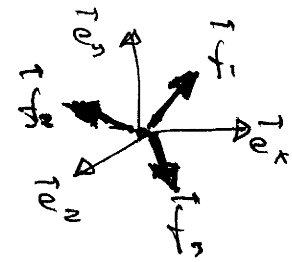
Change of Basis

Two coordinate systems (same origo), given by two ortonormal bases

1) $\vec{e}_x, \vec{e}_y, \vec{e}_z$

2) $\vec{f}_1, \vec{f}_2, \vec{f}_3$



[Orthonormal : vectors have unit length and are pairwise orthogonal, i.e.

$$\vec{e}_a \cdot \vec{e}_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$
]

We know $\vec{f}_i$'s expressed in $\vec{e}$'s system :

$$\vec{f}_1 = f_{1x} \cdot \vec{e}_x + f_{1y} \cdot \vec{e}_y + f_{1z} \cdot \vec{e}_z$$
$$\vec{f}_2 = f_{2x} \cdot \vec{e}_x + f_{2y} \cdot \vec{e}_y + f_{2z} \cdot \vec{e}_z$$
$$\vec{f}_3 = f_{3x} \cdot \vec{e}_x + f_{3y} \cdot \vec{e}_y + f_{3z} \cdot \vec{e}_z$$

{The coordinates of the $\vec{f}$'s in system 1) }

Given a vector $\vec{x}$ (or point), we can consider its coordinates in each system.

Let $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be the coordinates in system 1)

and let $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ ———— " ———————— 2).

We then have :

$$\vec{x} = w_1 \cdot \vec{f}_1 + w_2 \cdot \vec{f}_2 + w_3 \cdot \vec{f}_3$$

$$= w_1 \cdot \Big(\underbrace{f_{1x} \cdot \vec{e}_x + f_{1y} \cdot \vec{e}_y + f_{1z} \cdot \vec{e}_z}_{\vec{f}} \Big)$$

$$+ w_2 \Big(\underbrace{f_{2x} \cdot \vec{e}_x + f_{2y} \cdot \vec{e}_y + f_{2z} \cdot \vec{e}_z}_{\vec{f}} \Big)$$

$$+ w_3 \Big(\underbrace{f_{3x} \cdot \vec{e}_x + f_{3y} \cdot \vec{e}_y + f_{3z} \cdot \vec{e}_z}_{\vec{f}} \Big)$$

$$= \vec{e}_x \cdot (w_1 \cdot f_{1x} + w_2 \cdot f_{2x} + w_3 \cdot f_{3x})$$
$$+ \vec{e}_y \cdot (w_1 \cdot f_{1y} + w_2 \cdot f_{2y} + w_3 \cdot f_{3y})$$
$$+ \vec{e}_z \cdot (w_1 \cdot f_{1z} + w_2 \cdot f_{2z} + w_3 \cdot f_{3z})$$

So for $F = \begin{bmatrix} f_{1x} & f_{2x} & f_{3x} \\ f_{1y} & f_{2y} & f_{3y} \\ f_{1z} & f_{2z} & f_{3z} \end{bmatrix}$

$\vec{f}_1 \quad \vec{f}_2 \quad \vec{f}_3 \quad \Leftarrow$ Expressed in system 1)

we have

$$\boxed{F \cdot \vec{w} = \vec{v}} \quad *)$$

$\vec{x}$ expressed in system 2) $\qquad$ $\vec{x}$ expressed in system 1)

Note that since the $\vec{f}$'s are orthogonal, we have

$$F^T \cdot F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Hence, $F^T \cdot (F \cdot \vec{w}) = F^T \cdot \vec{v}$

$$\begin{aligned} F^T \cdot (F \cdot \vec{w}) &= (F^T \cdot F) \cdot \vec{w} \\ &= I \cdot \vec{w} \\ &= \vec{w} \end{aligned}$$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{x}$ expressed in system 1)

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \vec{x}$ expressed in system 2)

that is,

$$\boxed{\vec{w} = F^T \cdot \vec{v}} \qquad **)$$

The two equations *) and **) describe how to change between a point (or vector) $\vec{x}$'s coordinates in the two systems.

So, to eg. rotate around any axis $\vec{r}$ (not just the z-axis), create coord. system with its z-axis parallel to $\vec{r}$, move to this system, rotate, and then move back:

$$F \cdot (Rot_z \cdot (F^T \cdot \vec{v})) = \underbrace{(F \cdot Rot_z \cdot F^T)}_{\text{Matrix to rotate around } \vec{r}} \cdot \vec{v}$$