Transformations

Recall the Graphics Pipeline:

- Compose the scene from the provided models. This requires *scaling*, *rotation*, and *translation* of the models.
- Define a *camera position*.
- *Project* the points of the models of the scene onto a 2D plane (given by the camera position). The plane is identified with the screen.
- For each triangle perform *rasterization*: find the pixels it covers in the screen. *Interpolate* any extra data (color, etc.) in the three vertices to find data values for the pixels.
- For each of these pixels: calculate a color (*shading*).
- Apply the color to the pixel on screen in a suitable way (e.g. if other triangles are closer, discard the color (using a *z-buffer*)).

Until rasterization, our data are essentially points.

We will see the details of all the terms in italics (and many more) later.

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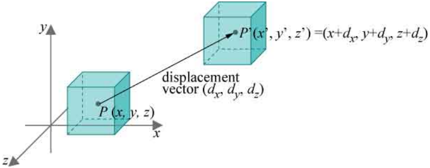
Composing the Scene

A model (box, car, building, character,. . .) is defined in one position (often centered around origo), size and orientation.

Will be needed in another *position* in the scene, possibly in another *size* and *orientation*. Translation, scaling, rotation (= model transformations).

E.g. translation:



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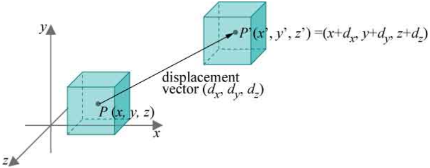

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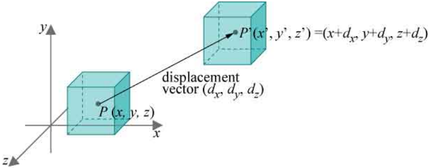

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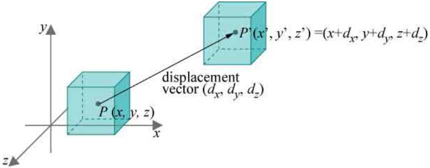

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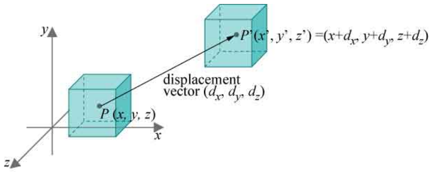
Task: find the functions that we need for model transformations.

Translation



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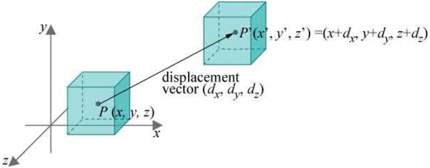
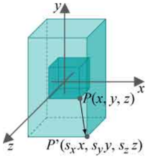


$$f(x, y, z) = \begin{pmatrix} x + d_x \\ y + d_y \\ z + d_z \end{pmatrix}$$

Note: (d_x, d_y, d_r) is a fixed translation vector, the point (x, y, z) is a vertex of the model and is the input to f.

Scaling



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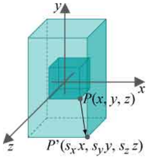
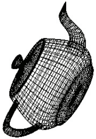


$$f(x, y, z) = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \\ s_z \cdot z \end{pmatrix}$$

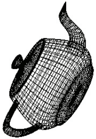
Uniform scaling: $s_x = s_y = s_z$

Rotation



Euler [1775]: for every orientation of a model there is a line l through (0,0,0) and an angle ϕ, such that rotating ϕ around l gives this orientation.

Rotation



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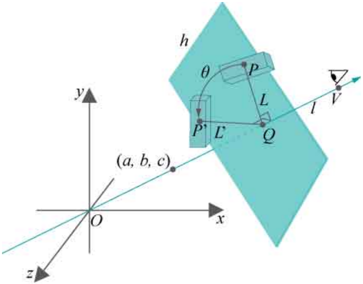
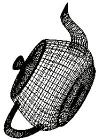


Rotation



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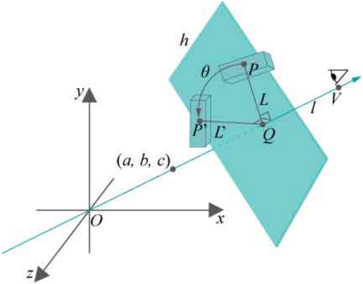
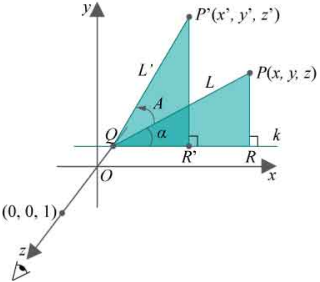


$$f(x, y, z) = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$$

Rotation

Simpler case: Rotation around z-axis.



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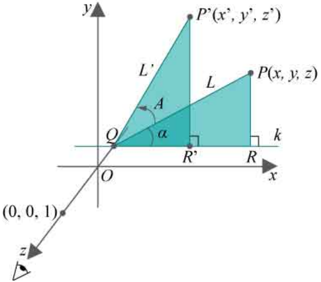

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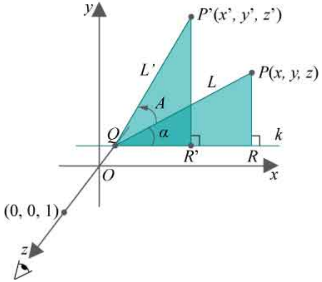

Use formula for rotating ϕ round origin in 2D:

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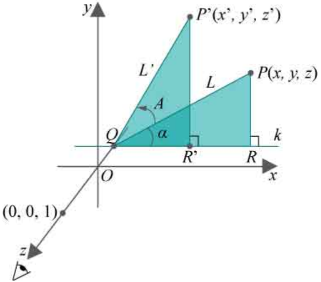


Use formula for rotating ϕ round origin in 2D:

$$f(x, y, z) = \begin{pmatrix} x \cdot \cos\phi - y \cdot \sin\phi \\ x \cdot \sin\phi + y \cdot \cos\phi \\ z \end{pmatrix}$$

Rotation

Similar: Rotation around x-axis and y-axis.

$$f(x,y,z) = \begin{pmatrix} x \\ y \cdot \cos\phi - z \cdot \sin\phi \\ y \cdot \sin\phi + z \cdot \cos\phi \end{pmatrix}$$

$$f(x,y,z) = \begin{pmatrix} z \cdot \sin\phi + x \cdot \cos\phi \\ y \\ z \cdot \cos\phi - x \cdot \sin\phi \end{pmatrix}$$

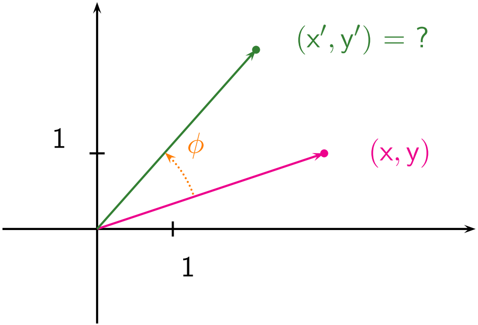
Euler Angles

Theorem [Euler, 1775]: any orientation can be created as three succesive rotations around the three coordinate axes.

The angles of the coordinate axis rotations are called Euler angles.

Using Euler angles to specify generic rotations is often intuitive, but also has drawbacks. We will return to that later.

Proof of Formula for Rotation in 2D



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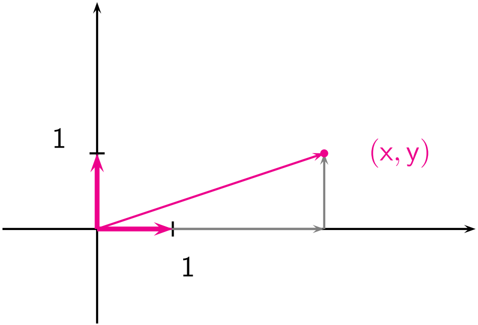

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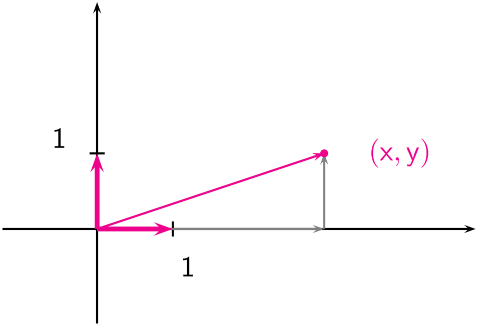

$$(x, y) = x \cdot (1, 0) + y \cdot (0, 1)$$
$$= x \cdot \vec{v}_1 + y \cdot \vec{v}_2$$

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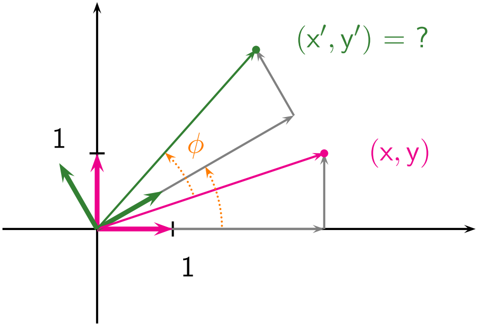

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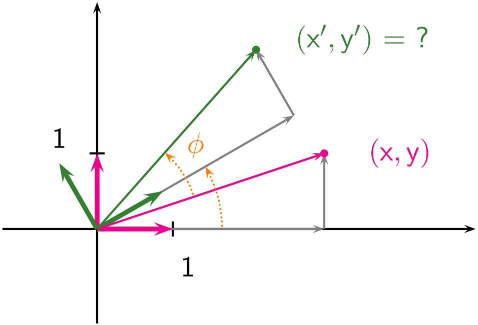

$$(x', y') = x \cdot \vec{v'}_1 + y \cdot \vec{v'}_2$$

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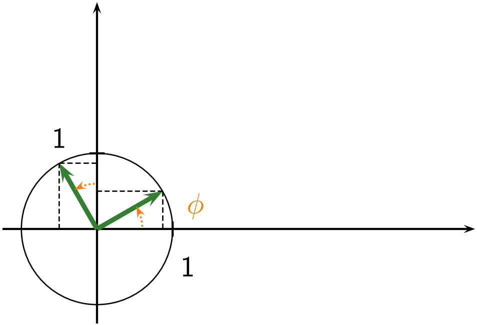

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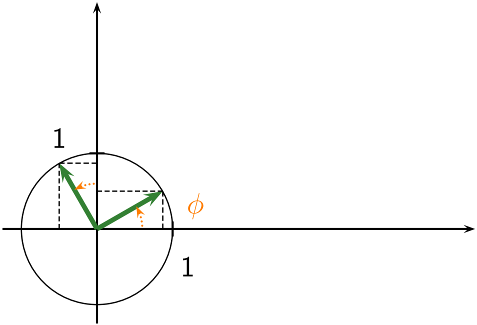

$$\vec{v'}_1 = (\cos(\phi), \sin(\phi))$$
$$\vec{v'}_2 = (-\sin(\phi), \cos(\phi))$$

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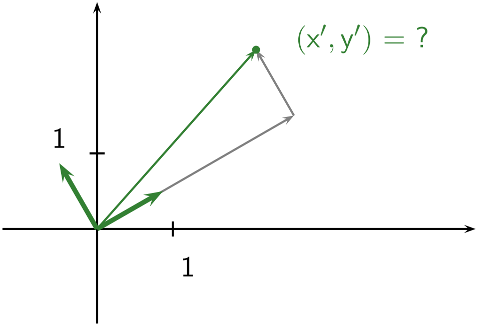

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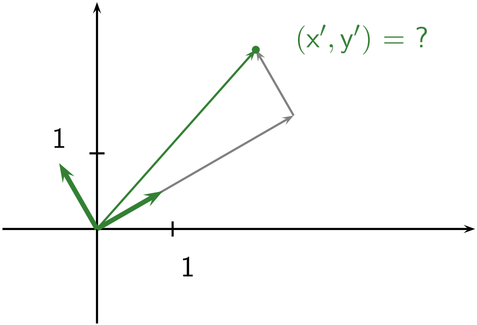

$$
\begin{aligned}
(x', y') &= x \cdot \vec{v'}_1 + y \cdot \vec{v'}_2 \\
&= x \cdot (\cos(\phi), \sin(\phi)) + y \cdot (-\sin(\phi), \cos(\phi)) \\
&= (x \cdot \cos(\phi) - y \cdot \sin(\phi),\ x \cdot \sin(\phi) + y \cdot \cos(\phi))
\end{aligned}
$$

Interlude

$$2 + 3 + 4 = ?$$

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Doesn't matter, addition is associative:

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Interlude

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Doesn't matter, multiplication is associative:

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Interlude

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Does matter, subtraction is not associative:

$$2 - (3 - 4) = 3$$

$$(2 - 3) - 4 = -5$$

Recap of Matrices

Matrix = 2D table of numbers:

$$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 7 \\ 9 & 1 & 1 & 0 \end{bmatrix}$$

Above is a 3×4 matrix.

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Other examples:

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 9 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 7 & 2 & 6 & 5 \end{bmatrix}$$

Recap of Matrices

Matrix addition:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 6+2 & 4+1 \\ 2+4 & 5+3 & 7+2 \\ 9+5 & 1+4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 5 \\ 6 & 8 & 9 \\ 14 & 5 & 4 \end{bmatrix}$$

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Matrix multiplication:

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 5 & 7 \\ 9 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

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$$33 = 2 \cdot 1 + 5 \cdot 2 + 7 \cdot 3$$

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$$f\left(x, y, z\right) = A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{pmatrix}$$

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Saves calculations: 3D object = many triangles = many points. All points go through the same sequence of 5-15 transformations. Calculate the matrix product $G = ((((A \cdot B) \cdot C) \cdot E) \cdot F)$ once.

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Question: can all our needed functions be expressed as matrices?

Transformations as Matrices

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- Scaling

$$f(x, y, z) = \begin{pmatrix} s_x x \\ s_y y \\ s_z z \end{pmatrix} =$$

Transformations as Matrices

- Scaling

$$f(x, y, z) = \begin{pmatrix} s_x x \\ s_y y \\ s_z z \end{pmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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$$f(x,y,z) = \begin{pmatrix} x + d_x \\ y + d_y \\ z + d_z \end{pmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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No. For (non-trivial) translation we have $f(0,0,0) \neq (0,0,0)$, but all functions induced by matrices have $f(0,0,0) = (0,0,0)$.

Homogeneous Coordinates

Go to 4D:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

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And back:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \to \begin{pmatrix} x/w \\ y/w \\ z/w \end{pmatrix}$$

Homogeneous Coordinates

Translations (in 3D) can now be expressed as matrix multiplication:

$$\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{pmatrix}$$

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All 3x3 matrices are still available (incl. scaling and rotation):

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 1x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \\ 1 \end{pmatrix}$$

Projection

Projection to screen: $f : \mathbb{R}^3 \to \mathbb{R}^2$.

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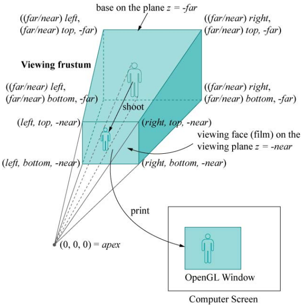

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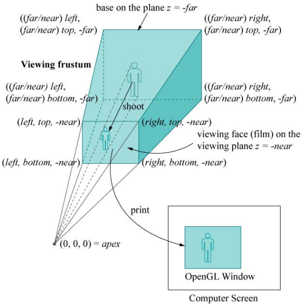

Expressed as 4x4 matrix multiplication ($d = -$near):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix} \to \begin{pmatrix} xd/z \\ yd/z \\ d \end{pmatrix}$$

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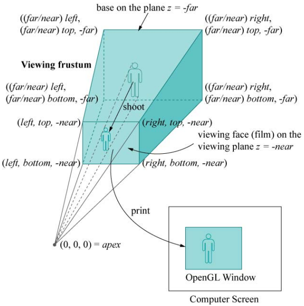

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So also proj. can be expressed via matrices (and homogeneous coord.).

Transformations in Practice

Usually, a library (such as `Graphicslib3D` from the textbook), offering easy and intuitive generation of the relevant matrices, is used.

When multiplying matrices, that last mentioned is usually the rightmost, meaning the one first affecting the vertices. Cf. the math notation $f(g(h(x)))$ (where h is applied first to x, then g, then f).

Examples from `Graphicslib3D`:

```
M1.concatenate(M2)
M1.rotateZ(45.0)
```

This means $M_1 \cdot M_2$ and $M_1 \cdot R$, where R is the matrix rotating 45 degrees around the z-axis. Recall that on a vertex $\vec{x}$, we have

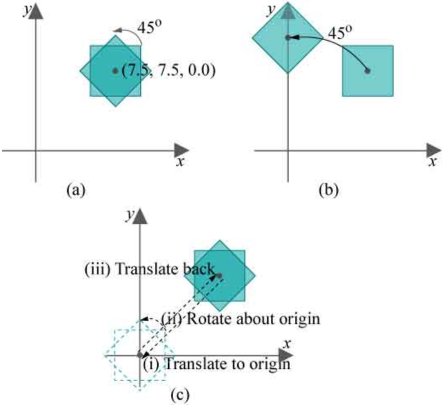
$$(M_1 \cdot M_2) \cdot \vec{x} = M_1 \cdot (M_2 \cdot \vec{x})$$

So rightmost matrix in the product is applied first.

Often using a stack of matrices works well with hierarchical scenes (scene where position of objects defined relative to each other in a hierarchical fashion).

A Standard Consideration

Note that rotations are always around origo. To get the effect of a), a single rotation will not work, but will give the effect of b). Instead, do as in c) (translate to origo, rotate, translate back).



Similar considerations relate to scaling.