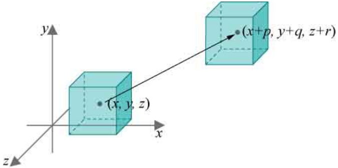
Transformations

Moving Objects

We need to move our objects in 3D space.



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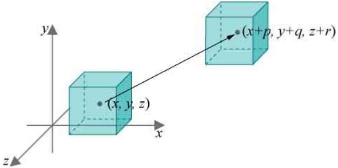

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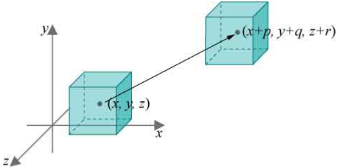

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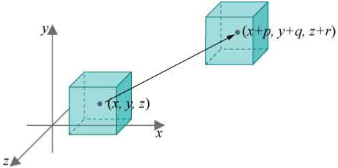

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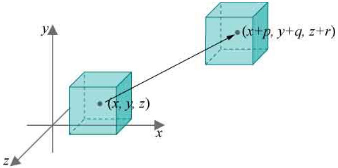

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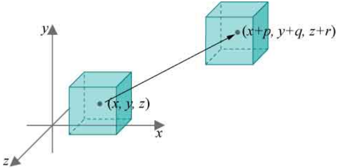

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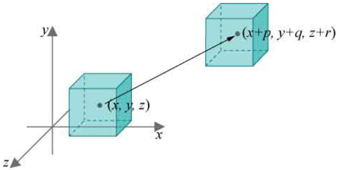
Move model $\Leftrightarrow$ move triangles $\Leftrightarrow$ move points (vertices) $\Leftrightarrow f : \mathbb{R}^3 \to \mathbb{R}^3$

Translation



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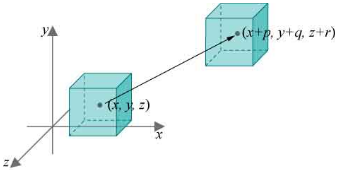
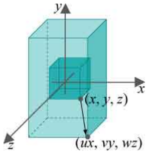


$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+p \\ y+q \\ z+r \end{pmatrix}$$

Scaling



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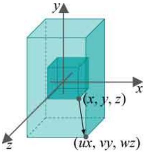
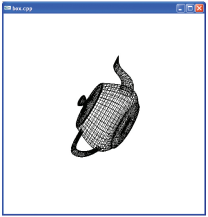


$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \cdot x \\ v \cdot y \\ w \cdot z \end{pmatrix}$$

Rotation



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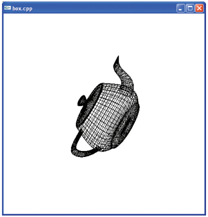
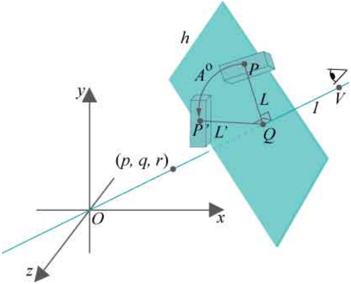


Rotation around line through origin:



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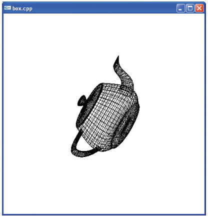

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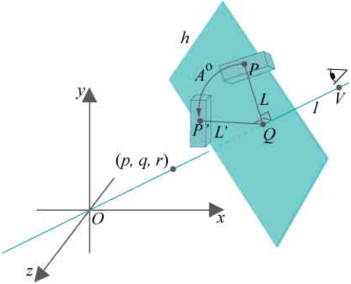
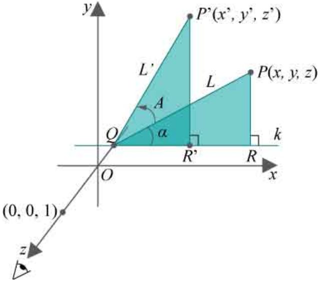


$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$$

Rotation

Simpler case: Rotation around z-axis.



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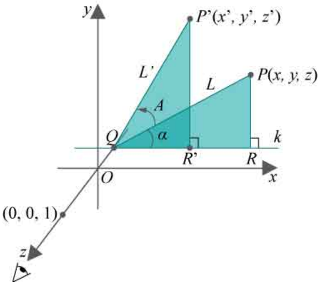

From formula for rotation in 2D (known from high school):

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ z \end{pmatrix}$$

Rotation

Similar: Rotation around x-axis and y-axis.

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\cos\phi - z\sin\phi\\y\sin\phi + z\cos\phi\end{pmatrix}$$

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\sin\phi + x\cos\phi\\y\\z\cos\phi - x\sin\phi\end{pmatrix}$$

Euler

Theorem (Euler, 1775): any rotation with axis through origo can be created as three succesive rotations around the three coordinate axes.

The angles of the three coordinate axis rotations are called Euler angles.

Using Euler angles to specify generic rotations is often intuitive, but also has drawbacks. We will return to that later.

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Question: can all our needed transformations be expressed as matrices?

Transformations as Matrices

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- Scaling

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No. Translation is not linear: $f(\vec{x_1}+\vec{x_2})\neq f(\vec{x_1})+f(\vec{x_2})$.

Homogeneous Coordinates

Go to 4D:

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And back:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \rightarrow \begin{pmatrix} x/w \\ y/w \\ z/w \end{pmatrix}$$

Homogeneous Coordinates

Translations (in 3D) can now be expressed as matrix multiplication:

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All 3x3 matrices are still available (incl. skaling and rotation):

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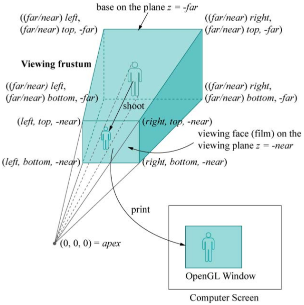
Prespective projection:



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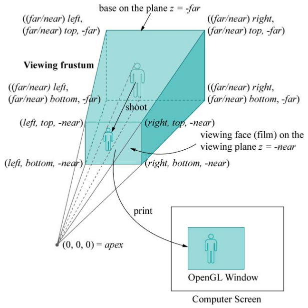

Expressed as 4x4 matrix multiplication ($d = -$near):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix} \to \begin{pmatrix} xd/z \\ yd/z \\ d \end{pmatrix}$$

Transformations in OpenGL

OpenGL uses 4x4-matrices/homogeneous coordinates internally. Matrices are normally created by more intuitive commands:

- `glTranslatef(dx,dy,dz)`
- `glScalef(sx,sy,sz)`
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Each command generates the corresponding matrix, and right-multiplies it on the current matrix.

So last transformaton specified in code is first applied to vertices.

Cf. the math notation $f(g(h(x)))$ (where h is applied first to x, then g, then f).

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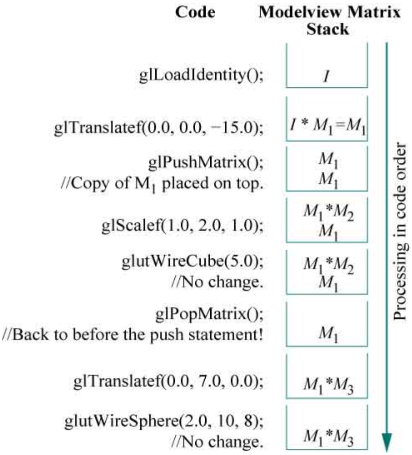
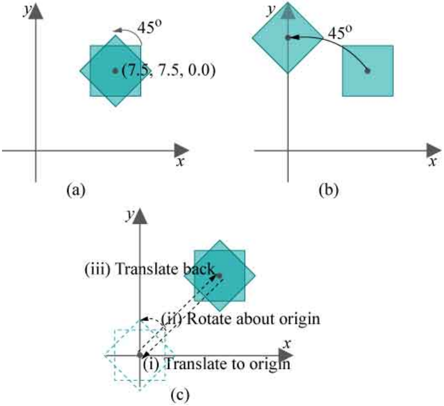
There is a current matrix for model-view transformations, for projections, and for textures. Each has a stack.

Matrix Stack



"The Trick"



y
45°
(7.5, 7.5, 0.0)
x
(a)
y
45°
x
(b)
y
(iii) Translate back
(ii) Rotate about origin
x
(i) Translate to origin
(c)

Example Program