Cryptography, Number Theory, and RSA

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Outline

- Symmetric key cryptography
- Public key cryptography
- Introduction to number theory
- RSA
- Digital signatures with RSA
- Combining symmetric and public key systems
- Modular exponentiation
- Greatest common divisor
- Primality testing
- Correctness of RSA

Cryptology

Cryptology = cryptography + cryptanalysis

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Cryptography is necessary for security, but not sufficient

Caesar cipher (With key = 3)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

P	Q	R	S	T	U	V	W	X	Y	Z	Æ	Ø	Å
15	16	17	18	19	20	21	22	23	24	25	26	27	28
S	T	U	V	W	X	Y	Z	Æ	Ø	Å	A	B	C
18	19	20	21	22	23	24	25	26	27	28	0	1	2

$$E(m) = m\ +\ 3 \pmod{29}$$

Symmetric key systems

Suppose the following was encrypted using a Caesar cipher and the Danish alphabet. The key is unknown. What does it say?

ZQOØQOØ, RI.

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What does this say about how many keys should be possible?

Symmetric key systems

- Caesar Cipher
-
-
- Enigma
- DES
- Blowfish
- IDEA
- Triple DES
- AES

Public key cryptography

Bob — 2 keys -PK_B,SK_B

PK_B — Bob's public key
SK_B — Bob's private (secret) key

For Alice to send m to Bob,
Alice computes: $c = E(m, PK_B)$.

To decrypt c, Bob computes:
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It must be "hard" to compute SK_B from PK_B.

Definition. Suppose $a, b \in \mathbb{Z}$, $a > 0$.
Suppose $\exists c \in \mathbb{Z}$ s.t. $b = ac$. Then a *divides* b.
$a \mid b$.
a is a *factor* of b.
b is a *multiple* of a.
$e \nmid f$ means e does not divide f.

Theorem. $a, b, c \in \mathbb{Z}$. Then

1. if $a|b$ and $a|c$, then $a|(b + c)$
2. if $a|b$, then $a|bc$ $\forall c \in \mathbb{Z}$
3. if $a|b$ and $b|c$, then $a|c$.

Definition. $p \in \mathbb{Z}$, $p > 1$.
p is *prime* if 1 and p are the only positive integers which divide p.
$2, 3, 5, 7, 11, 13, 17, \ldots$
p is *composite* if it is not prime.
$4, 6, 8, 9, 10, 12, 14, 15, 16, \ldots$

Theorem. $a \in \mathbb{Z}$, $d \in \mathbb{N}$
$\exists$ unique q, r, $0 \leq r < d$ s.t. $a = dq + r$

d – divisor
a – dividend
q – quotient
r – remainder $=$ a mod d

Definition. $gcd(a, b) =$ greatest common divisor of a and b
$=$ largest $d \in \mathbb{Z}$ s.t. $d|a$ and $d|b$

If $gcd(a, b) = 1$, then a and b are *relatively prime*.

Definition. $a \equiv b \pmod{m}$ — a is congruent to b modulo m if $m \mid (a - b)$.

$m \mid (a - b) \;\Rightarrow\; \exists k \in \mathbb{Z}$ s.t. $a = b + km$.

Theorem. $a \equiv b \pmod{m} \qquad c \equiv d \pmod{m}$
Then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$.

Proof.(of first) $\exists k_1, k_2$ s.t.
$a = b + k_1 m \qquad c = d + k_2 m$

$$\begin{aligned} a + c &= b + k_1 m + d + k_2 m \\ &= b + d + (k_1 + k_2) m \end{aligned} \qquad \square$$

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Examples.

1. $15 \equiv 22 \pmod{7}$? $\qquad 15 = 22 \pmod{7}$?
2. $15 \equiv 1 \pmod{7}$? $\qquad 15 = 1 \pmod{7}$?
3. $15 \equiv 37 \pmod{7}$? $\qquad 15 = 37 \pmod{7}$?
4. $58 \equiv 22 \pmod{9}$? $\qquad 58 = 22 \pmod{9}$?

RSA — a public key system

$N_A = p_A \cdot q_A$, where p_A, q_A prime.
$gcd(e_A, (p_A - 1)(q_A - 1)) = 1$.
$e_A \cdot d_A \equiv 1 \pmod{(p_A - 1)(q_A - 1)}$.

- $PK_A = (N_A, e_A)$
- $SK_A = (N_A, d_A)$

To encrypt: $c = E(m, PK_A) = m^{e_A} \pmod{N_A}$.
To decrypt: $r = D(c, SK_A) = c^{d_A} \pmod{N_A}$.
$r = m$.

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Example: $p = 5$, $q = 11$, $e = 3$, $d = 27$, $m = 8$.
Then $N = 55$. $e \cdot d = 81$. So $e \cdot d \equiv 1 \pmod{4 \cdot 10}$.
To encrypt m: $c = 8^3 \pmod{55} = 17$.
To decrypt c: $r = 17^{27} \pmod{55} = 8$.

Digital Signatures with RSA

Suppose Alice wants to sign a document m such that:

- No one else could forge her signature
- It is easy for others to verify her signature

Note m has arbitrary length.
RSA is used on fixed length messages.
Alice uses a cryptographically secure hash function h, such that:

- For any message m', $h(m')$ has a fixed length (512 bits?)
- It is "hard" for anyone to find 2 messages (m_1, m_2) such that $h(m_1) = h(m_2)$.

Digital Signatures with RSA

Then Alice "decrypts" $h(m)$ with her secret RSA key (N_A, d_A)

$$s = (h(m))^{d_A} \pmod{N_A}$$

Bob verifies her signature using her public RSA key (N_A, e_A) and h:

$$c = s^{e_A} \pmod{N_A}$$

He accepts if and only if

$$h(m) = c$$

.

This works because $s^{e_A} \pmod{N_A} =$

$$((h(m))^{d_A})^{e_A} \pmod{N_A} = ((h(m))^{e_A})^{d_A} \pmod{N_A} = h(m).$$

Combining symmetric and public key systems

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To encrypt a message m to send to Bob:

- Choose a random *session key* k for a symmetric key system (AES?)
- Encrypt k with Bob's public key — Result k_e
- Encrypt m with k — Result m_e
- Send k_e and m_e to Bob

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How does Bob decrypt? Why is this efficient?

Security of RSA

The primes p_A and q_A are kept secret with d_A.

Suppose Eve can factor N_A.

Then she can find p_A and q_A.
From them and e_A, she finds d_A.

Then she can decrypt just like Alice.

Factoring must be hard!

Factoring

Theorem. N composite $\Rightarrow$ N has a prime divisor $\leq \sqrt{N}$

Factor(N)

for $i = 2$ to $\sqrt{N}$ **do**
  check if i divides N
  if it does **then** output $(i, N/i)$
endfor
output -1 if divisor not found

Corollary There is an algorithm for factoring N (or testing primality) which does $O(\sqrt{N})$ tests of divisibility.

Factoring

Check all possible divisors between 2 and $\sqrt{N}$.
Not finished in your grandchildren's life time for N with 3072 bits.

Problem The length of the input is $n = \lceil \log_2(N+1) \rceil$. So the running time is $O(2^{n/2})$ — exponential.

Open Problem Does there exist a polynomial time factoring algorithm?

Use primes which are at least 1024 (or 1536) bits long.
So $2^{1023} \leq p_A, q_A < 2^{1024}$.
So $p_A \approx 10^{308}$.

RSA

How do we implement RSA?

We need to find: p_A, q_A, N_A, e_A, d_A.
We need to encrypt and decrypt.

RSA — encryption/decryption

We need to encrypt and decrypt: compute $a^k \pmod{n}$.

$a^2 \pmod{n} \equiv a \cdot a \pmod{n}$ — 1 modular multiplication

Modular Exponentiation

Theorem. For all nonnegative integers, b, c, m,
$b \cdot c \pmod{m} = (b \pmod{m}) \cdot (c \pmod{m}) \pmod{m}$.

Example: $a \cdot a^2 \pmod{n} = (a \pmod{n})(a^2 \pmod{n}) \pmod{n}$.

$$\begin{aligned}
8^3 \pmod{55} &= 8 \cdot 8^2 \pmod{55} \\
&= 8 \cdot 64 \pmod{55} \\
&= 8 \cdot (9 + 55) \pmod{55} \\
&= 72 + (8 \cdot 55) \pmod{55} \\
&= 17 + 55 + (8 \cdot 55) \pmod{55} \\
&= 17
\end{aligned}$$

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Guess: $k - 1$ modular multiplications.

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This is too many!
$e_A \cdot d_A \equiv 1 \pmod{(p_A - 1)(q_A - 1)}$.
p_A and q_A have ≥ 1024 bits each.
So at least one of e_A and d_A has ≥ 1024 bits.

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To either encrypt or decrypt would need $\geq 2^{1023} \approx 10^{308}$ operations (more than number of atoms in the universe).

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How do you calculate $a^4 \pmod{n}$ in less than 3?

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In general: $a^{2s} \pmod{n}$?

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In general: $a^{2s+1} \pmod{n}$?

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$a^{2s+1} \pmod{n} \equiv a \cdot ((a^s \pmod{n})^2 \pmod{n}) \pmod{n}$

Modular Exponentiation

$\text{Exp}(a, k, n)$ { Compute $a^k \pmod{n}$ }

if $k < 0$ **then** report error
if $k = 0$ **then** return(1)
if $k = 1$ **then** return($a \pmod{n}$)
if k is odd **then** return($a \cdot \text{Exp}(a, k-1, n) \pmod{n}$)
if k is even **then**
    $c \leftarrow \text{Exp}(a, k/2, n)$
    return($(c \cdot c) \pmod{n}$)

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To compute $3^6 \pmod{7}$: $\text{Exp}(3, 6, 7)$
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$c \leftarrow 3 \cdot 2 \pmod{7} \leftarrow 6$

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Divide exponent by 2 every other time.
How many times can we do that?

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How many modular multiplications?

Divide exponent by 2 every other time.
How many times can we do that?

$\lfloor \log_2(k) \rfloor$
So at most $2\lfloor \log_2(k) \rfloor$ modular multiplications.

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Try using $N = 35$, $e = 11$ to create keys for RSA.
What is d? Try $d = 11$ and check it.
Encrypt 4. Decrypt the result.

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What is d? Try $d = 11$ and check it.
Encrypt 4. Decrypt the result.
Did you get $c = 9$? And $r = 4$?

RSA

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Greatest Common Divisor

We need to find: e_A, d_A.
$gcd(e_A, (p_A - 1)(q_A - 1)) = 1$.
$e_A \cdot d_A \equiv 1 \pmod{(p_A - 1)(q_A - 1)}$.

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Choose random e_A.
Check that $gcd(e_A, (p_A - 1)(q_A - 1)) = 1$.
Find d_A such that $e_A \cdot d_A \equiv 1 \pmod{(p_A - 1)(q_A - 1)}$.

The Extended Euclidean Algorithm

Theorem. $a, b \in \mathbb{N}$. $\exists\ s, t \in \mathbb{Z}$ s.t. $sa + tb = gcd(a, b)$.
Proof. Let d be the smallest positive integer in
$D = \{xa + yb \mid x, y \in \mathbb{Z}\}$.
$d \in D \ \Rightarrow\ d = x'a + y'b$ for some $x', y' \in \mathbb{Z}$.
$gcd(a, b)|a$ and $gcd(a, b)|b$, so $gcd(a, b)|x'a$, $gcd(a, b)|y'b$, and
$gcd(a, b)|(x'a + y'b) = d$. We will show that $d|gcd(a, b)$, so
$d = gcd(a, b)$. Note $a \in D$.
Suppose $a = dq + r$ with $0 \le r < d$.

$$\begin{aligned} r &= a - dq \\ &= a - q(x'a + y'b) \\ &= (1 - qx')a - (qy')b \end{aligned}$$

$\Rightarrow\ r \in D$
$r < d \ \Rightarrow r = 0 \ \Rightarrow d|a$.
Similarly, one can show that $d|b$.
Therefore, $d|gcd(a, b)$. □

The Extended Euclidean Algorithm

How do you find d, s and t?

Let $d = gcd(a, b)$. Write b as $b = aq + r$ with $0 \leq r < a$.
Then, $d|b \;\Rightarrow\; d|(aq + r)$.
Also, $d|a \;\Rightarrow\; d|(aq) \;\Rightarrow\; d|((aq + r) - aq) \;\Rightarrow\; d|r$.

Let $d' = gcd(a, b - aq)$.
Then, $d'|a \;\Rightarrow\; d'|(aq)$
Also, $d'|(b - aq) \;\Rightarrow\; d'|((b - aq) + aq) \;\Rightarrow\; d'|b$.

Thus, $gcd(a, b) = gcd(a, b \pmod{a})$
$= gcd(b \pmod{a}, a)$. This shows how to reduce to a "simpler" problem and gives us the Extended Euclidean Algorithm.

The Extended Euclidean Algorithm

{ Initialize}

$d_0 \leftarrow b \qquad s_0 \leftarrow 0 \qquad t_0 \leftarrow 1$

$d_1 \leftarrow a \qquad s_1 \leftarrow 1 \qquad t_1 \leftarrow 0$

$n \leftarrow 1$

{ Compute next d}

while $d_n > 0$ **do**

begin

$n \leftarrow n + 1$

{ Compute $d_n \leftarrow d_{n-2} \pmod{d_{n-1}}$}

$q_n \leftarrow \lfloor d_{n-2}/d_{n-1} \rfloor$

$d_n \leftarrow d_{n-2} - q_n d_{n-1}$

$s_n \leftarrow s_{n-2} - q_n s_{n-1}$

$t_n \leftarrow t_{n-2} - q_n t_{n-1}$

end

$s \leftarrow s_{n-1} \qquad t \leftarrow t_{n-1}$

$gcd(a, b) \leftarrow d_{n-1}$

The Extended Euclidean Algorithm

Finding **multiplicative inverses** modulo m:

Given a and m, find x s.t. $a \cdot x \equiv 1 \pmod{m}$.

Should also find a k, s.t. $ax = 1 + km$.
So solve for an s in an equation $sa + tm = 1$.

This can be done if $gcd(a, m) = 1$.
Just use the Extended Euclidean Algorithm.

If the result, s, is negative, add m to s.
Now $(s - m)a + tm \equiv 1 \pmod{m}$.

Examples

Calculate the following:

1. $gcd(6, 9)$
2. s and t such that $s \cdot 6 + t \cdot 9 = gcd(6, 9)$
3. $gcd(15, 23)$
4. s and t such that $s \cdot 15 + t \cdot 23 = gcd(15, 23)$

RSA

$N_A = p_A \cdot q_A$, where p_A, q_A prime.
$gcd(e_A, (p_A - 1)(q_A - 1)) = 1$.
$e_A \cdot d_A \equiv 1 \pmod{(p_A - 1)(q_A - 1)}$.

- $PK_A = (N_A, e_A)$
- $SK_A = (N_A, d_A)$

To encrypt: $c = E(m, PK_A) = m^{e_A} \pmod{N_A}$.
To decrypt: $r = D(c, SK_A) = c^{d_A} \pmod{N_A}$.
$r = m$.

Primality testing

We need to find: p_A, q_A — large primes.

Choose numbers at random and check if they are prime?

Questions

1. How many random integers of length 1024 are prime?

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Prime Number Theorem: About $\frac{x}{\ln x}$ numbers $< x$ are prime, so about $\frac{2^{1024}}{709}$

So we expect to test about 709 before finding a prime with 1024 bits.

(This holds because the expected number of tries until a "success", when the probability of "success" is p, is $1/p$.)

Questions

1. How many random integers of length 1024 are prime?

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Questions

1. How many random integers of length 1024 are prime?

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So we expect to test about 709 before finding a prime.

2. How fast can we test if a number is prime?

Quite fast, using randomness.

Method 1

Sieve of Eratosthenes:
Lists:

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

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Sieve of Eratosthenes:

Lists:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	3		5		7		9		11		13		15		17		19

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					7				11		13				17		19

10^{308} — more than number of atoms in universe
So we cannot even write out this list!

Method 2

CheckPrime(n)

for $i = 2$ to $n-1$ **do**
    check if i divides n
    if it does **then** output i
endfor
output -1 if divisor not found

Check all possible divisors between 2 and n (or $\sqrt{n}$).
Our sun will die before we're done!

Rabin–Miller Primality Testing

In practice, use a randomized primality test.

Miller–Rabin primality test:
Starts with Fermat test:

> $2^{14} \pmod{15} \equiv 4 \neq 1$.
> So 15 is not prime.

Fermat's Little Theorem. Suppose p is a prime. Then for all $1 \leq a \leq p-1$, $a^{p-1} \pmod{p} = 1$.

Rabin–Miller Primality Test

Fermat test:

Prime(n)

repeat r times
  Choose random $a \in \{1, 2, \ldots, n-1\}$
  if $a^{n-1} \pmod{n} \not\equiv 1$ **then** return(Composite)
end repeat
return(Probably Prime)

Carmichael Numbers Composite n.
For all $a \in \{1, 2, \ldots, n-1\}$ s.t. $\gcd(a, n) = 1$, $a^{n-1} \pmod{n} \equiv 1$.
Example: $561 = 3 \cdot 11 \cdot 17$

Theorem.
If p is prime, $\sqrt{1} \pmod{p} = \{x \mid x^2 \pmod{p} = 1\} = \{1, p-1\}$.
If p has > 1 distinct factors, 1 has at least 4 square roots.

Example: $\sqrt{1} \pmod{15} = \{1, 4, 11, 14\}$

Rabin–Miller Primality Test

Taking square roots of 1 (mod 561):

$50^{560} \pmod{561} \equiv 1$
$50^{280} \pmod{561} \equiv 1$
$50^{140} \pmod{561} \equiv 1$
$50^{70} \pmod{561} \equiv 1$
$50^{35} \pmod{561} \equiv 560$

$2^{560} \pmod{561} \equiv 1$
$2^{280} \pmod{561} \equiv 1$
$2^{140} \pmod{561} \equiv 67$

2 is a witness that 561 is composite.

Rabin–Miller Primality Test

Miller–Rabin(n, r)

Calculate odd m such that $n - 1 = 2^s \cdot m$
repeat r times
    Choose random $a \in \{1, 2, \ldots, n-1\}$
    if $a^{n-1} \pmod{n} \not\equiv 1$ **then** return(Composite)
    if $a^{(n-1)/2} \pmod{n} \equiv n-1$ **then** continue
    if $a^{(n-1)/2} \pmod{n} \not\equiv 1$ **then** return(Composite)
    if $a^{(n-1)/4} \pmod{n} \equiv n-1$ **then** continue
    if $a^{(n-1)/4} \pmod{n} \not\equiv 1$ **then** return(Composite)
        ....
    if $a^{m} \pmod{n} \equiv n-1$ **then** continue
    if $a^{m} \pmod{n} \not\equiv 1$ **then** return(Composite)
end repeat
return(Probably Prime)

Rabin–Miller Primality Test

Theorem. If n is composite, at most $1/4$ of the a's with $1 \leq a \leq n-1$ will not end in "return(Composite)" during an iteration of the **repeat**-loop.

This means that with r iterations, a composite n will survive to "return(Probably Prime)" with probability at most $(1/4)^r$. For e.g. $r = 100$, this is less than $(1/4)^{100} = 1/2^{200} < 1/10^{60}$.

A prime n will always survive to "return(Probably Prime)".

Conclusions about primality testing

1. Miller–Rabin is a practical primality test
2. There is a less practical deterministic primality test
3. Randomized algorithms are useful in practice
4. Algebra is used in primality testing
5. Number theory is not useless

Why does RSA work?

Thm (**The Chinese Remainder Theorem**) Let $n_1, n_2, ..., n_k$ be pairwise relatively prime. For any integers $x_1, x_2, ..., x_k$, there exists $x \in \mathbb{Z}$ s.t. $x \equiv x_i \pmod{n_i}$ for $1 \leq i \leq k$, and this integer is uniquely determined modulo the product $N = n_1 n_2 ... n_k$.

We consider the special case where $n_1 = p$ and $n_2 = q$ are two primes (hence $N = pq$), and where $x_1 = x_2 = m$.

Clearly, $m \equiv m \pmod{p}$ and $m \equiv m \pmod{q}$ for any m. So if x fulfills $x \equiv m \pmod{p}$ and $x \equiv m \pmod{q}$, then $x \equiv m \pmod{N}$.

In particular, $0 \leq x, m \leq N - 1$, so we must have $x = m$.

Fermat's Little Theorem

Why does RSA work? CRT +

Fermat's Little Theorem: p is a prime, $p \not| a$.
Then $a^{p-1} \equiv 1 \pmod{p}$ and $a^p \equiv a \pmod{p}$.

RSA

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Correctness of RSA

Consider $x = D(E(m, PK_A), SK_A)$.
Note $\exists k$ s.t. $e_A d_A = 1 + k(p_A - 1)(q_A - 1)$.
$x \equiv (m^{e_A} \pmod{N_A})^{d_A} \pmod{N_A} \equiv m^{e_A d_A} \equiv m^{1+k(p_A-1)(q_A-1)} \pmod{N_A}$.

Consider $x \pmod{p_A}$.
$x \equiv m^{1+k(p_A-1)(q_A-1)} \equiv m \cdot (m^{(p_A-1)})^{k(q_A-1)} \equiv m \cdot 1^{k(q_A-1)} \equiv m \pmod{p_A}$.

Consider $x \pmod{q_A}$.
$x \equiv m^{1+k(p_A-1)(q_A-1)} \equiv m \cdot (m^{(q_A-1)})^{k(p_A-1)} \equiv m \cdot 1^{k(p_A-1)} \equiv m \pmod{q_A}$.

Apply the Chinese Remainder Theorem:
$gcd(p_A, q_A) = 1, \Rightarrow x \equiv m \pmod{N_A}$.
So $D(E(m, PK_A), SK_A) = m$.