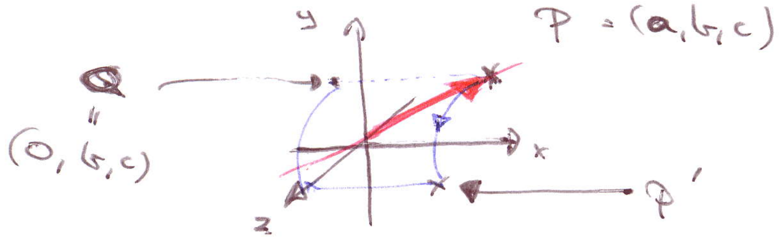
Rotation about arbitrary radial axis

← θ angle

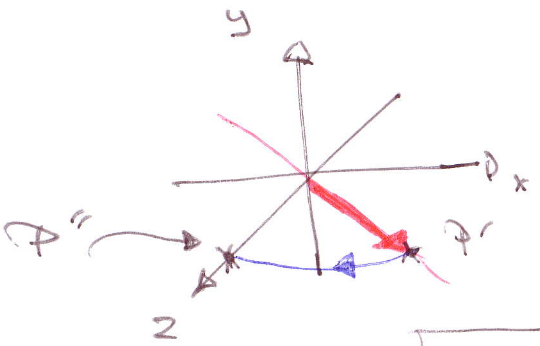
← P direction vector for axis

(Idé: reducér til rot. on coord. syst. axes)

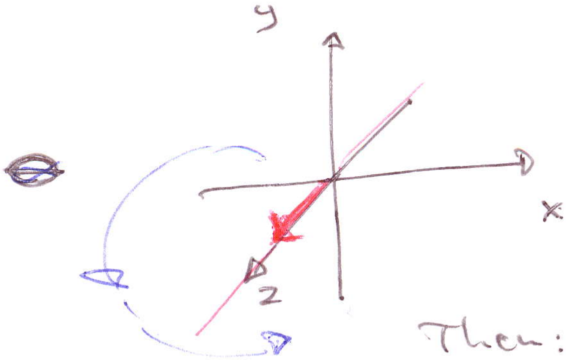


Assume wlog |P| = 1
(else: normalize)

↓ | ROT on x-axis | α |



↓ | ROT on y-axis | −β |



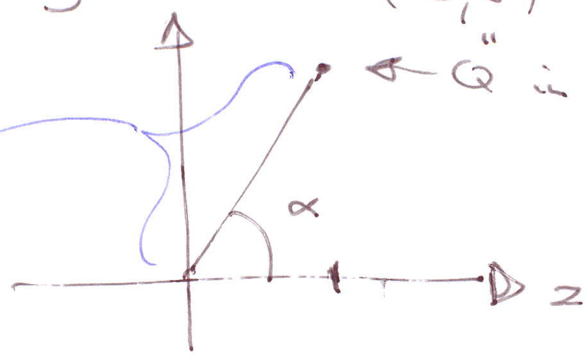
Now:

| ROT on z-axis | θ |

Then:

↓ go back (ROT +β on y-axis, ROT −α on x-axis)

α :



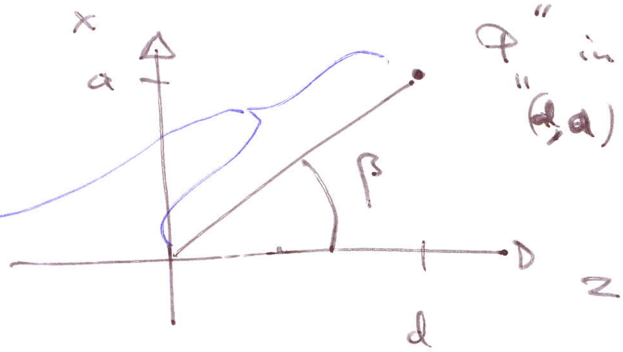
$$d = \sqrt{c^2 + b^2}$$

$$\cos(\alpha) = \frac{c}{d}$$

$$\sin(\alpha) = \frac{b}{d}$$

(NB: $d = 0 \Leftrightarrow Q = (a,0,0) \Leftrightarrow$ Rotation is already around x-axis)

β :



$$d' = \sqrt{a^2 + d^2} = \sqrt{a^2 + b^2 + c^2} = 1$$

(as $|Q| = 1$)

$$\cos(\beta) = \frac{d}{1} = d$$

$$\sin(\beta) = \frac{a}{1} = a$$

$$\Downarrow$$

$$\cos(-\beta) = d$$

$$\sin(-\beta) = -a$$

Recall (see slides, or book) rot. matrices for coord. syst. axes:

$$R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$R_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$R_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, rot. angle Θ around axis with direction vector $P = (a, b, c)$ can be found as:

$$R_{a,b,c}(\Theta) = R_x(-\alpha) \cdot R_y(\beta) \cdot R_z(\Theta) \cdot R_y(-\beta) \cdot R_x(\alpha)$$

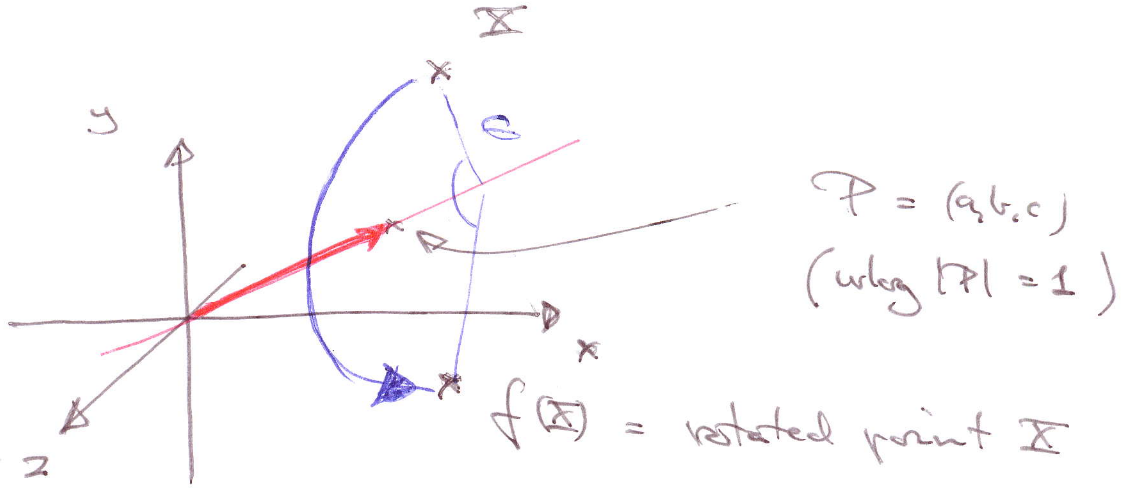
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c/d & b/d \\ 0 & -b/d & c/d \end{bmatrix} \cdot \begin{bmatrix} d & 0 & a \\ 0 & 1 & 0 \\ -a & 0 & d \end{bmatrix}$$

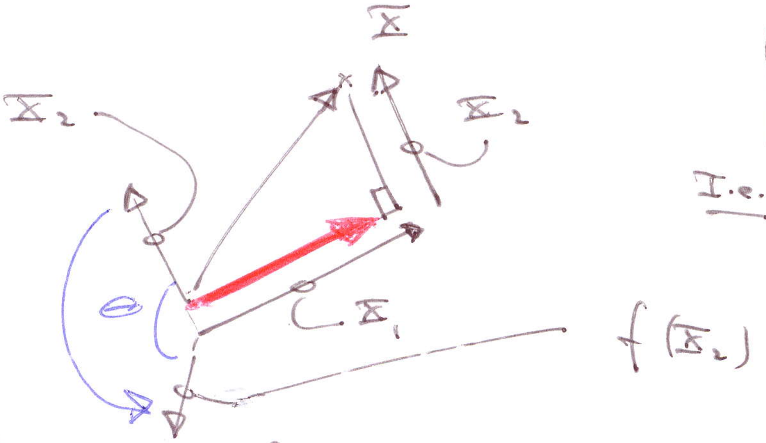
$$\cdot \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d & 0 & -a \\ 0 & 1 & 0 \\ a & 0 & d \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c/d & -b/d \\ 0 & b/d & c/d \end{bmatrix}$$

Rotation about arbitrary radial axis

(Version 2)





$$X = X_1 + X_2$$

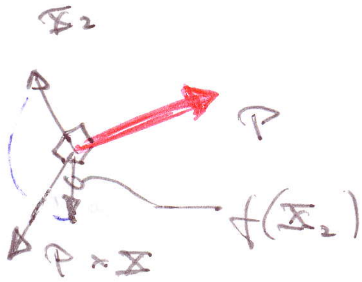
I.e.: $X_1 = (X \cdot P) \cdot P$ ← (proj)

$X_2 = X - X_1$

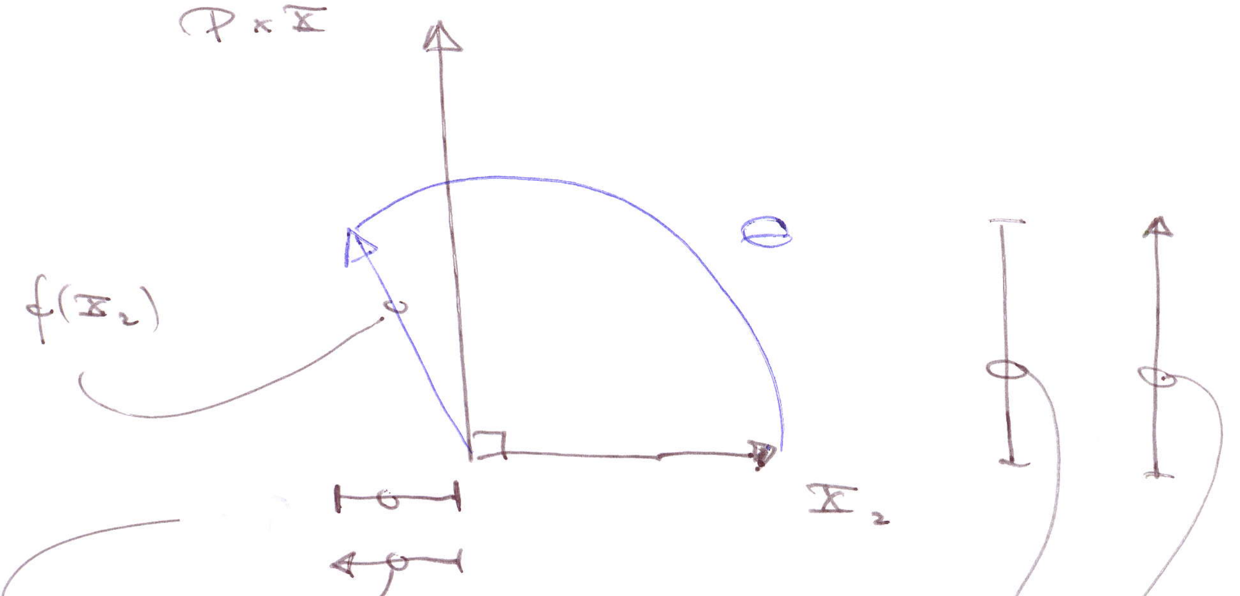
$$f(X) = f(X_1) + f(X_2) = X_1 + f(X_2) \quad (*)$$

$P \times X$ is vector perp. to P and X, hence perp. to X_2 (as X_2 is linear comb. of P and X)

So picture is:



Or in $P \times X$ and X_2 plane :



i) $\cos(\theta) \cdot |f(X_2)|$

ii) $\cos(\theta) \cdot |f(X_2)| \cdot \frac{X_2}{|X_2|} = \cos(\theta) \cdot X_2$ (as $|X_2| = |f(X_2)|$)

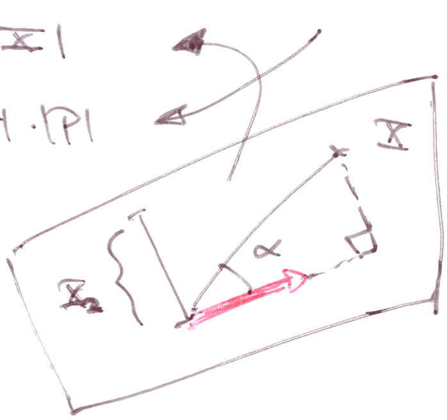
iii) $\sin(\theta)\,|f(X_2)| = \sin(\theta)\,|X_2|$

$= \sin(\theta) \cdot |\sin(\alpha)| \cdot |X|$

$= \sin(\theta) \cdot |\sin(\alpha)|\,|X| \cdot |P|$

$= \sin(\theta) \cdot |P \times X|$

$|P| = 1$



iv) $\sin(\theta) \cdot |P \times X| \cdot \frac{P \times X}{|P \times X|}$

$= \sin(\theta) \cdot (P \times X)$

So $f(\underline{X}_2) = ii) + iv)$

$$= \cos(\theta)\cdot \underline{X}_2 + \sin\theta(P \times \underline{X})$$

From $*)$ we get $f(\underline{X}) = \underline{X}_1 + f(\underline{X}_2)$

$$= \underline{X}_1 + \underline{X}_2 \cos(\theta) + \sin(\theta)\cdot(P \times \underline{X})$$

$(\underline{X}_2 = \underline{X} - \underline{X}_1)$ ⟶

$$= \underline{X}_1 + (\underline{X} - \underline{X}_1)\cos(\theta) + \sin(\theta)(P \times \underline{X})$$

$$= \underline{X}\cdot\cos(\theta) + \underline{X}_1(1-\cos(\theta)) + (P \times \underline{X})\sin(\theta)$$

$$= \boxed{\underline{X}\cdot\cos(\theta) + (\underline{X}\cdot P)\cdot P\,(1-\cos(\theta)) + (P \times \underline{X})\sin\theta}$$

"Rodrigues' rotation formula"

(Olinde Rodrigues, France, 1795–1851)

v) $(X \cdot P) \cdot P = P \cdot (X \cdot P)$

vector prod. ; scalar mult.

$= P \cdot (P \cdot X)$

$= P \cdot (P^T \cdot X)$

matrix prod. : 

$= (P \cdot P^T) \cdot X$

$P = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$= \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} \cdot X$$

vi) $P \times X = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = \begin{bmatrix} bz - yc \\ cx - az \\ ay - bx \end{bmatrix}$

$$= \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \cdot X$$

From v), vi), and Rodrigues' formula:

$$f(\underline{X}) = \left(\cos(\theta) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1 - \cos(\theta)) \cdot \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} + \sin(\theta) \cdot \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \right) \cdot \underline{X}$$

$$= \left(\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} + \cos(\theta) \begin{bmatrix} 1-a^2 & -ab & -ac \\ -ab & 1-b^2 & -bc \\ -ac & -bc & 1-c^2 \end{bmatrix} + \sin(\theta) \cdot \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \right) \cdot \underline{X}$$

$\square$