Propositional Calculus

Atomic sentence: p, q, r ...

Boolean operators:

$\neg$p — not p.
p$\wedge$q — p and q.
p$\vee$q — p or q.
p$\Rightarrow$q — if p then q
p$\Leftrightarrow$q — p if and only if q.

Sentence: Either an atomic sentence or a Boolean operator applied to sentences.
Examples:

p.
p $\vee$ q
$\neg$p $\Leftrightarrow$ (q $\vee$ p).

A *literal* is either an atomic sentence or the negation of an atomic sentence.
Examples: p, q, $\neg$p, $\neg$q.

A sentence is in *conjunctive normal form* (CNF) if it is the disjunction of literals. A set of sentences is in CNF if each sentence is in CNF.
Example: The following set of sentences is in CNF.

p.
$\neg$p $\vee$ q $\vee$ r.
q $\vee$ $\neg$r.

Converting a sentence to CNF:

1. Replace every occurrence of $\alpha \Leftrightarrow \beta$ by $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
   When this is complete, the sentence will have no occurrence of $\Leftrightarrow$.
2. Replace every occurrence of $\alpha \Rightarrow \beta$ by $\neg\alpha \vee \beta$. When this is complete, the only Boolean operators will be $\vee$, $\neg$, and $\wedge$.
3. Replace every occurrence of $\neg(\alpha \vee \beta)$ by $\neg\alpha \wedge \neg\beta$; every occurrence of $\neg(\alpha \wedge \beta)$ by $\neg\alpha \vee \neg\beta$; and every occurrence of $\neg\neg\alpha$ by α. Repeat as long as applicable. When this is done, all negations will be next to an atomic sentence.
4. Replace every occurrence of $(\alpha \wedge \beta) \vee \gamma$ by $(\alpha \vee \gamma) \wedge (\beta \vee \gamma)$, and every occurrence of $\alpha \vee (\beta \wedge \gamma)$ by $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$. Repeat as long as applicable. When this is done, all conjunctions will be at top level.
5. Break up the top-level conjunctions into separate sentences. That is, replace $\alpha \wedge \beta$ by the two sentences α and β. When this is done, the set will be in CNF.

Example:

Start: $(p \Rightarrow q) \Leftrightarrow r$.
After step 1: $((p \Rightarrow q) \Rightarrow r) \land (r \Rightarrow (p \Rightarrow q))$.
After step 2: $(\neg(\neg p \lor q) \lor r) \land (\neg r \lor (\neg p \lor q))$.
Step 3(a): $((\neg\neg p \land \neg q) \lor r) \land (\neg r \lor (\neg p \lor q))$.
After step 3: $((p \land \neg q) \lor r) \land (\neg r \lor (\neg p \lor q))$.
After step 4: $((p \lor r) \land (\neg q \lor r)) \land (\neg r \lor (\neg p \lor q))$.
After step 5: { $p\lor r$.
$\neg q \lor r$.
$\neg r \lor \neg p \lor q$. }

Resolution: Rules of Inference

Rule 1: If a literal appears more than once in a CNF sentence, all but the first may be dropped.
Example: From $p \lor \neg q \lor p$, infer $p \lor \neg q$.

Rule 2: If a CNF sentence contains both an atom and its negation, then the sentence is useless, and may be ignored.
Example: The sentence $p \lor \neg q \lor \neg p$ is useless.

Rule 3: Let S1 be the sentence $\alpha_1 \lor \ldots \lor \alpha_k$ and let S2 be the sentence $\beta_1 \lor \ldots \lor \beta_n$, where each α and each β is a literal. Suppose that for some particular i and j, α_i is the negation of β_j. Then it is possible to infer a new sentence S3 which is the disjunction of all the α's except α_i with all the β's except β_j.

Examples:

If S1 is $p \lor q \lor r$ and S2 is $\neg q \lor \neg s$, infer $p \lor r \lor \neg s$.

If S1 is $\neg p \lor s$ and S2 is $r \lor p$, infer $r \lor s$.

If S1 is p and S2 is $\neg p \lor q$, infer q.

If S1 is p and S2 is $\neg p$, infer the empty sentence.

Resolution: Proof Technique

To prove sentence ϕ from a set of axioms Γ:

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begin Set Δ = Γ ∪ {¬φ};
    Convert Δ to CNF;
    For each sentence S ∈ Δ, apply rule 1 or rule 2 if applicable.
    Loop Find two sentences S1 and S2 in Δ where rule 3 applies, but has not been previously used;
        If there are no such two sentences, then return "φ cannot be proven."
        Apply rule 3 to get a new sentence S3;
        If S3 is the empty sentence, then return "φ has been proven."
        If either rule 1 or rule 2 applies to S3, then apply it;
        Add S3 to Δ
    Endloop
end.
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Example:

Given: 1. p $\Leftrightarrow$ (r $\vee$ s).
2. r $\Rightarrow$ $\neg$p.
3. s $\Rightarrow$ $\neg$p.
Prove: H. $\neg$p $\wedge$ $\neg$r $\wedge$ $\neg$s.

Negation of H: 4. $\neg$($\neg$p $\wedge$ $\neg$r $\wedge$ $\neg$s).

Converted to CNF.
1a. $\neg$p $\vee$ r $\vee$ s.
1b. $\neg$r $\vee$ p.
1c. $\neg$s $\vee$ p.
2. $\neg$r $\vee$ $\neg$p.
3. $\neg$s $\vee$ $\neg$p.
4. p $\vee$ r $\vee$ s.

From 4 and 1a, infer	5. r $\vee$ s $\vee$ r $\vee$ s.
From 5 using rule 1, infer	6. r $\vee$ s.
From 6 and 1b, infer	7. p $\vee$ s.
From 7 and 1c, infer	8. p $\vee$ p.
From 8, using rule 1, infer	9. p.
From 9 and 2, infer	10. $\neg$r.
From 9 and 3, infer	11. $\neg$s.
From 11 and 6, infer	12. r.
From 12 and 10, infer	13. $\emptyset$.

Syntax of Predicate Calculus

The predicate calculus uses the following types of symbols:

Constants: A constant symbol denotes a particular entity. E.g. "john", "muriel" "1".

Functions: A function symbol denotes a mapping from a number of entities to a single entities: E.g. "father_of" is a function with one argument. "plus" is a function with two arguments. "father_of(john)" is some person. "plus(2,7)" is some number.

Predicates: A predicate denotes a relation on a number of entities. e.g. "married" is a predicate with two arguments. "odd" is a predicate with one argument. "married(john, sue)" is a sentence that is true if the relation of marriage holds between the people John and Sue. 'odd(plus(2,7))" is a true sentence.

Variables: These represent some undetermined entity. Examples: "X" "$S1$" ...

Boolean operators: $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$.

Quantifiers: The symbols $\forall$ (for all) and $\exists$ (there exists).

Grouping symbols: The open and close parentheses and the comma.

A *term* is either

1. A constant symbol; or
2. A variable symbol; or
3. A function symbol applied to terms.

Examples: "john", "X", "father_of(john)", "plus(X,plus(1,3))".

An *atomic formula* is a predicate symbol applied to terms.
Examples: "odd(X)" "odd(plus(2,2))", "married(sue,father_of(john))".

A *formula* is either

1. An atomic formula; or
2. The application of a Boolean operator to formulas; or
3. A quantifier followed by a variable followed by a formula.

Examples: "odd(X)," "odd(X) $\vee$ $\neg$odd(plus(X, X))," "$\exists_X$ odd(plus(X, Y)),"
"$\forall_X$ odd(X) $\Rightarrow$ $\neg$odd(plus(X,3))."

A *sentence* is a formula with no free variables. (That is, every occurrence of every variable is associated with some quantifier.)

A *literal* is either an atomic formula or the negation of an atomic formula.

Examples: odd(3). ¬odd(plus(X,3)). married(sue,Y).

A *clause* is the disjunction of literals. Variables in a clause are interpreted as universally quantified with the largest possible scope.

Example: odd(X) ∨ odd(Y) ∨ ¬odd(plus(X, Y)) is interpreted as $\forall_{X,Y}$ odd(X) ∨ odd(Y) ∨ ¬odd(plus(X, Y)).

Converting a sentence to clausal form.

1. Replace every occurrence of $\alpha \Leftrightarrow \beta$ by $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$. When this is complete, the sentence will have no occurrence of ⇔.

2. Replace every occurrence of $\alpha \Rightarrow \beta$ by $\neg\alpha \vee \beta$. When this is complete, the only Boolean operators will be ∨, ¬, and ∧.

3. Replace every occurrence of $\neg(\alpha \vee \beta)$ by $\neg\alpha \wedge \neg\beta$; every occurrence of $\neg(\alpha \wedge \beta)$ by $\neg\alpha \vee \neg\beta$; and every occurrence of $\neg\neg\alpha$ by α.

   New step: Also, replace every occurrence of $\neg\exists_\mu \alpha$ by $\forall_\mu \neg\alpha$ and every occurrence of $\neg\forall_\mu \alpha$ by $\exists_\mu \neg\alpha$.

   Repeat as long as applicable. When this is done, all negations will be next to an atomic sentence.

4. (New Step: Skolemization). For every existential quantifier $\exists_\mu$ in the formula, do the following: If the existential quantifier is not inside the scope of any universal quantifiers, then

   i. Create a new constant symbol γ.
   ii. Replace every occurrence of the variable μ by γ.
   iii. Drop the existential quantifier.

   If the existential quantifier is inside the scope of universal quantifiers with variables $\Delta_1 \dots \Delta_k$,then

   i. Create a new function symbol γ.
   ii. Replace every occurrence of the variable μ by the term $\gamma(\Delta_1 \dots \Delta_k)$
   iii. Drop the existential quantifier.

   Example. Change "$\exists_X$ blue(X)" to "blue(sk1)".
   Change $\forall_X \exists_Y$ odd(plus(X, Y)) to $\forall_X$ odd(plus(X,sk2(X)).
   Change $\forall_{X,Y} \exists_Z \forall_A \exists_B$ p(X, Y, Z, A, B) to p(X, Y,sk3(X, Y),A,sk4(X, Y, A)).

5. New step: Elimination of universal quantifiers:
   Part 1. Make sure that each universal quantifier in the formula uses a variable with a different name, by changing variable names if necessary.
   Part 2. Drop all univeral quantifiers.

   Example. Change [$\forall_X$ p(X)] ∨ [$\forall_X$ q(X)] to p(X) ∨ q($X1$).

6. (Same as step 4 of CNF conversion.) Replace every occurrence of $(\alpha \wedge \beta) \vee \gamma$ by $(\alpha \vee \gamma) \wedge (\beta \vee \gamma)$, and every occurrence of $\alpha \vee (\beta \wedge \gamma)$ by $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$. Repeat as long as applicable. When this is done, all conjunctions will be at top level.

7. (Same as step 5 of CNF conversion.) Break up the top-level conjunctions into separate sentences. That is, replace $\alpha \wedge \beta$ by the two sentences α and β. When this is done, the set will be in CNF.

Example:

Start. $\forall_X$ [even(X) $\Leftrightarrow$ [$\forall_Y$ even(times(X, Y))]]

After Step 1: $\forall_X$ [[even(X) $\Rightarrow$ [$\forall_Y$ even(times(X, Y))]] $\wedge$
[[$\forall_Y$ even(times(X, Y))] $\Rightarrow$ even(X)]].

After step 2: $\forall_X$ [[$\neg$even(X) $\vee$ [$\forall_Y$ even(times(X, Y))]] $\wedge$
[$\neg$[$\forall_Y$ even(times(X, Y))] $\vee$ even(X)]].

After step 3: $\forall_X$ [[$\neg$even(X) $\vee$ [$\forall_Y$ even(times(X, Y))]] $\wedge$
[[$\exists_Y$ $\neg$even(times(X, Y))] $\vee$ even(X)]].

After step 4: $\forall_X$ [[$\neg$even(X) $\vee$ [$\forall_Y$ even(times(X, Y))]] $\wedge$
[$\neg$even(times(X,sk1(X))) $\vee$ even(X)]].

After step 5: [$\neg$even(X) $\vee$ even(times(X, Y))] $\wedge$
[$\neg$even(times(X,sk1(X))) $\vee$ even(X)].

Step 6 has no effect.

After step 7: $\neg$even(X) $\vee$ even(times(X, Y)).
$\neg$even(times(X,sk1(X))) $\vee$ even(X).