Online Algorithms

a topic in

DM573 – Introduction to Computer Science

Kim Skak Larsen

Department of Mathematics and Computer Science (IMADA)
University of Southern Denmark (SDU)

kslarsen@imada.sdu.dk
https://imada.sdu.dk/u/kslarsen/

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Ski Rental – a simplest online problem

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- A highly skilled, successful computer scientist is rewarded a ski vacation until her company desperately needs her again, at which point she is called home immediately, being notified when she wakes up in the morning.
- At the ski resort, skis cost 10 units in the local currency.
- One can rent skis for 1 unit per day.
- Of course, if she buys, she doesn't have to rent anymore.
- Which algorithm does she employ to minimize her spending?

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- What is a good algorithm?

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- What is a good algorithm?
- How do we measure if it's a good algorithm?

Competitive Analysis

Competitive analysis is one way to measure the quality of an online algorithm.

Idea:

Let $\textsc{Opt}$ denote an *optimal offline algorithm*.

"Offline" means getting the entire input before having to compute.

This corresponds to knowing the future!

We calculate how well we perform compared to $\textsc{Opt}$.

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Notation:

$\textsc{Alg}(I)$ denote the result of running an algorithm $\textsc{Alg}$ on the input sequence I.

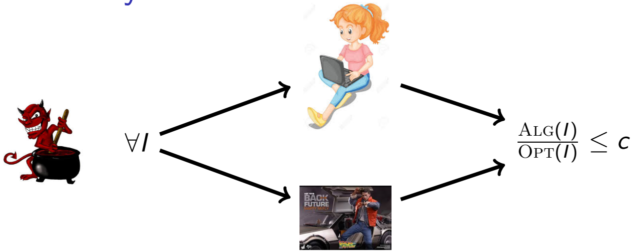
Thus, $\textsc{Opt}(I)$ is the result of running $\textsc{Opt}$ on I.

Competitive Analysis

$\forall I$

$$\frac{\textsc{Alg}(I)}{\textsc{Opt}(I)} \leq c$$

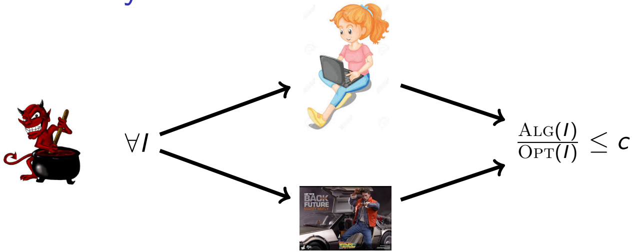
Competitive Analysis



An algorithm, ALG, is *c-competitive* if

$$\forall I : \frac{\text{ALG}(I)}{\text{OPT}(I)} \leq c.$$

Competitive Analysis



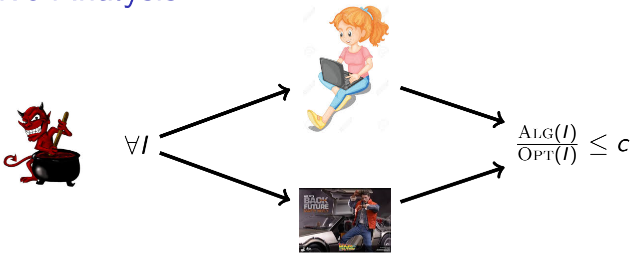
An algorithm, ALG, is *c-competitive* if

$$\forall I : \frac{\text{ALG}(I)}{\text{OPT}(I)} \leq c.$$

ALG has *competitive ratio c* if

c is the *best* (smallest) c for which ALG is c-competitive.

Competitive Analysis



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Technically, the definition of being c-competitive is that $\exists b \, \forall I : \text{ALG}(I) \leq c \, \text{OPT}(I) + b$, but the additive term, b, does not become relevant for what we consider. Also not relevant today, a *best* c does not necessarily exist, so the competitive ratio is $\inf \{c \mid \text{ALG is } c\text{-competitive}\}$.

Ski Rental – a simplest online problem

We have a very simple request sequence with very simple responses:

input sequence	decision
it's a new day	

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We have a very simple request sequence with very simple responses:

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come home	go home

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- The optimal offline algorithm, $\textsc{Opt}$:

  d = the number of days we get to stay
  if $d < 10$:
      rent every day
  else: $\# \; d \geq 10$
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  - Our cost: $d < 10$.
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  - Our cost: $9 + 10$.
  - OPT's cost: 10.

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  - Our cost: $d < 10$.
  - $\textsc{Opt}$'s cost: d.
- In the worst case, we are called home at (or any time after) day 10:
  - Our cost: $9 + 10$.
  - $\textsc{Opt}$'s cost: 10.
- Result: we perform at most $\frac{19}{10} < 2$ times worse than $\textsc{Opt}$.

Competitive Analysis

In general, we want to *define* good algorithms for online problems and *prove* that they are good.

The ratio ($\frac{19}{10}$) from ski rental is a guarantee:

> *That algorithm never performs worse than* $\frac{19}{10}$ *times* $\textsc{Opt}$.

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Online Problems

So, what characterizes an *online problem*?

- Input arrives *one request at a time*.
- For each request, we have to make an *irrevocable decision*.
- We want to *minimize cost*.

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Sometimes we want to maximize a profit instead of minimizing a cost and there are some technicalities in adjusting definitions to accommodate that possibility.

Machine Scheduling

- $m \geq 1$ machines.
- n jobs of varying sizes arriving one at a time to be assigned to a machine.
- The goal is to *minimize makespan*, i.e., finish all jobs as early as possible.

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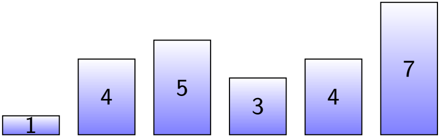
- n jobs of varying sizes arriving one at a time to be assigned to a machine.
- The goal is to *minimize makespan*, i.e., finish all jobs as early as possible.
- Algorithm List Scheduling (LS): place next job on the least loaded machine.

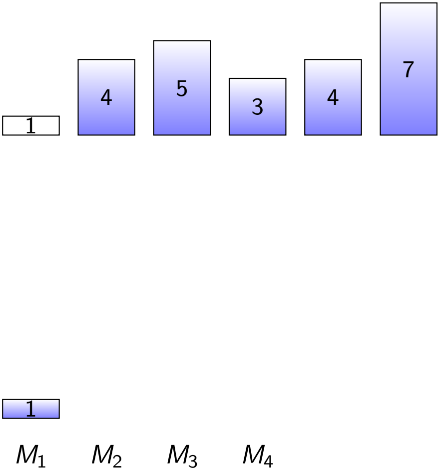
Machine Scheduling – List Scheduling example



M_1 M_2 M_3 M_4

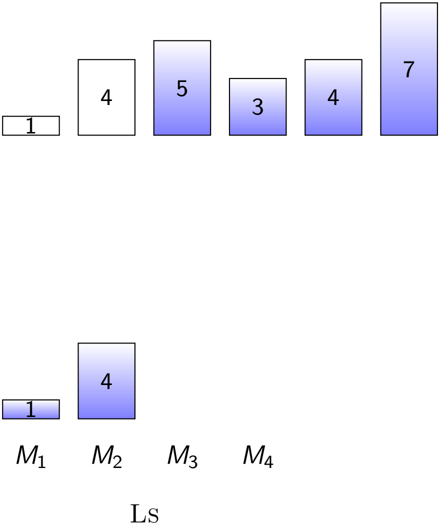
LS

Machine Scheduling – List Scheduling example



Ls

Machine Scheduling – List Scheduling example



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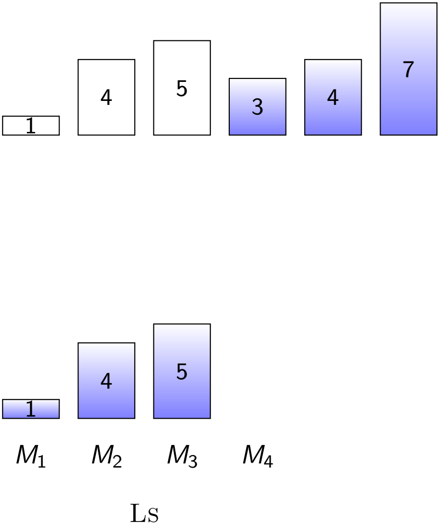

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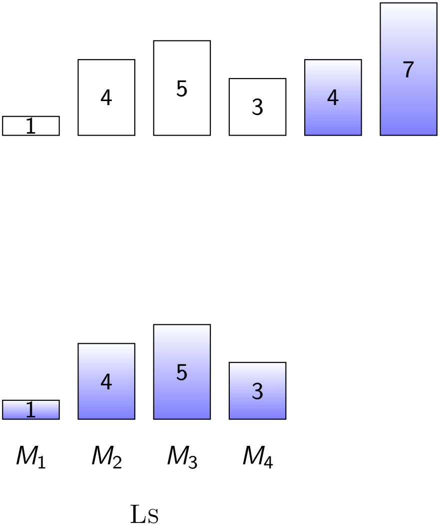

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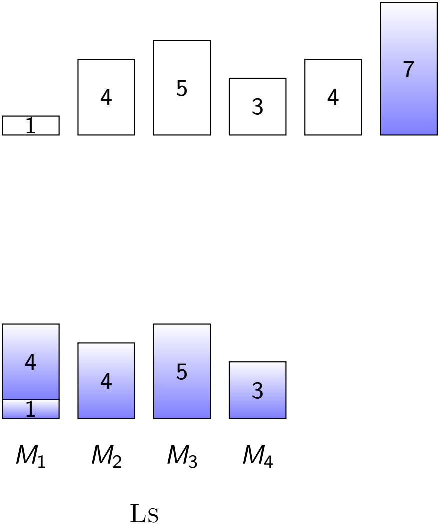

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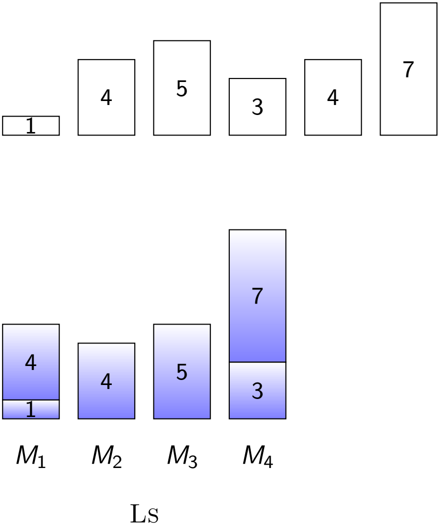

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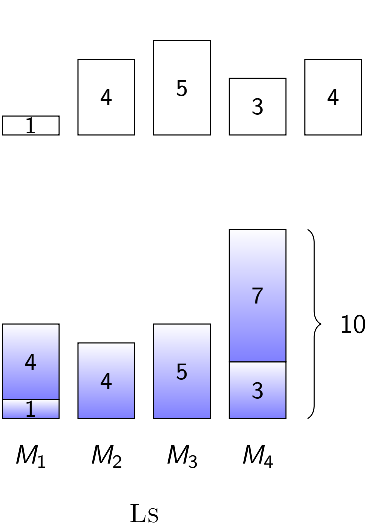

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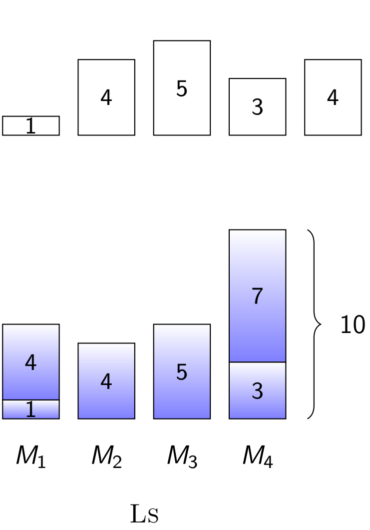


Is 10 a good result?

Machine Scheduling – List Scheduling example

On some sequences, 10 is a good result.

On some sequences, 1.000.000 is a good result.

Sometimes 1000 is a bad result.

It depends on what is possible, i.e., how large or “difficult” the input is.

Competitive Analysis – why $\textsc{Opt}$?

$\textsc{Opt}$ is not an online algorithm.

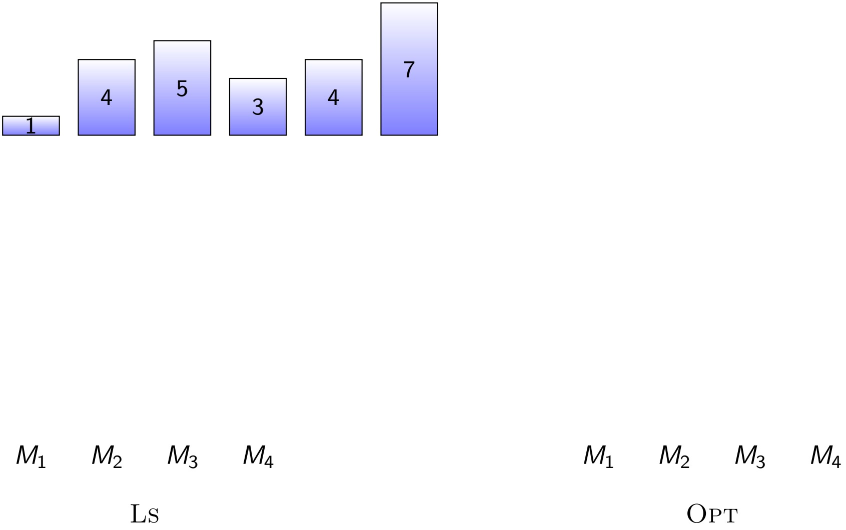
However, it is a natural *reference point*:

- Our online algorithms cannot perform better than $\textsc{Opt}$.
- As input sequences I get harder, $\textsc{Opt}(I)$ typically grows, so a comparison to $\textsc{Opt}$ remains somewhat reasonable.

Formally:

- For any $\textsc{Alg}$ and any I, $\dfrac{\textsc{Alg}(I)}{\textsc{Opt}(I)} \geq 1$.
- We want to design algorithms with smallest possible competitive ratio.

Machine Scheduling – List Scheduling example



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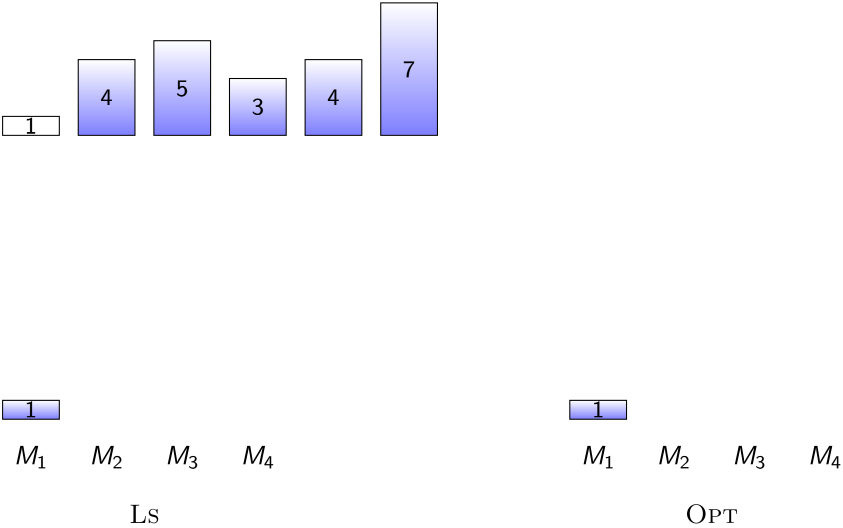

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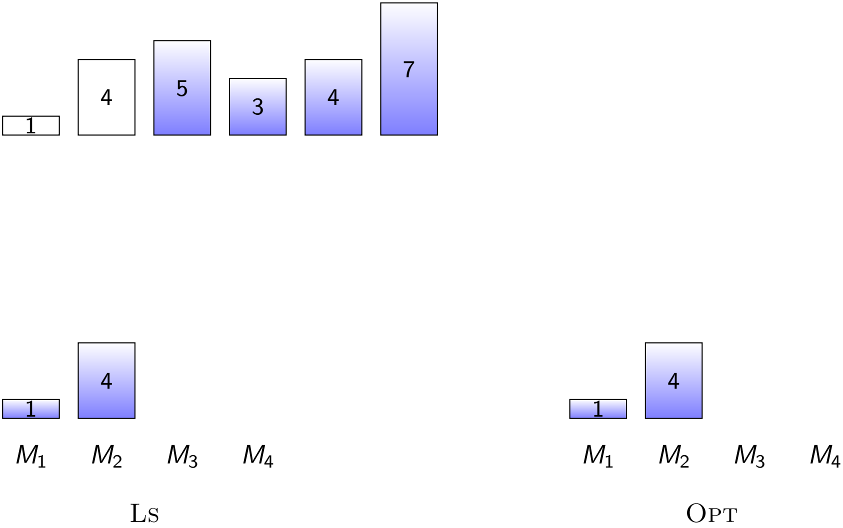

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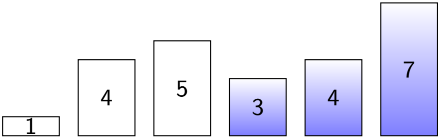
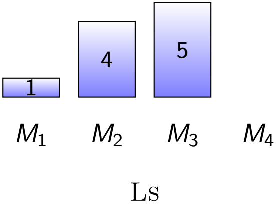
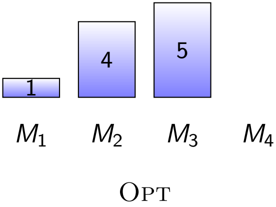


What does OPT do now?





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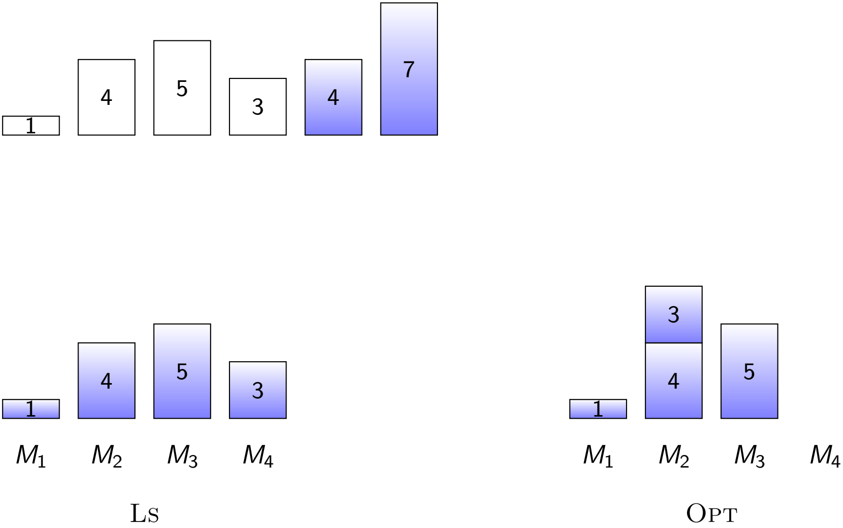

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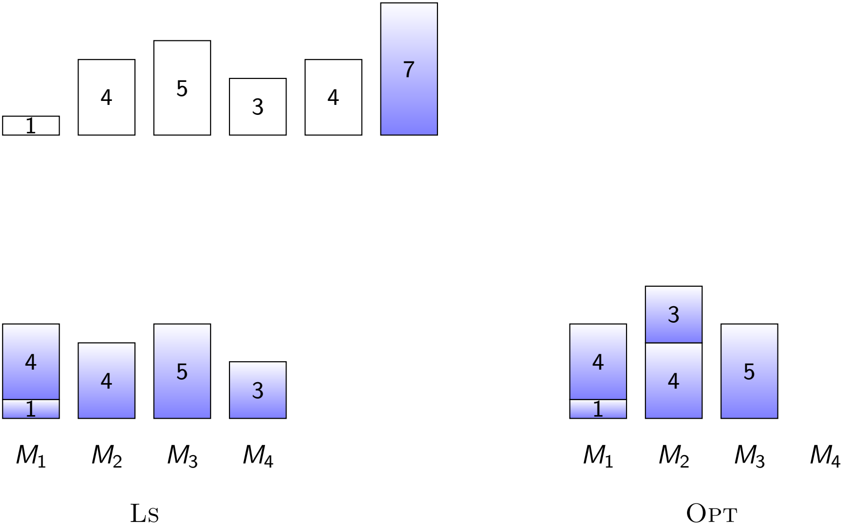

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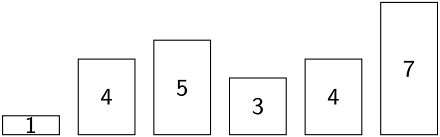
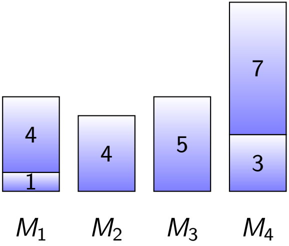
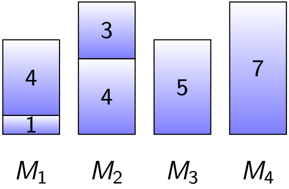




Ls



Opt

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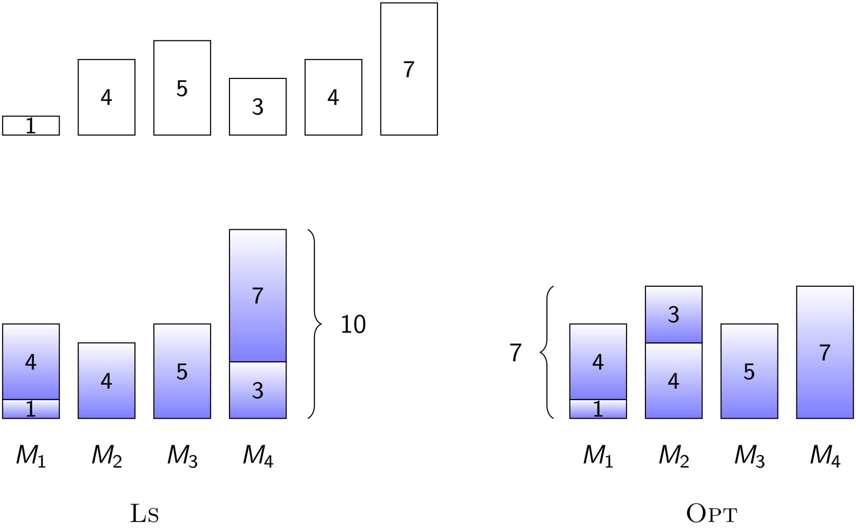
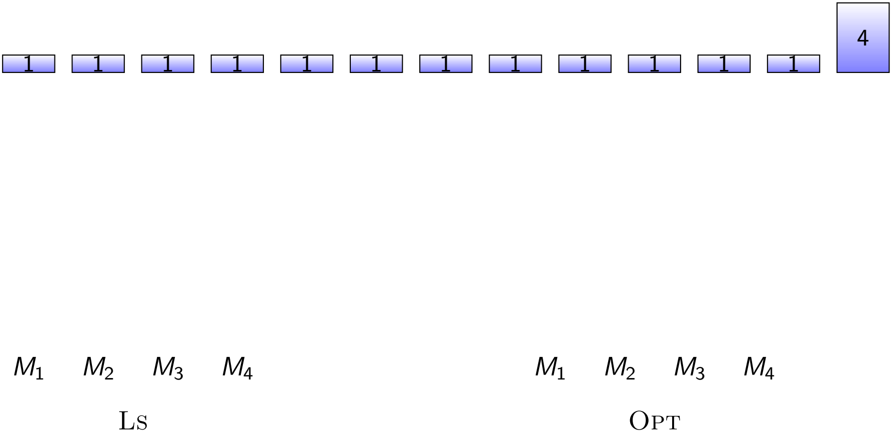


Machine Scheduling – Ls is at best $(2 - \frac{1}{m})$-competitive



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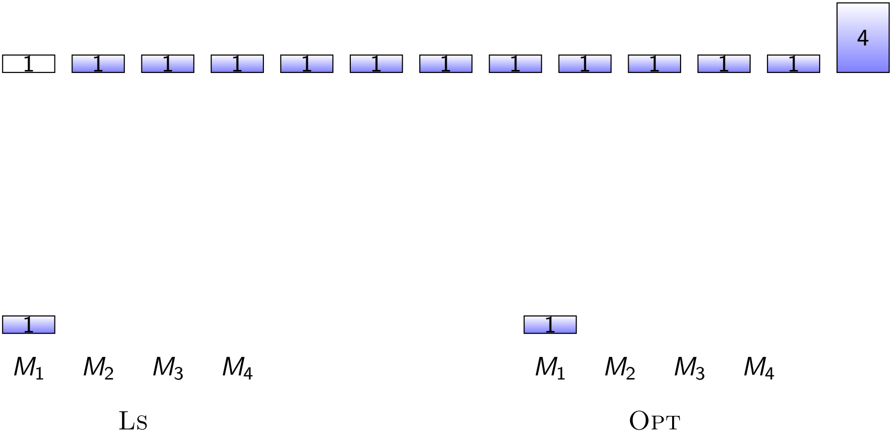
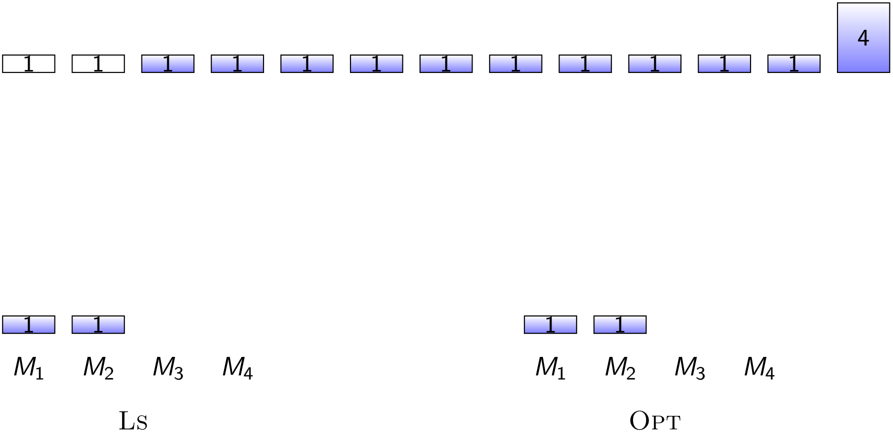


Machine Scheduling – LS is at best $(2 - \frac{1}{m})$-competitive



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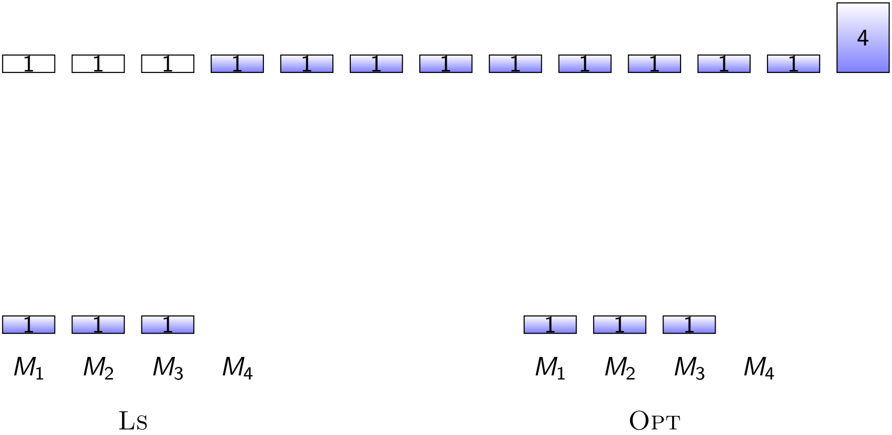

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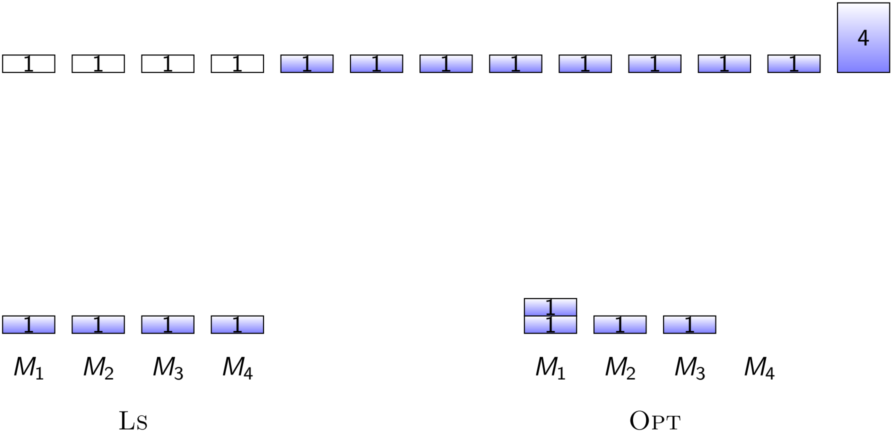

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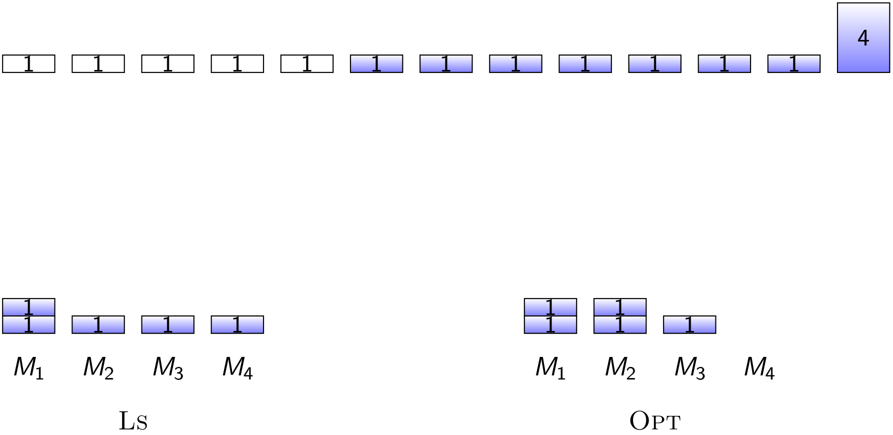

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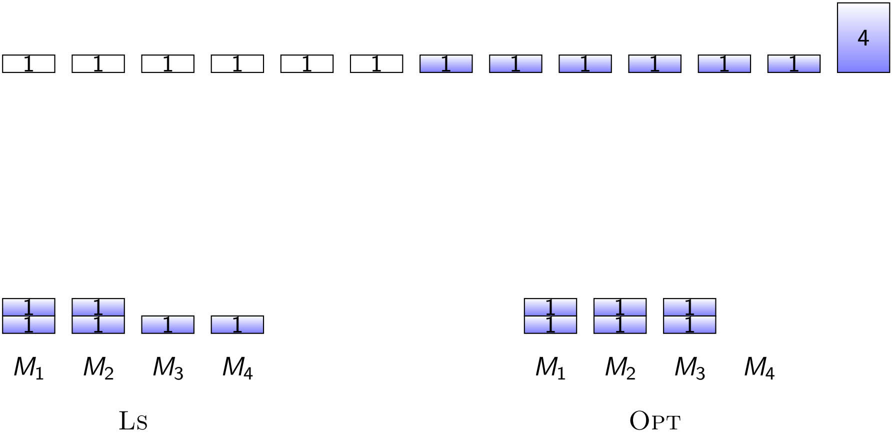

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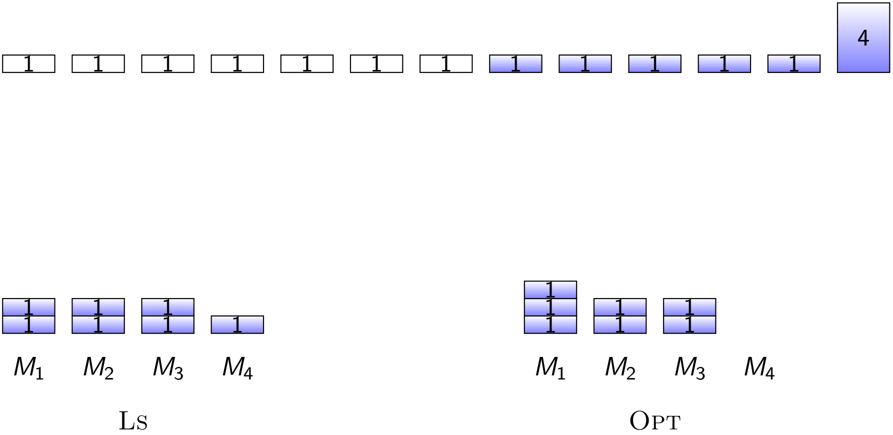

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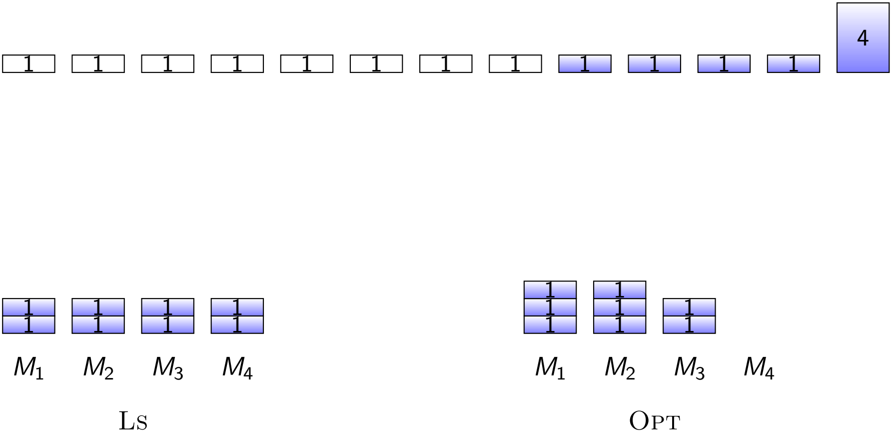

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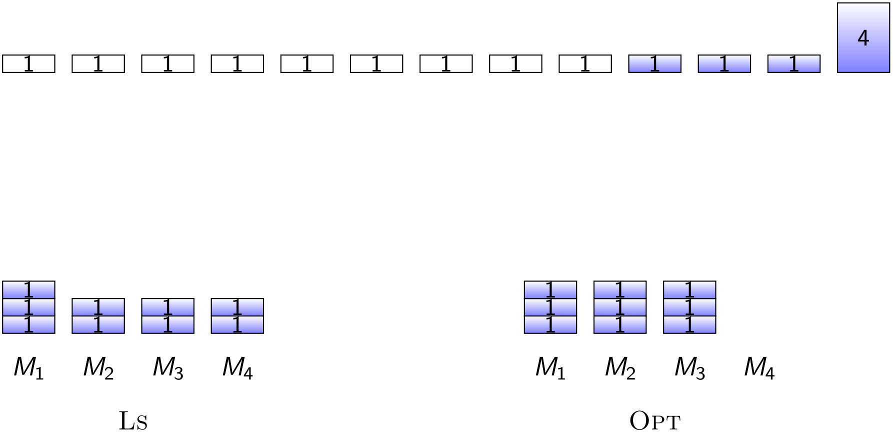

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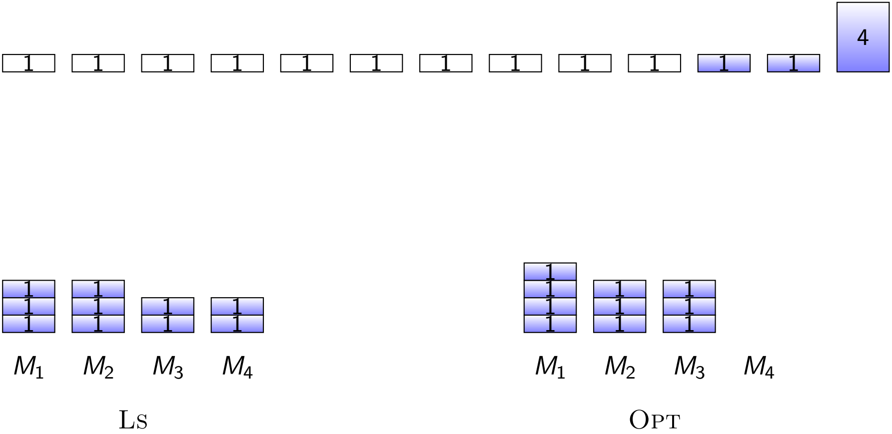

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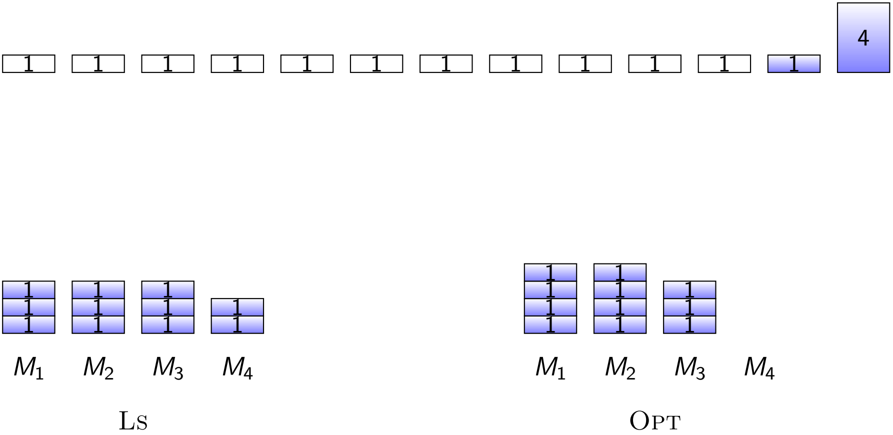

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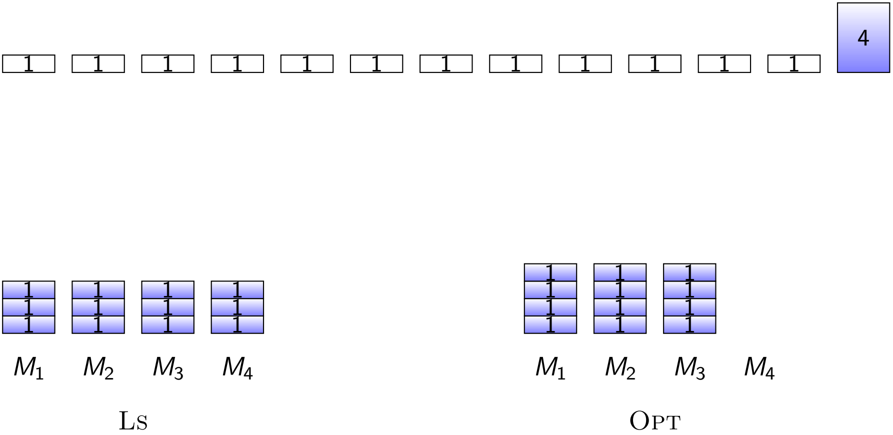

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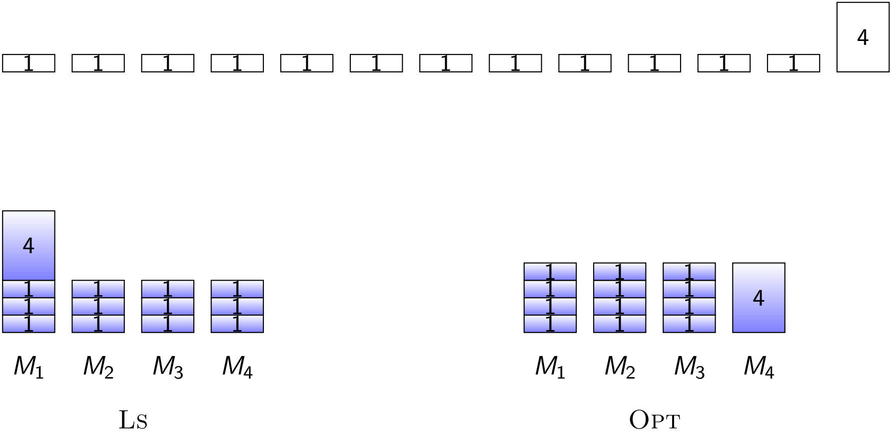

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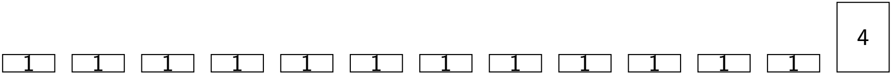
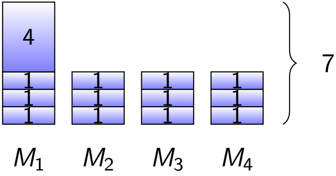
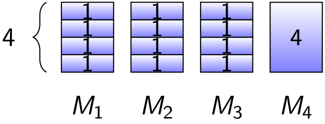




Ls

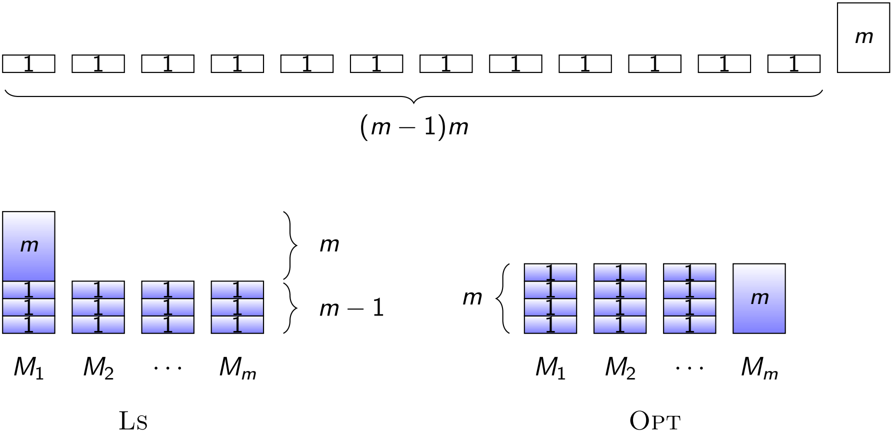


Opt

Thus,

$$\frac{\text{Ls}(I)}{\text{Opt}(I)} = \frac{7}{4} = 2 - \frac{1}{4}$$

Machine Scheduling – Ls is at best $(2 - \frac{1}{m})$-competitive



In general,

$$\frac{\text{Ls}(I)}{\text{Opt}(I)} \geq \frac{(m-1)+m}{m} = \frac{2m-1}{m} = 2 - \frac{1}{m}$$

Machine Scheduling - proof techniques

You have just seen a

lower bound proof

In our context, that is relatively easy: Just demonstrate an example (family of examples), where the algorithm performs worse than some ratio.

Machine Scheduling - proof techniques

You have just seen a

lower bound proof

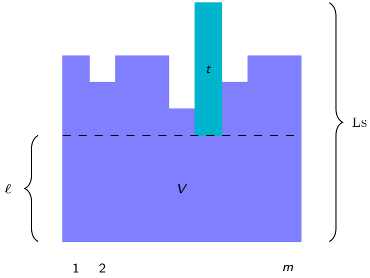
In our context, that is relatively easy: Just demonstrate an example (family of examples), where the algorithm performs worse than some ratio.

Now we want to show the much harder

upper bound proof

We must show that it is *never* worse than a given ratio for *any* of the (potentially infinitely many) possible input sequences.

Machine Scheduling – Ls is $(2 - \frac{1}{m})$-competitive



t is the length of the job starting at ℓ and ending at the makespan

T is the total length of all jobs

Define $V = \ell \cdot m$

Now conclude:

$\text{Opt} \geq T/m$ and $\text{Opt} \geq t$

$T \geq V + t$, due to Ls's choice for t

$$
\begin{aligned}
\text{Ls} &= \ell + t \\
&\leq \frac{T-t}{m} + t, \text{ since } \ell = \frac{V}{m} \text{ and } V \leq T - t \\
&= \frac{T}{m} + (1 - \frac{1}{m})t \\
&\leq \text{Opt} + (1 - \frac{1}{m})\,\text{Opt}, \text{ from Opt inequalities above} \\
&= (2 - \frac{1}{m})\,\text{Opt}
\end{aligned}
$$

Machine Scheduling

Since we have both an upper and lower bound, we have shown the following:

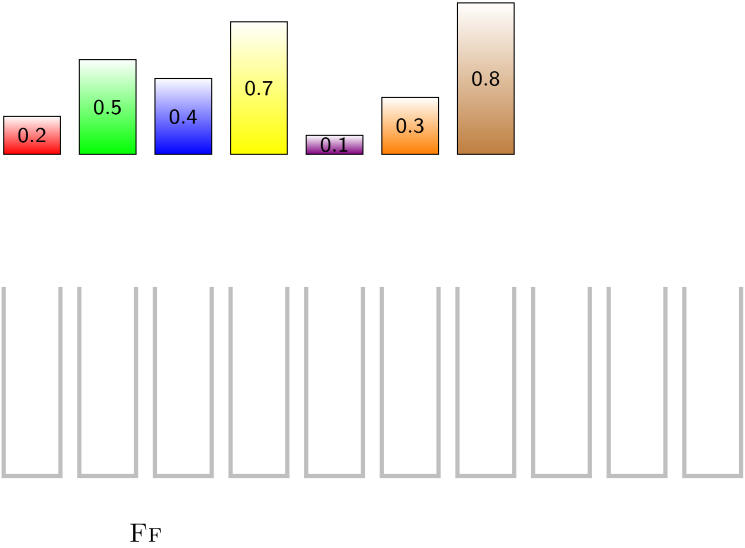
Theorem

The algorithm Ls for minimizing makespan in machine scheduling with $m \geq 1$ machines has competitive ratio $2 - \frac{1}{m}$.

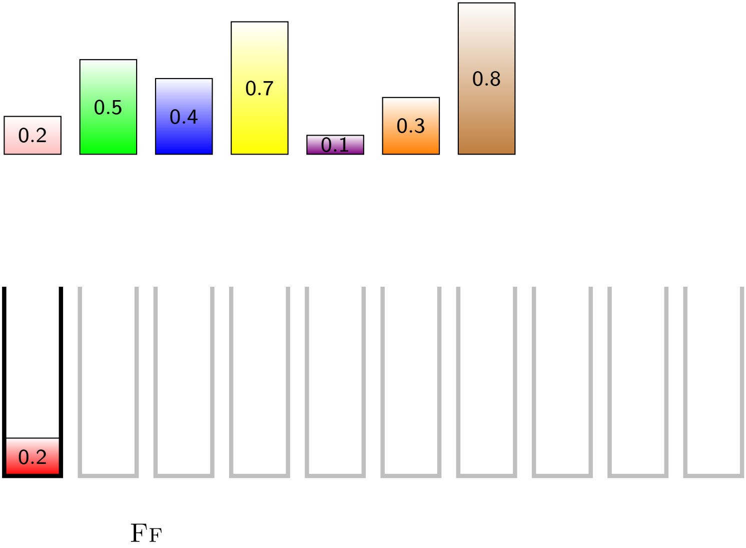
Bin Packing

- Unbounded supply of bins of size 1.
- n items, each of size strictly between zero and one, to be placed in a bin.
- Obviously, the items placed in any given bin cannot be larger than 1 in total.
- The goal is to *minimize the number of bins used*.
- Algorithm First-Fit ($\textsc{Ff}$): place next item in the first bin with enough space. The ordering of the bins is determined by the first time they receive an item.

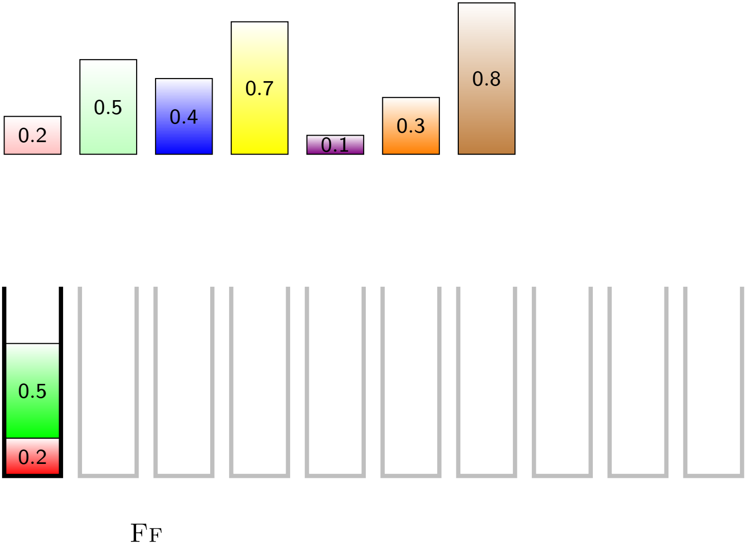
Bin Packing – FF example



Bin Packing – FF example



Bin Packing – FF example



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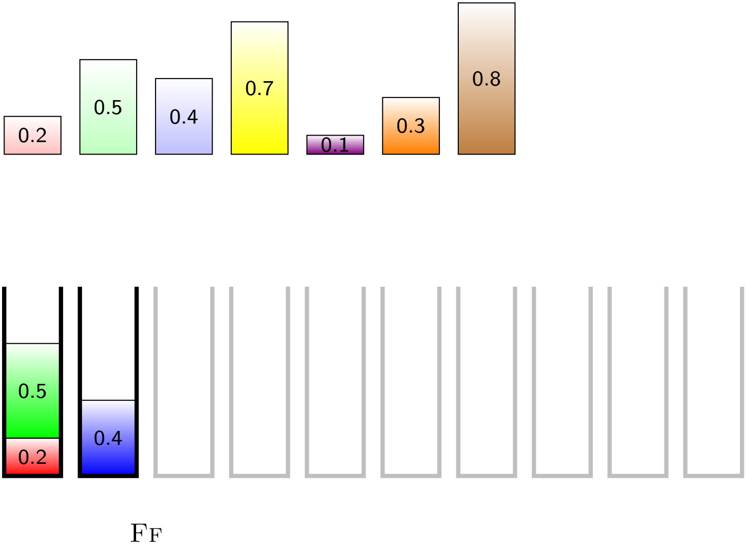

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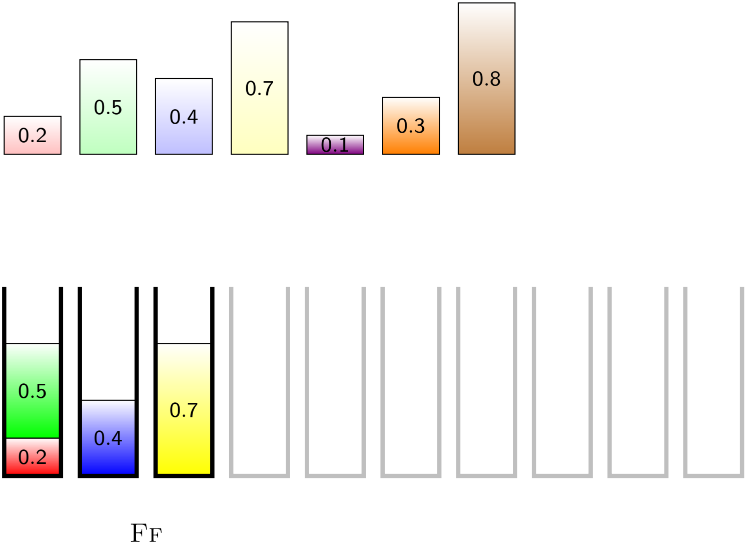
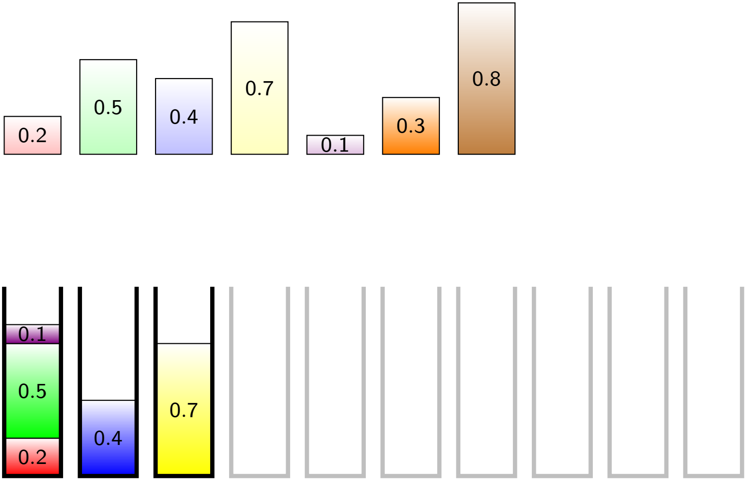


Bin Packing – FF example



FF

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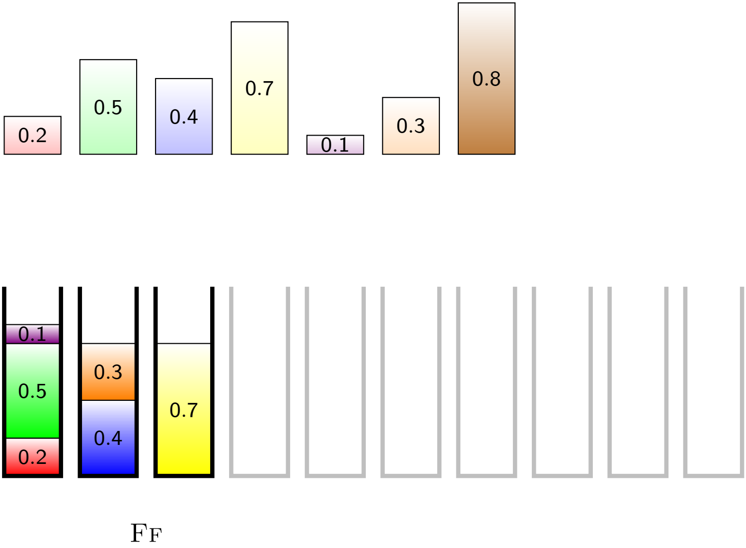

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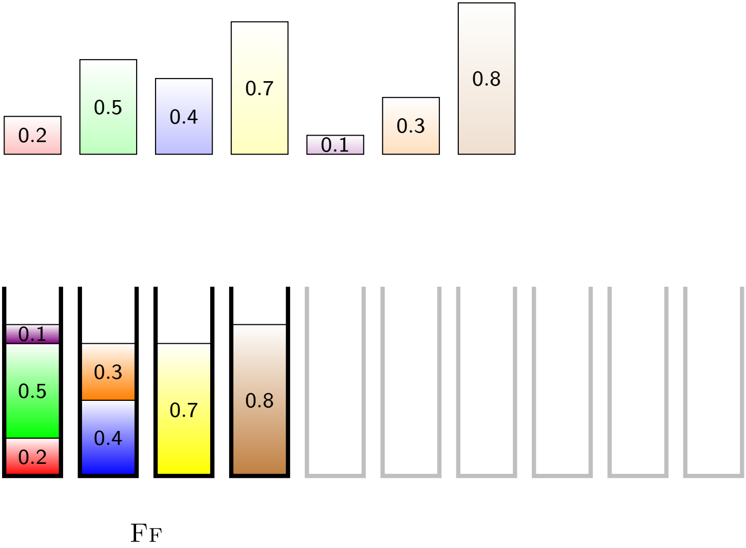

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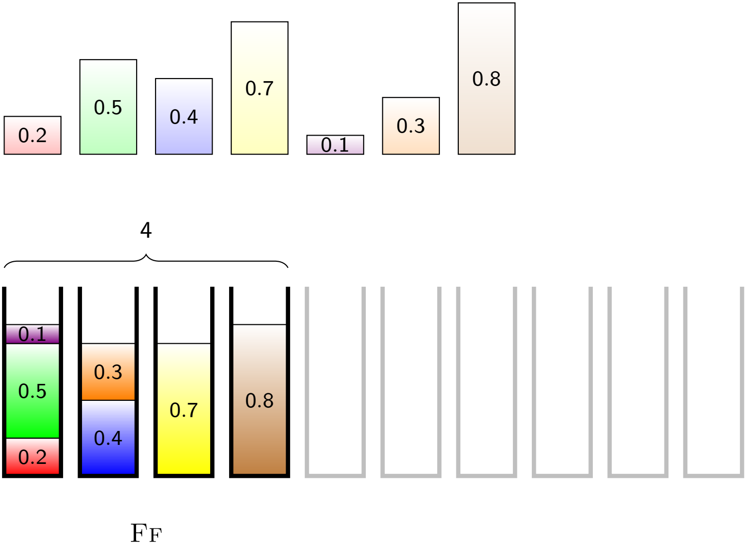
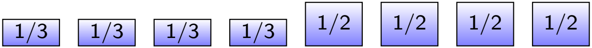


Bin Packing – simple lower bound against FF

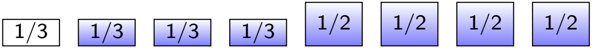


All items are slightly larger than the indicated fraction, e.g., $\frac{1}{3} + \frac{1}{1000}$.

FF

OPT

Bin Packing – simple lower bound against FF



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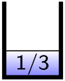
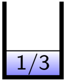


FF



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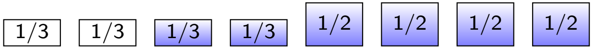

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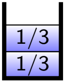

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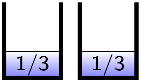

OPT

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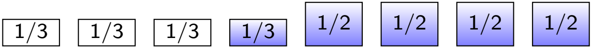

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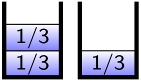
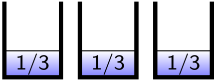
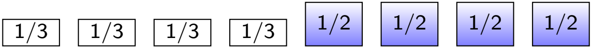


FF



OPT

Bin Packing – simple lower bound against $\textsc{Ff}$



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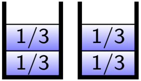
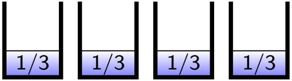
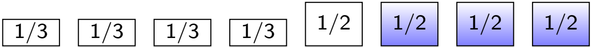


$\textsc{Ff}$



$\textsc{Opt}$

Bin Packing – simple lower bound against FF



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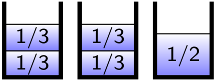

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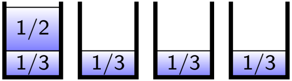

OPT

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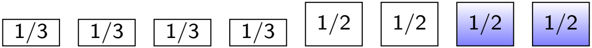

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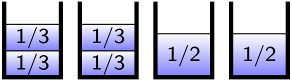
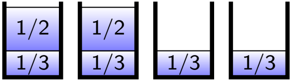
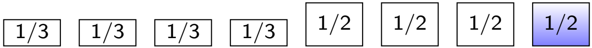


Ff



Opt

Bin Packing – simple lower bound against FF



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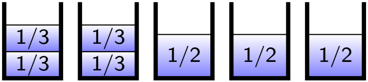

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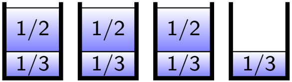

OPT

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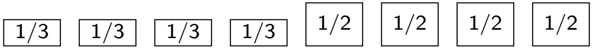

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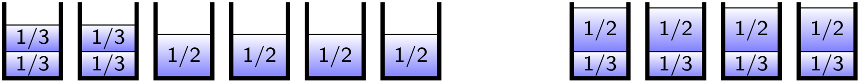
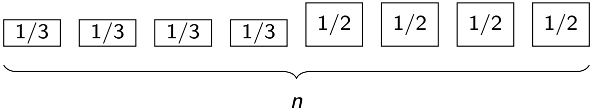


FF

OPT

Bin Packing – simple lower bound against FF



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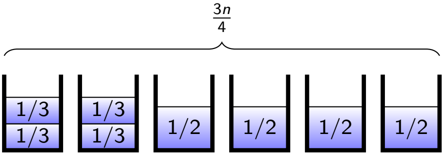
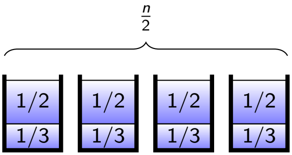


FF



OPT

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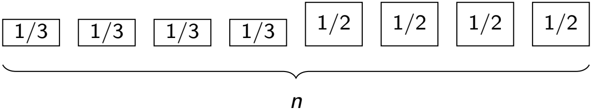

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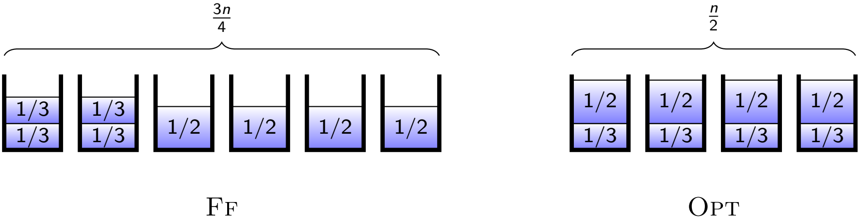


Thus,

$$\frac{\text{FF}(I)}{\text{OPT}(I)} \geq \frac{\frac{n}{4} + \frac{n}{2}}{\frac{n}{2}} = \frac{3}{2}$$

Bin Packing – First-Fit

- FF has been shown to be 1.7-competitive [hard].
  Thus, the competitive ratio is *at most* 1.7.
- We have just shown that the competitive ratio is *at least* 1.5.
- So, the competitive ratio of FF is in the interval $[1.5, 1.7]$.
- We'll approach the precise value in the exercises.

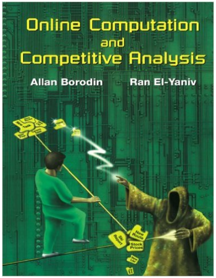
How To Learn More

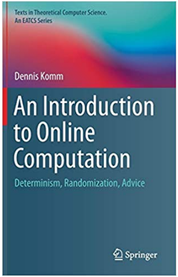
You'll meet online algorithms in courses, without necessarily being told that they are *online* problems.

A formal treatment of this topic is somewhat abstract and heavy on proofs, and more appropriate for the MS level. At that point, you can take the course

DM860: Online Algorithms

or read one of

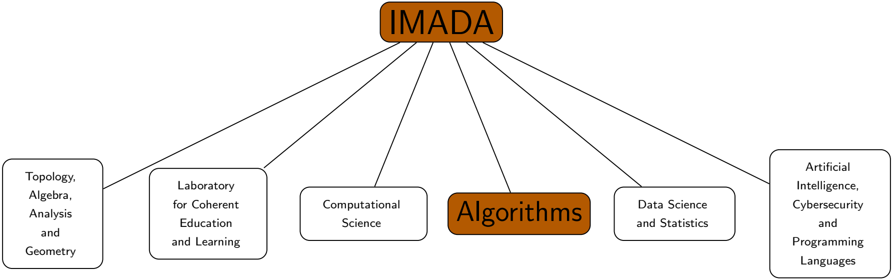




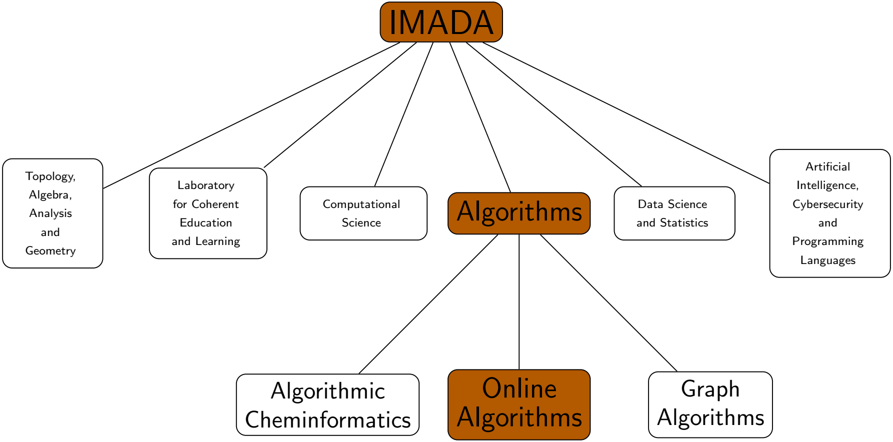
IMADA People in Online Algorithms

Online Algorithms		
	Joan Boyar	Online algorithms, combinatorial optimization, cryptology, computational complexity
	Lene Monrad Favrholdt	Online algorithms, graph algorithms
	Kim Skak Larsen	Online algorithms, algorithms and data structures, database systems, semantics
	Kevin Schewior	Online algorithms, algorithms under uncertainty, approximation algorithms
	Magnus Berg	Online algorithms

IMADA Research Structure



IMADA Research Structure



Online Algorithms Group

SDU

Online Algorithms

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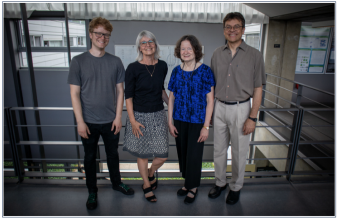

Group photo: Kevin Schewior, Lene M. Favrholdt, Joan Boyar, Kim S. Larsen.

Online algorithms deal with a particularly challenging branch of the algorithms area. Many real world problems have an online component, some decisions that have to be made immediately, before all related data is available. An example of an online problem is the seat reservation problem, where, as with the IC trains in Denmark, a given train has a fixed number of seats and passengers can make seat reservations before their trip and be given seat numbers immediately. In contrast, if the problem were "offline", the seat reservation system could wait until it had all reservation requests before assigning any seats. The online property limits how well a system can perform in optimizing for its objective (maximizing the number of passengers able to buy tickets, for example). However, it may be possible for a system to perform quite well, even if the problem is online, as long as the correct online algorithm (method of solving the online problem) is used. The study of online algorithms is aimed at designing and analyzing online algorithms.

In analyzing algorithms, it is very useful to have a theoretical tool which can be used in predicting how well algorithms will do in practice. The group has been working extensively on this, concentrating on alternatives to the standard quality measure for online algorithms, the competitive ratio. They have defined and are investigating the relative worst order ratio, which gives results superior to those obtained with competitive

Online Algorithms – Subgroup of Algorithms

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Group photo (left to right): Rolf Fagerberg, Yun Wang, Lene M. Favrholdt, Kevin Schewior, Jørgen Bang-Jensen, Joan Boyar, Anders Yeo, Jakob L. Andersen, Simon Erfurth, Kim S. Larsen, Daniel Merkle.

In recent years, the word "algorithms" has become part of everyone's vocabulary. When newscasters explain that some software is really amazing, they state that there is "intelligence" or "algorithms" in the product. The recent popularity of the word "algorithms" is well deserved. In fact, what is often referred to as "intelligence" in systems is really huge amounts of data combined with advanced algorithms for searching and computing based on this data.

Algorithms are at the core of Computer Science, so in addition to offering expertise in concrete application areas, a solid background in algorithms makes it easy to enter other subareas of Computer Science; also after

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a topic in

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Kim Skak Larsen

Department of Mathematics and Computer Science (IMADA)
University of Southern Denmark (SDU)

kslarsen@imada.sdu.dk
https://imada.sdu.dk/u/kslarsen/

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