# All-Pairs-Shortest-Path via Modified Matrix Multiplication

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This is a short note proving the following version of a theorem from the slides:

**Theorem.** *If $G$ is a weighted graph with no negative cycles, $W$ is its edge weight matrix, $k$ is a positive integer, and matrix multiplication denotes min-plus matrix multiplication, then the $ij$-th entry of $W^k$ is the length of the shortest path from node $i$ to node $j$ that uses at most $k$ edges.*

*Proof.* Recall that the edge weight matrix $W$ is defined by its $ij$-th entry $w_{ij}$ being

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{weight of edge } (i,j) & \text{if } i \neq j \text{ and edge } (i,j) \text{ exists} \\ \infty & \text{else} \end{cases}$$

Let $d_{ij}^k$ be the length of the shortest path from node $i$ to node $j$ that uses at most $k$ edges. We first remark that since there are no negative cycles, no path from a node $i$ to itself can be shorter than the path with zero edges (which is defined to have length zero). Thus, $d_{ii}^k = 0$ for all $i$ and $k$.

We now prove the theorem by induction on $k$. The basis is $k = 1$, where $W^k = W^1 = W$ and our task is to prove $w_{ij} = d_{ij}^1$. For $i = j$, we have $d_{ij}^1 = 0$ by the above remark, hence $w_{ij} = d_{ij}^1$ by the first line in the definition of $W$. For $i \neq j$, any path from node $i$ to node $j$ must contain at least one edge. Since $k = 1$, we are only considering paths with at most one edge. Thus, $w_{ij} = d_{ij}^1$, by the second and third lines in the definition of $W$.

For the inductive step, assume the statement holds for some $k$. We are to prove it for $k + 1$. Let $b_{ij}$ denote the $ij$-th entry of $W^k$ and let $c_{ij} = \min_l \{w_{il} + b_{lj}\}$ be the $ij$-th entry of $W^{k+1} = W \cdot W^k$, as defined by min-plus matrix multiplication. Thus, the inductive hypothesis is $b_{ij} = d_{ij}^k$ and our task is to prove $c_{ij} = d_{ij}^{k+1}$. We will do this by proving first $d_{ij}^{k+1} \leq c_{ij}$ and then $d_{ij}^{k+1} \geq c_{ij}$.

Consider any node $l$. If $w_{il} < \infty$ and $b_{lj} < \infty$, the value $w_{il}+b_{lj} = d_{il}^1+d_{il}^k$ is the length of a path constructed by first following a shortest path from node $i$ to node $l$ using at most one edge and then following a shortest path from node $l$ to node $j$ using at most $k$ edges. This path has at most $k+1$ edges, hence it cannot be shorter than the shortest path from node $i$ to node $j$ using at most $k+1$ edges. In other words, $d_{ij}^{k+1} \leq w_{il} + b_{lj}$. If $w_{il} = \infty$ or $b_{lj} = \infty$, then $w_{il} + b_{lj} = \infty$, hence $d_{ij}^{k+1} \leq w_{il} + b_{lj}$ also in this case. We can conclude $d_{ij}^{k+1} \leq \min_l\{w_{il} + b_{lj}\} = c_{ij}$, as we have proven $d_{ij}^{k+1}$ smaller than or equal to all the numbers that we minimize over.

Conversely, consider a shortest path $P$ from node $i$ to node $j$ using at most $k+1$ edges. If no such path exist at all, $d_{ij}^{k+1} = \infty$, hence $d_{ij}^{k+1} \geq c_{ij}$. Otherwise, $d_{ij}^{k+1}$ is the length of $P$. In case $i \neq j$, the path uses at least one edge. We can then consider $P$ composed of a path of one edge from node $i$ to the first node $l$ plus the rest, which is a path of at most $k$ edges from node $l$ to node $j$. Hence, the length of $P$ cannot be smaller than $d_{il}^1 + d_{lj}^k$, so we have $d_{ij}^{k+1} \geq d_{il}^1 + d_{lj}^k = w_{il} + b_{lj}$. In case $i = j$, we know (see remark in the beginning) that $d_{ij}^{k+1} = 0 \geq 0 + 0 = d_{ii}^1 + d_{ij}^k = w_{ii} + b_{ij}$. In both cases, we can conclude $d_{ij}^{k+1} \geq \min_l\{w_{il} + b_{lj}\} = c_{ij}$, as we have proven $d_{ij}^{k+1}$ larger than or equal to one of the numbers that we minimize over. $\square$