Institut for Matematik og Datalogi
Syddansk Universitet, Odense

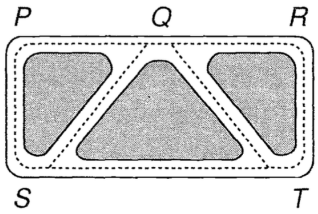
4. december 2022
DM/RF/JFB

Opgaver DM573 uge 49/50

Husk at læse de relevante sider i slides og noter før du/I forsøger at løse en opgave.

I: Løses i løbet af øvelsestimerne i uge 49

1. Draw the graph representing the road system in the figure below, and write down the number of vertices, the number of edges and the degree of each vertex.



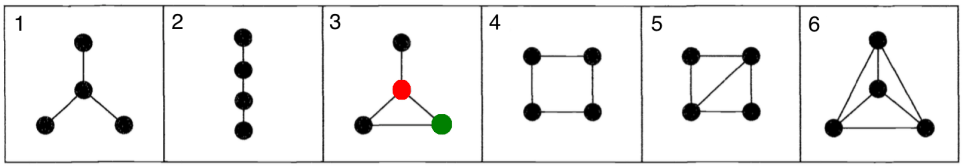
2. On Twitter:

   (a) John follows Joan, Jean and Jane; Joe follows Jane and Joan; Jean and Joan follow each other. Draw a digraph illustrating these follow-relationships between John, Joan, Jean, Jane and Joe.

   (b) Twitter has $\approx$ 313 million active users (June 2016, based on Twitter Inc.). Imagine you would like to store the digraph for the follow-relationships in an adjacency matrix that uses 4 bytes per entry on your new laptop which has 64 GB of RAM. Is this feasible?

(c) The municipality of Odense has a population of ≈ 200000 people. Let G be the graph where the meaning of an edge from vertex i to j is *"person i is friends with person j"*. Imagine you would like to store the adjacency matrix for this graph for the relationships in a matrix representation that uses 4 bytes per entry on your new laptop which has 64 GB of RAM. Is this feasible?

3. Consider the following six graphs (note that the nodes do not have labels).



(a) How many walks of length 3 from the red vertex to the green vertex are there in graph 3?

(b) How many paths from the red vertex to the green vertex are there in graph 3?

(c) How many shortest paths from the red vertex to the green vertex are there in graph 3?

(d) For each of the graphs: what is the longest of all pairwise shortest paths?

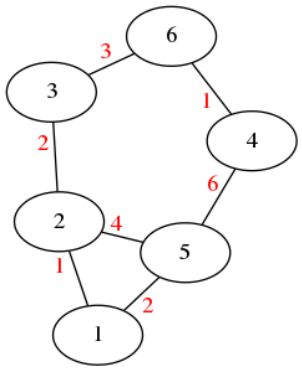
(e) Give an adjacency matrix for graph 1. Can there be different adjacency matrices for the same graph? If so, name a second adjacency matrix for graph 1. Can you find two different adjacency matrices for graph 6?

4. Let A be an adjacency matrix. In the lecture you learned that the ij-entry of A^k is the number of different walks from vertex i to vertex j using exactly k edges.

(a) What is the interpretation of ij-entry of the matrix $A^1+A^2+A^3$?

(b) Complete the following sentence with the missing expression: In a graph G with adjacency matrix A, vertex i and j are connected if and only if $\ldots > 0$.

5. Let the following weighted graph G (from the lecture slides, weights are depicted in red) be given:



It has the following distance matrix D:

$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 3 & 7 & 2 & 6 \\ 1 & 0 & 2 & 6 & 3 & 5 \\ 3 & 2 & 0 & 4 & 5 & 3 \\ 7 & 6 & 4 & 0 & 6 & 1 \\ 2 & 3 & 5 & 6 & 0 & 7 \\ 6 & 5 & 3 & 1 & 7 & 0 \end{pmatrix} \end{array}$$

(a) How many shortest path in G are of length 6? Name them.

(b) How long is the longest of all pairwise shortest paths in the graph? Are there several longest shortest paths?

(c) How many paths in G are of length 6? (Note: a path does not necessarily need to be a shortest path.) Name them.

6. Assume in this exercise that all weights on edges are non-negative values.

(a) In a graph G with $n = 6$ vertices, how many matrix-matrix multiplication operations are needed in the worst case in order to compute the distance matrix D, when the method of repeated squaring is used to compute D?

(b) In a graph G with $n = 200$ vertices, how many matrix-matrix multiplications are needed in the worst case in order to compute the distance matrix D, when the method of repeated squaring is used to compute D?

(c) Can you find a graph G with $n = 6$ vertices, for which $W^4 \neq W^5$? If so, depict it.

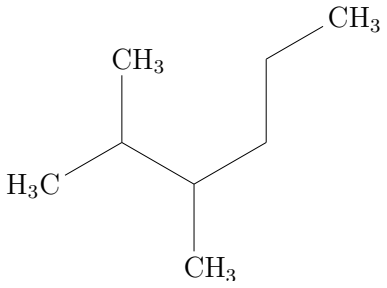
(d) Can you find a graph G with $n = 6$ vertices, for which $W^5 \neq W^6$? If so, depict it.

(e) Can you find a graph G with $n = 6$ vertices, for which $W^1 = W^2$? If so, depict it.

(f) What is the computational runtime in order to compute the distance matrix D for a graph G with n vertices if the method of repeated squaring is used to compute D?

II: Løses hjemme inden øvelsestimerne i uge 48

1. Consider the following molecule (it's called 2,3-Dimethylhexane, see `https://en.wikipedia.org/wiki/2,3-Dimethylhexane`):



(a) How many carbon atoms does this molecule have?

(b) Draw the graph G corresponding to the carbon backbone of the molecule.

(c) Give the edge weight matrix W for the graph G.

(d) Use your brain or the Python program `shortestPaths.py` to infer the distance matrix. [Hint: the graph is rather simple, you won't need a program for that.]

(e) What is the Wiener Index $\mathcal{W}(G)$?

(f) How many shortest paths of length 3 $i \to \ldots \to j$ with $i < j$ are in G?

(g) Using Wiener's method for predicting the boiling point, what is your prediction for 2,3-Dimethylhexane?

2. Try breaking these two encrypted messages:

   (a) This English message was encrypted using a Caesar cipher. Decrypt it.

   YMNX HWDUYTLWFR NX JFXD YT IJHNUMJW.

   Discuss which techniques you used. [Hint: You may want to write a simple program to help you try out things.]

   (b) This was entitled "Cold Country". It was encrypted using a monoalphabetic substitution cipher. A monoalphabetic substitution cipher works similarly to a Caesar cipher. However, instead of just shifting the alphabet cyclically by a fixed amount to get the mapping defined for each letter, the alphabet is permuted (reordered) arbitrarily. In other words, in such a cipher the key is a permutation of the alphabet which tells what letter "A" maps to, what letter "B" maps to, etc. If the alphabet has 29 letters, the number of keys is now 29! Why? The original message here was in English, so there are only 26 letters. How many possible keys are there?

   TOWWJPHJC ZY RXW PHOTWYR ZYPHJC ZJ RXW SFOPC. UFYR FB ZR ZY QFIWOWC SZRX ZQW RXFMYHJCY FB BWWR CWWD.

   Discuss which techniques you used. [Hint: Use knowledge (or good guesses) of frequencies of letters in English. You may want to write a simple program to help you try out things.]