Institut for Matematik og Datalogi
Syddansk Universitet, Odense

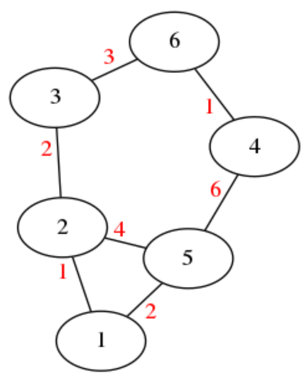
10. december 2021
DM/RF

Opgaver DM534 uge 50

Husk at læse de relevante sider i slides og noter før du/I forsøger at løse en opgave.

I: Løses i løbet af øvelsestimerne i uge 50

1. Let the following weighted graph G (from the lecture slides, weights are depicted in red) be given:



It has the following distance matrix D:

$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 3 & 7 & 2 & 6 \\ 1 & 0 & 2 & 6 & 3 & 5 \\ 3 & 2 & 0 & 4 & 5 & 3 \\ 7 & 6 & 4 & 0 & 6 & 1 \\ 2 & 3 & 5 & 6 & 0 & 7 \\ 6 & 5 & 3 & 1 & 7 & 0 \end{pmatrix} \end{array}$$

(a) How many shortest path in G are of length 6? Name them.

(b) How long is the longest of all pairwise shortest paths in the graph? Are there several longest shortest paths?

(c) How many paths in G are of length 6? (Note: a path does not necessarily need to be a shortest path.) Name them.

2. Assume in this exercise that all weights on edges are non-negative values.

   (a) In a graph G with $n = 6$ vertices, how many matrix-matrix multiplication operations are needed in the worst case in order to compute the distance matrix D, when the method of repeated squaring is used to compute D?

   (b) In a graph G with $n = 200$ vertices, how many matrix-matrix multiplications are needed in the worst case in order to compute the distance matrix D, when the method of repeated squaring is used to compute D?

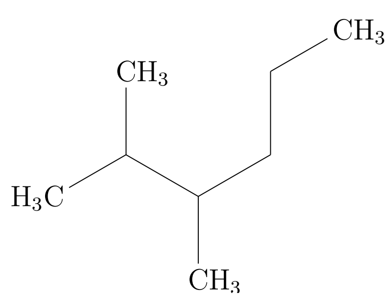
   (c) Can you find a graph G with $n = 6$ vertices, for which $W^4 \neq W^5$? If so, depict it.

   (d) Can you find a graph G with $n = 6$ vertices, for which $W^5 \neq W^6$? If so, depict it.

   (e) Can you find a graph G with $n = 6$ vertices, for which $W^1 = W^2$? If so, depict it.

   (f) What is the computational runtime in order to compute the distance matrix D for a graph G with n vertices if the method of repeated squaring is used to compute D?

3. Consider the following molecule (it's called 2,3-Dimethylhexane, see `https://en.wikipedia.org/wiki/2,3-Dimethylhexane`):



(a) How many carbon atoms does this molecule have?

(b) Draw the graph G corresponding to the carbon backbone of the molecule.

(c) Give the edge weight matrix W for the graph G.

(d) Use your brain or the Java program `ShortestPaths.java` to infer the distance matrix. [Hint: the graph is rather simple, you won't need a program for that.]

(e) What is the Wiener Index $\mathcal{W}(G)$?

(f) How many shortest paths of length 3 $i \to \ldots \to j$ with $i < j$ are in G?

(g) Using Wiener's method for predicting the boiling point, what is your prediction for 2,3-Dimethylhexane?

4. Assume in this exercise that all weights on edges are non-negative values. Prove the following theorem stated on the slides:

<u>*Theorem:*</u>
If G is a weighted graph with edge weight matrix W, and vertices with indices $1, \ldots, n$ then for each positive integer k the ij-th entry of

$$W^k = \underbrace{W \odot W \odot \ldots \odot W}_{k \text{ times}}$$

is the length of the shortest path from i to j using maximally k edges.

Prove this theorem by induction over k. [Hint: Use the induction proof for the theorem on counting different walks in a graph (page 33 of the slides) as a guidance. During this, you will need that if $i \to \cdots \to s \to \cdots \to t \to \cdots \to j$ is a shortest path between i and j, then the $s \to \cdots \to t$ part of the path must be a shortest path between s and t (give a contradiction based proof for this first).]

II: Løses hjemme

Ingen (der er ikke flere øvelsestimer i kurset).