A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = C_{21}$$

$$A_{21} \cdot B_{12} + A_{22} \cdot B_{22} = C_{22}$$

Matrix addition: $O(n^2)$

Matrix multiplikation: Rekursivt kald til matrixmultiplikation på $n/2 \times n/2$ matricer. (Base case: $n = 1 \Rightarrow$ multiplikation af tal.)

Rekursiv algoritme for multiplikation

$$\begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array} = \begin{array}{|c|c|}\hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline \end{array}$$

$$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = C_{11}$$

$$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$$

$$A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = C_{21}$$

$$A_{21} \cdot B_{12} + A_{22} \cdot B_{22} = C_{22}$$

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Matrix multiplikation: Rekursivt kald til matrixmultiplikation på $n/2 \times n/2$ matricer. (Base case: $n = 1 \Rightarrow$ multiplikation af tal.)

$$T(n) = 8T(n/2) + n^2$$

Rekursiv algoritme for multiplikation

$$T(n) = 8\,T(n/2) + n^2$$

Rekursiv algoritme for multiplikation

$$T(n) = 8T(n/2) + n^2$$

Master theorem:

- $\alpha = \log_b(a) = \log_2(8) = 3$
- $f(n) = n^2$

Rekursiv algoritme for multiplikation

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- $\alpha = \log_b(a) = \log_2(8) = 3$
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$n^2 = O(n^{\alpha - 0.1}) \Rightarrow$ Case 1

$$T(n) = \Theta(n^{\alpha}) = \Theta(n^3)$$

Rekursiv algoritme for multiplikation

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$$T(n) = \Theta(n^\alpha) = \Theta(n^3)$$

Det samme som den almindelige algoritme. Øv.

Strassen [1969]

Beregn:

$$S_1 = B_{12} - B_{22} \qquad S_6 = B_{11} + B_{22}$$

$$S_2 = A_{11} + A_{12} \qquad S_7 = A_{12} - A_{22}$$

$$S_3 = A_{21} + A_{22} \qquad S_8 = B_{21} + B_{22}$$

$$S_4 = B_{21} - B_{11} \qquad S_9 = A_{11} - A_{21}$$

$$S_5 = A_{11} + A_{22} \qquad S_{10} = B_{11} + B_{12}$$

Strassen [1969]

Beregn:

$$
\begin{aligned}
S_1 &= B_{12} - B_{22} & \qquad S_6 &= B_{11} + B_{22} \\
S_2 &= A_{11} + A_{12} & \qquad S_7 &= A_{12} - A_{22} \\
S_3 &= A_{21} + A_{22} & \qquad S_8 &= B_{21} + B_{22} \\
S_4 &= B_{21} - B_{11} & \qquad S_9 &= A_{11} - A_{21} \\
S_5 &= A_{11} + A_{22} & \qquad S_{10} &= B_{11} + B_{12}
\end{aligned}
$$

Tid: $O(n^2)$, da både addition og subtraktion tager denne tid.

Strassen [1969]

Beregn:

$P_1 = A_{11} \cdot S_1$

$P_2 = S_2 \cdot B_{22}$

$P_3 = S_3 \cdot B_{11}$

$P_4 = A_{22} \cdot S_4$

$P_5 = S_5 \cdot S_6$

$P_6 = S_7 \cdot S_8$

$P_7 = S_9 \cdot S_{10}$

Strassen [1969]

Beregn:

$P_1 = A_{11} \cdot S_1$

$P_2 = S_2 \cdot B_{22}$

$P_3 = S_3 \cdot B_{11}$

$P_4 = A_{22} \cdot S_4$

$P_5 = S_5 \cdot S_6$

$P_6 = S_7 \cdot S_8$

$P_7 = S_9 \cdot S_{10}$

7 rekursive kald til matrixmultiplikation på $n/2 \times n/2$ matricer.

Strassen [1969]

Check nu at der gælder:

$$\begin{aligned}
P_5 + P_4 - P_2 + P_6 &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
P_1 + P_2 &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
P_3 + P_4 &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
P_5 + P_1 - P_3 - P_7 &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}
\end{aligned}$$

Strassen [1969]

Check nu at der gælder:

$$\begin{aligned} P_5 + P_4 - P_2 + P_6 &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ P_1 + P_2 &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ P_3 + P_4 &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ P_5 + P_1 - P_3 - P_7 &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Dvs. output kan beregnes i $O(n^2)$ tid ud fra $P_1, \ldots, P_7$, eftersom

$A_{11} \cdot B_{11} + A_{12} \cdot B_{21} = C_{11}$

$A_{11} \cdot B_{12} + A_{12} \cdot B_{22} = C_{12}$

$A_{21} \cdot B_{11} + A_{22} \cdot B_{21} = C_{21}$

$A_{21} \cdot B_{12} + A_{22} \cdot B_{22} = C_{22}$

Strassen [1969]

Check nu at der gælder:

$$\begin{aligned} P_5+P_4-P_2+P_6 &= A_{11}\cdot B_{11}+A_{12}\cdot B_{21} \\ P_1+P_2 &= A_{11}\cdot B_{12}+A_{12}\cdot B_{22} \\ P_3+P_4 &= A_{21}\cdot B_{11}+A_{22}\cdot B_{21} \\ P_5+P_1-P_3-P_7 &= A_{21}\cdot B_{12}+A_{22}\cdot B_{22} \end{aligned}$$

Dvs. output kan beregnes i $O(n^2)$ tid ud fra $P_1,\ldots,P_7$, eftersom

$A_{11}\cdot B_{11}+A_{12}\cdot B_{21}=C_{11}$

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$A_{21}\cdot B_{11}+A_{22}\cdot B_{21}=C_{21}$

$A_{21}\cdot B_{12}+A_{22}\cdot B_{22}=C_{22}$

$$T(n)=7T(n/2)+n^2$$

Strassen [1969]

$$T(n) = 7\,T(n/2) + n^2$$

Strassen [1969]

$$T(n) = 7T(n/2) + n^2$$

Master theorem:

- $\alpha = \log_b(a) = \log_2(7) = 2.80735\ldots$
- $f(n) = n^2$

Strassen [1969]

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- $f(n) = n^2$

$n^2 = O(n^{\alpha-0.1}) \Rightarrow$ Case 1

$$T(n) = \Theta(n^\alpha) = O(n^{2.81})$$

Strassen [1969]

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- $f(n) = n^2$

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Bedre end den almindelige algoritme!