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DM534 - Introduction to Computer Science
Study Groups - Week 45, Graph Theory

November 6, 2016

Exercise 1

A proof can be seen as a convincing argument of why you think something is true. In the lecture, you saw a proof of the fact that the ij-th entry of the k-th power of an adjacency matrix corresponds to the number of different walks from a vertex i to a vertex j. The task for you here is to prove the following theorem: (Assume in this exercise that all weights on edges are non-negative values.)

Theorem:
If G is a weighted graph with edge weight matrix W and vertices with indices $1, \ldots, n$, then for each positive integer k the ij-th entry of

$$W^k = \underbrace{W \odot W \odot \ldots \odot W}_{k \text{ times}}$$

is the length of the shortest path from i to j using maximally k edges.

The aim of this exercise is to prove this theorem by induction over k. (Hint: your can use the induction proof for the theorem on counting different walks in a graph (page 31-34 in the slide set) as a guidance). Of course, such a proof should eventually be written down in clear and concise language, similar to what has been done in the slide set. However, before writing down, one needs to be able to convince oneself and others about the correctness of one's arguments. The exercise goes as follows. First split up in two sub-groups and in each try to come up with the core elements of a proof. Next, each sub-group in turn should try to convince the other sub-group by explaining its arguments, while the other sub-group should try to find flaws and be critical.

Exercise 2

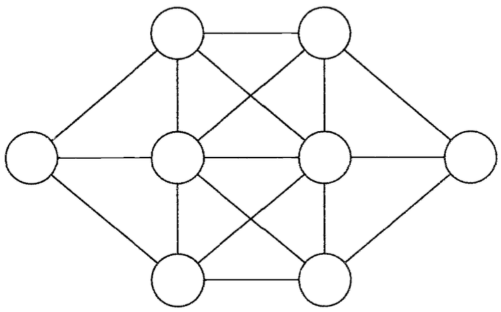
In order to show a mathematical fallacy, I will try to trick you: Here is an example of an induction proof for a quite surprising statement:
"Theorem":
Every graph G with $|V(G)| \geq 3$ vertices, for which all vertices are of degree ≥ 2, contains a cycle of length 3.

"Proof:" We proceed by induction on $|V(G)|$. As a base case, observe that the theorem is true when $|V(G)| = 3$, since any graph on three vertices with all vertices of degree ≥ 2 must be a cycle of length 3. To prove the inductive step, let G be a graph on n vertices for which the theorem holds, and construct a new graph G' on $n+1$ vertices by adding one new vertex to G and ≥ 2 edges incident with this new vertex. Since G contained a cycle of length 3, the graph G' also contains a cycle of length 3. This completes the proof.

Is this convincing? Discuss this proof!

Exercise 3

A puzzle. Who of you can solve the following problem fastest? Don't prepare this exercise, start only when your group meets. Your SGV should make sure you all start at the same time.



Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure above, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.

Exercise 4

Some perspective: This hasn't been discussed in the lecture, but graphs can be modified. To see an example of this, go to the webpage http://www.imada.sdu.dk/Employees/daniel/DM840-2016/assignment1/assign1-2016.html, and ignore everything but the description of the small game "Catalan" embedded in that webpage. In the Catalan game you are allowed to remove nodes and edges in a graph by clicking on any node n having exactly 3 neighbors. Applying such a "move" removes the 3 incident edges of n and collapses the three neighbor nodes n_1, n_2, and n_3 and the clicked node n to one node. That node has all the neighbors that n_1, n_2, and n_3 had before. The goal is to collapse the initial graph to a single node only.

The game can be addictive. Have fun! Which level did you reach? Can you imagine how "shortest path finding" in the "state space" can be used to solve this game? Discuss!