

Cryptography, Number Theory, and RSA

Slides by Joan Boyar

IMADA

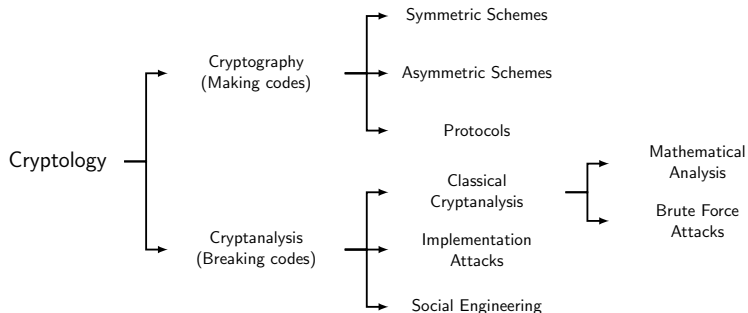
University of Southern Denmark

(with edits by Rolf Fagerberg)

Outline of lectures

- ▶ Cryptography vs. Cryptanalysis
- ▶ Symmetric key cryptography
- ▶ Public key cryptography
- ▶ Recap of number theory
- ▶ RSA
- ▶ Digital signatures with RSA
- ▶ Combining symmetric and public key systems
- ▶ Modular exponentiation
- ▶ Greatest common divisor
- ▶ Primality testing
- ▶ Correctness of RSA

Cryptography vs. Cryptanalysis



Symmetric key cryptography

Alice and Bob share a *single* secret key SK .

For Alice to send message m to Bob in encrypted form, Alice computes:

$$c = E(m, SK).$$

To decrypt c , Bob computes:

$$r = D(c, SK).$$

Of course, $r = m$ must be guaranteed by the pair of functions E and D constituting the cryptosystem.

Example of a symmetric key system: Caesar cipher

Idea: shift cyclically all letters of the alphabet by the same amount. The secret key SK is the shift. For $SK = 3$, encryption is given by the following table (A becomes D, B becomes E, etc.):

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

P	Q	R	S	T	U	V	W	X	Y	Z	Æ	Ø	Å
15	16	17	18	19	20	21	22	23	24	25	26	27	28
S	T	U	V	W	X	Y	Z	Æ	Ø	Å	A	B	C
18	19	20	21	22	23	24	25	26	27	28	0	1	2

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Suppose the following was encrypted using a Caesar cipher and the Danish alphabet. The key is unknown. How would you try to decrypt it?

ZQOØQOØ, RI.

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What does this say about how many keys should be possible?

Some symmetric key systems

- ▶ ~~Caesar Cipher (< 100 BC, unknown)~~
- ▶ ...
- ▶ ...
- ▶ ~~Enigma (1930-40, German army)~~
- ▶ ~~DES (1976, IBM)~~
- ▶ ~~Triple DES (1978-81, Walter Tuchman, Ralph Merkle and Martin Hellman)~~
- ▶ IDEA (1991, James Massey, Xuejia Lai)
- ▶ Blowfish (1993, Bruce Schneider)
- ▶ AES (2001, Joan Daemen and Vincent Rijmen)

Crossed out systems are considered broken by now.

Public key cryptography [Hellman, Diffie, Merkle, 1976]

Bob — two keys: PK_B, SK_B

PK_B — Bob's public key

SK_B — Bob's private (secret) key

For Alice to send message m to Bob, Alice computes:

$$c = E(m, PK_B).$$

To decrypt c , Bob computes:

$$r = D(c, SK_B).$$

Of course, $r = m$ must be guaranteed by the pair of functions E and D constituting the cryptosystem.

It must also be “hard” to compute SK_B from PK_B .

[Public key cryptography is also called asymmetric key cryptography.]

Recap of Number Theory

Definition. Suppose $a, b \in \mathbb{Z}$, $a > 0$.

The terminology/notation below all mean the same, namely that $\exists c \in \mathbb{Z}$ such that $b = ac$.

- ▶ a divides b
- ▶ $a \mid b$
- ▶ a is a *factor* of b
- ▶ b is a *multiple* of a

The notation $e \nmid f$ means e does not divide f .

Theorem. $a, b, c \in \mathbb{Z}$. Then

1. if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$
2. if $a \mid b$, then $a \mid bc \ \forall c \in \mathbb{Z}$
3. if $a \mid b$ and $b \mid c$, then $a \mid c$.

Recap of Number Theory

Definition. For $p \in \mathbb{Z}$, $p > 1$ we say that

- ▶ p is *prime* if 1 and p are the only positive integers which divide p .

2, 3, 5, 7, 11, 13, 17, ...

- ▶ p is *composite* if it is not prime.

4, 6, 8, 9, 10, 12, 14, 15, 16, ...

Recap of Number Theory

Theorem. $a \in \mathbb{Z}$, $d \in \mathbb{N}$

\exists unique q, r , $0 \leq r < d$ such that $a = dq + r$

d – divisor

a – dividend

q – quotient

r – remainder = $a \bmod d$

Definition. $\gcd(a, b)$ = greatest common divisor of a and b
= largest $d \in \mathbb{Z}$ such that $d|a$ and $d|b$

If $\gcd(a, b) = 1$, then a and b are *relatively prime*.

Recap of Number Theory

Definition. $a \equiv b \pmod{m}$ — a is congruent to b modulo m if $m \mid (a - b)$.

$m \mid (a - b) \Leftrightarrow \exists k \in \mathbb{Z}$ such that $a = b + km$.

Theorem. $a \equiv b \pmod{m}$ $c \equiv d \pmod{m}$
Then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$.

Proof.(of first) $\exists k_1, k_2$ such that

$$a = b + k_1m \quad c = d + k_2m$$

$$a + c = b + k_1m + d + k_2m$$

$$= b + d + (k_1 + k_2)m$$

□

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Examples.

1. $15 \equiv 22 \pmod{7}$?
2. $15 \equiv 1 \pmod{7}$?
3. $15 \equiv 37 \pmod{7}$?
4. $58 \equiv 22 \pmod{9}$?

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2. $15 \equiv 1 \pmod{7}$?
3. $15 \equiv 37 \pmod{7}$?
4. $58 \equiv 22 \pmod{9}$?

Note the difference to:

1. $15 = 1 \pmod{7}$?
2. $1 = 15 \pmod{7}$?

RSA — a public key system [Rivest, Shamir, Adleman, 1977]

Choose two primes p, q . Set $N = p \cdot q$.

Find $e > 1$ such that $\gcd(e, (p - 1)(q - 1)) = 1$.

Find d such that $e \cdot d \equiv 1 \pmod{(p - 1)(q - 1)}$.

▶ $PK = (N, e)$

▶ $SK = (N, d)$

To encrypt: $c = E(m, PK) = m^e \pmod{N}$.

To decrypt: $r = D(c, SK) = c^d \pmod{N}$.

One can prove that $r = m$ (if $0 \leq m < N$).

Here, m is the message, c is the ciphertext (the encrypted message). Note: Any message of less than $\log_2 N$ bits can be seen as a binary number m with $0 \leq m < N$. For longer messages, chop it up and encrypt each part individually.

RSA, example

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Example:

$p = 5, q = 11$ (hence $N = 55$), $e = 3, d = 27, m = 8$.

Then $\gcd(e, (p-1)(q-1)) = \gcd(3, 4 \cdot 10) = 1$, as required.

Then $e \cdot d = 81$, so $e \cdot d \equiv 1 \pmod{4 \cdot 10}$, as required.

To encrypt m : $c = 8^3 \pmod{55} = 17$.

To decrypt c : $r = 17^{27} \pmod{55} = 8$.

RSA, exercise

Recap:

$N = p \cdot q$, where p, q prime.

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Exercise:

Try using $N = 35$, $e = 11$ as keys.

Factor 35 and check the requirement on e .

What is d ? Try $d = 11$ and check the requirement on d .

Encrypt $m = 4$. Decrypt the result.

RSA, exercise

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Did you get $c = 9$? And $r = 4$?

An Application: Digital Signatures with RSA

Suppose Alice wants to sign a document m such that:

- ▶ No one else could forge her signature
- ▶ It is easy for others to verify her signature

Note m has arbitrary length.

RSA is used on fixed length messages.

Alice uses a cryptographically secure hash function h , meaning that:

- ▶ For any message m' , $h(m')$ has a fixed length (e.g., 2048 bits)
- ▶ It is conjectured “hard” for anyone to find 2 messages (m_1, m_2) such that $h(m_1) = h(m_2)$.

Digital Signatures with RSA

Then Alice “decrypts” $h(m)$ with her secret RSA key (N_A, d_A)

$$s = (h(m))^{d_A} \pmod{N_A}$$

and publishes the document m and the signature s .

Bob verifies her signature using her public RSA key (N_A, e_A) and h :

$$c = s^{e_A} \pmod{N_A}$$

He accepts it and only if

$$h(m) = c$$

. This works because $s^{e_A} \pmod{N_A} =$

$$((h(m))^{d_A})^{e_A} \pmod{N_A} = ((h(m))^{e_A})^{d_A} \pmod{N_A} = h(m).$$

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I.e., to encrypt a message m to send to Bob:

- ▶ Choose a random *session key* k for a symmetric key system (e.g., AES)
- ▶ Encrypt k with Bob's public key — Result k_e
- ▶ Encrypt m with k — Result m_e
- ▶ Send k_e and m_e to Bob

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How does Bob decrypt? Why is this efficient?

Security of RSA

The primes p and q are kept secret along with d .

Suppose Eve can factor N .

Then she can find p and q (as these are the factors of N). From them and e , she can find d (using the same method as Alice, to be described later).

Then she can decrypt just like Alice!

So **factoring must be hard**, or RSA will be insecure (here, hard means very time-consuming).

Also, N **must be sufficiently big** to use hardness of factoring. Current recommendations are to choose p and q with at least 1024 bits each, making $N = p \cdot q$ have at least 2048 bits.

Factoring (naive approach)

Theorem N composite $\Rightarrow N$ has a prime divisor $\leq \sqrt{N}$

Factor(N)

for $i = 2$ to \sqrt{N}

 check if i divides N

if it does **then** output divisor i and stop

output "Prime" if divisor not found

Corollary There is an algorithm for factoring N (or verifying primality) which does $O(\sqrt{N})$ tests of divisibility.

Complexity of factoring

Assume that we use primes which are at least 1024 bits long (the current recommendation for RSA). The naive approach does up to $\sqrt{N} = \sqrt{2^{2048}} = (2^{2048})^{1/2} = 2^{2048/2} = 2^{1024} = (10^{\log_{10} 2})^{1024} = (10^{0.301\dots})^{1024} = (10^{1024 \cdot 0.301\dots}) > 10^{308}$ tests of divisibility.

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The input length in bits is $n = \log_2(N)$. So the running time above is $O(\sqrt{N}) = O(\sqrt{2^n}) = O((2^n)^{1/2}) = O(2^{n/2}) = O((2^{1/2})^n) = O((\sqrt{2})^n) = O((1.4142\dots)^n)$. This is *exponential* in n .

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Open Problem: Does there exist a factoring algorithm with running time *polynomial* in n ?

(Note: if large enough quantum computers can be built, the answer is yes [Peter Shor, 1994].)

RSA, implementation details

How do we implement RSA?

- ▶ We need to find p, q
- ▶ We need to find e, d
- ▶ To encrypt and decrypt, we need to compute $a^k \pmod n$ for large a, k , and n

We will now discuss how this is done (going backwards through the list of tasks).

RSA encryption/decryption, space usage

Computing $a^k \pmod n$:

$$e \cdot d \equiv 1 \pmod{(p-1)(q-1)},$$

where $e > 1$ and where p and q have ≥ 1024 bits each (using current recommendations). So at least one of e and d has ≥ 1024 bits.

We want to compute $a^k \pmod n$ for $a = m$, $n = N$ and $k = d, e$. The message m usually has the same number of bits as $N = p \cdot q$, which is at least 2048 bits.

Hence, we are dealing with a value of a^k on the order of $(2^{2048})^{2^{1024}}$ which has $\log_2((2^{2048})^{2^{1024}}) = 2^{1024} \log_2(2^{2048}) = 2^{1024} \cdot 2048$ bits, which is more than 10^{311} bits. This number cannot be stored even if using all the RAM existing in the world.

Keeping sizes of numbers down

Theorem [From DM549]

For all nonnegative integers, b, c, m :

$$b \cdot c \pmod{n} = (b \pmod{n}) \cdot (c \pmod{n}) \pmod{n}$$

Example: $a \cdot a^2 \pmod{n} = (a \pmod{n})(a^2 \pmod{n}) \pmod{n}$.

This allows us to take \pmod{n} after every multiplication without changing the result. This will ensure that all numbers dealt with have approximately the same number of bits as n (e.g., 2048 bits).

Space (RAM) problem solved.

A multiplication followed by \pmod{n} is called a modular multiplication.

RSA encryption/decryption, time usage

We need to encrypt and decrypt: compute $a^k \pmod{n}$.

$a^2 \pmod{n} = a \cdot a \pmod{n}$: 1 modular multiplication

$a^3 \pmod{n} = a \cdot (a \cdot a \pmod{n}) \pmod{n}$: 2 mod mults

⋮

$a^k \pmod{n}$: $k - 1$ modular multiplications.

This is way too many:

$$e \cdot d \equiv 1 \pmod{(p-1)(q-1)},$$

where $e > 1$ and where p and q have ≥ 1024 bits each (using current recommendations). So at least one of e and d has ≥ 1024 bits, and we want to compute both $a^e \pmod{n}$ and $a^d \pmod{n}$.

So to either encrypt or decrypt, the above method needs $\geq 2^{1024} - 1 \approx 10^{308}$ operations. Much more time than the age of the universe.

Keeping time down

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How do you calculate $a^4 \pmod n$ in less than 3 mod mults?

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In general: $a^{2^s+1} \pmod n$?

$$a^{2^s+1} \pmod n = a \cdot (a^{2^s} \pmod n) \pmod n$$

Modular Exponentiation

Resulting algorithm:

$\text{Exp}(a, k, n)$ { Compute $a^k \pmod n$ }

if $k < 0$ **then** report error

if $k = 0$ **then** return(1)

if $k = 1$ **then** return($a \pmod n$)

if k is odd **then** return($a \cdot \text{Exp}(a, k - 1, n) \pmod n$)

if k is even **then**

$c = \text{Exp}(a, k/2, n)$

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How many modular multiplications?

We divide exponent by 2 every other time. How many times can we do that? $\lfloor \log_2(k) \rfloor$.

So at most $2 \lfloor \log_2(k) \rfloor$ modular multiplications in total. Time problem solved, since $2 \lfloor \log_2(k) \rfloor \approx 2 \log_2(2^{1024}) = 2 \cdot 1024 = 2048$.

Finding e and d

We need to find: e, d .

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Choose random e .

Check that $\gcd(e, (p-1)(q-1)) = 1$. If not, repeat.

Find d such that $e \cdot d \equiv 1 \pmod{(p-1)(q-1)}$.

Finding multiplicative inverses modulo n

Finding **multiplicative inverses** modulo n :

Given e and n , find d such that $e \cdot d \equiv 1 \pmod{n}$.

Solved if we can find s and t fulfilling $s \cdot e + t \cdot n = 1$ (we can then use s as our d).

This can be done via the [Extended Euclidean Algorithm](#) if $\gcd(e, n) = 1$.

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The (Extended) Euclidean Algorithm is fast: it runs in time $\log(\max(a, b))$ [Lamé, 1844].

The extended Euclidean algorithm (two-pass)

Euclidean algorithm by example: Find $\gcd(75, 42)$

$$d_0 = 75$$

$$d_1 = 42 \quad (75 = 1 \cdot 42 + 33)$$

$$d_2 = 33 \quad (42 = 1 \cdot 33 + 9)$$

$$d_3 = 9 \quad (33 = 3 \cdot 9 + 6)$$

$$d_4 = 6 \quad (9 = 1 \cdot 6 + 3)$$

$$d_5 = 3 \quad (6 = 2 \cdot 3 + 0)$$

$$d_6 = 0 \quad \text{Stop and return previous } d \text{ (here } d_5) \text{ as gcd.}$$

The extended Euclidean algorithm (two-pass)

Euclidean algorithm by example: Find $\gcd(75, 42)$

$$d_0 = 75$$

$$d_1 = 42 \quad (75 = 1 \cdot 42 + 33)$$

$$d_2 = 33 \quad (42 = 1 \cdot 33 + 9)$$

$$d_3 = 9 \quad (33 = 3 \cdot 9 + 6)$$

$$d_4 = 6 \quad (9 = 1 \cdot 6 + 3)$$

$$d_5 = 3 \quad (6 = 2 \cdot 3 + 0)$$

$$d_6 = 0 \quad \text{Stop and return previous } d \text{ (here } d_5) \text{ as gcd.}$$

Extension: Find s and t using the equations from bottom to top:

$$\begin{aligned} \gcd(75, 42) &= 3 \\ &= 9 - 6 = 9 - (33 - 3 \cdot 9) = -33 + 4 \cdot 9 \\ &= -33 + 4 \cdot (42 - 33) = 4 \cdot 42 - 5 \cdot 33 = 4 \cdot 42 - 5 \cdot (75 - 42) \\ &= -5 \cdot 75 + 9 \cdot 42 = s \cdot 75 + t \cdot 42 \end{aligned}$$

The extended Euclidean algorithm (single pass)

{ Initialize}

$$d_0 = b \quad s_0 = 0 \quad t_0 = 1$$

$$d_1 = a \quad s_1 = 1 \quad t_1 = 0$$

$$i = 1$$

{ Compute next d }

while $d_i > 0$ **do**

begin

$$i = i + 1$$

{ Compute $d_i = d_{i-2} \pmod{d_{i-1}}$ }

$$q_i = \lfloor d_{i-2}/d_{i-1} \rfloor$$

$$d_i = d_{i-2} - q_i d_{i-1}$$

$$s_i = s_{i-2} - q_i s_{i-1}$$

$$t_i = t_{i-2} - q_i t_{i-1}$$

end

return: $\gcd(b, a) = d_{i-1}, \quad s = s_{i-1}, \quad t = t_{i-1}$

The extended Euclidean algorithm (single pass)

The single pass algorithm maintains the following invariant (from which correctness of s and t follows):

$$d_i = s_i a + t_i b$$

This invariant is proved by induction on i :

Initialization ($i = 0, i = 1$):

$$d_0 = b = 0 \cdot a + 1 \cdot b = s_0 a + t_0 b$$

$$d_1 = a = 1 \cdot a + 0 \cdot b = s_1 a + t_1 b$$

Step ($i \geq 2$):

$$\begin{aligned} d_i &= d_{i-2} - q_i d_{i-1} = (s_{i-2} a + t_{i-2} b) - q_i (s_{i-1} a + t_{i-1} b) \\ &= (s_{i-2} - q_i s_{i-1}) a + (t_{i-2} - q_i t_{i-1}) b = s_i a + t_i b \end{aligned}$$

Exercise

Calculate the following:

1. $\gcd(6, 9)$
2. s and t such that $s \cdot 6 + t \cdot 9 = \gcd(6, 9)$
3. $\gcd(15, 23)$
4. s and t such that $s \cdot 15 + t \cdot 23 = \gcd(15, 23)$

Primality testing

We also need to find the large primes p, q .

Plan: Choose numbers at random and test if they are prime.

Questions

1. How many random integers with 1024 bits are prime?

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2. How fast can we test if a number is prime?

Quite fast, it turns out (in practice using randomness). See the following pages.

Suggestion 1

Sieve of Eratosthenes:

Lists:

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

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We need to go up to $2^{1024} = 10^{308}$ — more than the number of atoms in universe.

So we cannot even write out the first list in the sieve of Eratosthenes. Not practical.

Suggestion 2

Use our naive factoring algorithm:

`CheckPrime(n)`

for $i = 2$ to \sqrt{n} **do**

 check if i divides n

if it does **then** return(Composite)

return(Prime)

As we saw earlier, this takes much more time than the age of the universe. Not practical.

Rabin–Miller Primality Testing

This is a practical, randomized primality test.

Starting point:

Fermat's Little Theorem: Suppose n is a prime. Then for all integers a where $1 \leq a \leq n - 1$, the following holds:

$$a^{n-1} \pmod{n} = 1.$$

A Fermat test based on $a = 2$:

$$2^{14} \pmod{15} \equiv 4 \neq 1.$$

So 15 is not prime.

We say that $a = 2$ is a Fermat **witness** that 15 is composite.

Rabin–Miller Primality Test

First attempt: repeat Fermat test with many a 's:

Fermat(n)

repeat k times

Choose random a with $1 \leq a \leq n - 1$

if $a^{n-1} \pmod n \neq 1$ **then** return(Composite)

return(Probably Prime)

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Unfortunately not efficient on all numbers. E.g. not on **Carmichael Numbers**: a *composite* n , where for all a with $1 \leq a \leq n - 1$ and a relatively prime to n , we have $a^{n-1} \pmod n \equiv 1$. This means that Carmichael numbers can have very few Fermat witnesses a .
Example of a Carmichael number: $561 = 3 \cdot 11 \cdot 17$

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Now add to the picture this motivating **Theorem**:

If n is *prime*, then $x^2 \pmod{n} \equiv 1$ implies $x \pmod{n} \in \{1, n - 1\}$.

If n is *composite*, odd, and has two distinct factors, then $x^2 \pmod{n} \equiv 1$ implies at least four values possible for $x \pmod{n}$.

Examples: $x^2 \pmod{11} \equiv 1 \Rightarrow x \in \{1, 10\}$

$x^2 \pmod{15} \equiv 1 \Rightarrow x \in \{1, 4, 11, 14\}$

Rabin–Miller Primality Test

Final version: Start with Fermat test for some a . Then, as long we have an x with $x^2 \pmod{n} \equiv 1$, we look at $x \pmod{n}$.

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Example with $n = 561$ and $a = 50$:

$$50^{560} \pmod{561} \equiv 1 \quad [\text{i.e., } (50^{280})^2 \pmod{561} \equiv 1 \text{ (so } x = 50^{280}\text{)}]$$

$$50^{280} \pmod{561} \equiv 1 \quad [\text{i.e., } (50^{140})^2 \pmod{561} \equiv 1 \text{ (so } x = 50^{140}\text{)}]$$

$$50^{140} \pmod{561} \equiv 1 \quad \vdots$$

$$50^{70} \pmod{561} \equiv 1$$

$$50^{35} \pmod{561} \equiv 560 \quad [\text{The process stops}]$$

[We must stop both because 35 is odd and because 560 is $n-1$, not 1.]

If n is *prime*, we can only end in $\equiv 1$ or $\equiv n - 1$ (for all a). Above, this (unfortunately) also happened for the *composite* $n = 561$ for $a = 50$.

Rabin–Miller Primality Test

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[We must stop both because 35 is odd and because 560 is $n-1$, not 1.]

If n is *prime*, we can only end in $\equiv 1$ or $\equiv n - 1$ (for all a). Above, this (unfortunately) also happened for the *composite* $n = 561$ for $a = 50$. Let's now try again with $a = 2$:

$$2^{560} \pmod{561} \equiv 1$$

$$2^{280} \pmod{561} \equiv 1$$

$$2^{140} \pmod{561} \equiv 67 \text{ [Not 1 or } n - 1. \text{ Busted, it's a composite!]}$$

We say that 2 is a Rabin–Miller **witness** that 561 is composite.

Rabin–Miller Primality Test

Resulting algorithm:

Miller–Rabin(n, k)

Calculate odd m such that $n - 1 = 2^s \cdot m$ [s = number of possible divisions by two]

repeat k times

 Choose random a with $1 \leq a \leq n - 1$

if $a^{n-1} \pmod n \neq 1$ **then** return(Composite)

if $a^{(n-1)/2} \pmod n \equiv n - 1$ **then** continue [\Rightarrow next iteration]

if $a^{(n-1)/2} \pmod n \neq 1$ **then** return(Composite)

if $a^{(n-1)/4} \pmod n \equiv n - 1$ **then** continue [\Rightarrow next iteration]

if $a^{(n-1)/4} \pmod n \neq 1$ **then** return(Composite)

 :

if $a^m \pmod n \equiv n - 1$ **then** continue [\Rightarrow next iteration]

if $a^m \pmod n \neq 1$ **then** return(Composite)

end repeat

return(Probably Prime) [None of the k iterations returned “composite”]

Rabin–Miller Primality Test

Theorem: If n is composite and odd, at most $1/4$ of the a 's with $1 \leq a \leq n - 1$ will not end in “return(Composite)” during an iteration of the **repeat**-loop.

This means that with k iterations, a **composite** odd n will survive to return(Probably Prime) (making the algorithm return a wrong answer) with probability at most $(1/4)^k$. Otherwise, the algorithm returns the correct answer “Composite”. Even numbers are always composite, so we don't need to test them.

For e.g. $k = 100$, the probability of a wrong answer is therefore less than $(1/4)^{100} = 1/4^{100} = 1/(10^{\log_{10} 4})^{100} < 1/(10^{0.602})^{100} = 1/(10^{0.602 \cdot 100}) < 1/10^{60}$.

A **prime** n will always survive to “return(Probably Prime)”, which is the correct answer.

Conclusions about primality testing

1. Miller–Rabin is a practical, randomized primality test
2. In 2002, a deterministic primality test was given [Agrawal, Kayal, Saxena]. It is less practical, though.
3. Randomized algorithms may be preferred over deterministic ones, even if they (with very low probability) can make errors.
4. Number theory has practical uses.

Why does RSA work?

Ingredient 1:

The Chinese Remainder Theorem (CRT) Let n_1, n_2, \dots, n_k be pairwise relatively prime. For any integers x_1, x_2, \dots, x_k , there exists $x \in \mathbb{Z}$ s.t. $x \equiv x_i \pmod{n_i}$ for $1 \leq i \leq k$. Also, x is *uniquely* determined modulo the product $N = n_1 n_2 \dots n_k$: If $x' \in \mathbb{Z}$ s.t. $x' \equiv x_i \pmod{n_i}$ for $1 \leq i \leq k$, then $x' \equiv x \pmod{N}$.

For RSA, we consider the special case where $n_1 = p$ and $n_2 = q$ are two primes (hence $N = pq$) and where $x = x_1 = x_2 = m$ (recall, m is the message in RSA)..

Clearly, $m \equiv x_1 \pmod{p}$ and $m \equiv x_2 \pmod{q}$, since $x_1 = x_2 = m$.

So if some x' fulfills $x' \equiv m \pmod{p}$ and $x' \equiv m \pmod{q}$, then $x' \equiv m \pmod{N}$, by the uniqueness part of the theorem (with m in x 's place).

Why does RSA work?

Ingredient 2:

Fermat's Little Theorem: If p is a prime and a is any integer for which $p \nmid a$, then

$$a^{p-1} \equiv 1 \pmod{p}$$

Recap:

$N = p \cdot q$, where p, q prime.

$\gcd(e, (p-1)(q-1)) = 1$.

$e \cdot d \equiv 1 \pmod{(p-1)(q-1)}$.

▶ $PK = (N, e)$

▶ $SK = (N, d)$

To encrypt: $c = E(m, PK) = m^e \pmod{N}$.

To decrypt: $r = D(c, SK) = c^d \pmod{N}$.

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To decrypt: $r = D(c, SK) = c^d \pmod{N}$.

We now show correctness of RSA, i.e., that $r = m$.

Correctness of RSA

Let $r = D(E(m, PK), SK)$, that is,
 $r = (m^e \pmod{N})^d \pmod{N} = m^{ed} \pmod{N}$.

Recall that $\exists k$ s.t. $ed = 1 + k(p-1)(q-1)$, by the choice of d .

If $p \nmid m$, then by Fermat's little theorem:

$$\begin{aligned} m^{ed} &= m^{1+k(p-1)(q-1)} = m \cdot (m^{(p-1)})^{k(q-1)} \\ &\equiv m \cdot 1^{k(q-1)} = m \pmod{p} \end{aligned}$$

Similarly, if $q \nmid m$:

$$\begin{aligned} m^{ed} &= m^{1+k(p-1)(q-1)} = m \cdot (m^{(q-1)})^{k(p-1)} \\ &\equiv m \cdot 1^{k(p-1)} = m \pmod{q} \end{aligned}$$

From CRT with $x' = m^{ed}$ we get $m^{ed} \equiv m \pmod{N}$.

Hence, $r = m^{ed} \pmod{N} = m \pmod{N} = m$, where the last equality holds if $0 \leq m < N$ (which we require in RSA).

Correctness of RSA

For the remaining cases: assume $p|m$

Then $m = pk$ for some k , so for any t we have $m^t = (pk)^t = pk'$ for some k' .

Hence, $m^{ed} \equiv 0 \equiv m \pmod{p}$.

If $q|m$, we can similarly show $m^{ed} \equiv 0 \equiv m \pmod{q}$.

Thus, in *all* cases, we have

$$m^{ed} \equiv m \pmod{p}$$

$$m^{ed} \equiv m \pmod{q}$$

and the argument at the bottom of the previous slide holds.