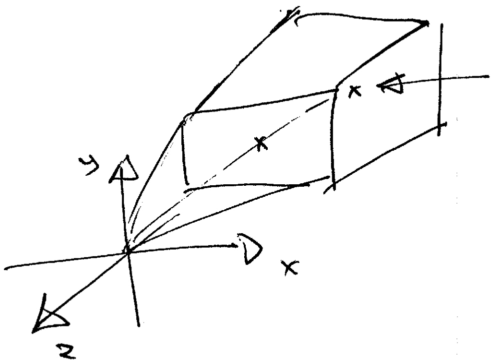
Perspective projection



View plane : $z = +n$

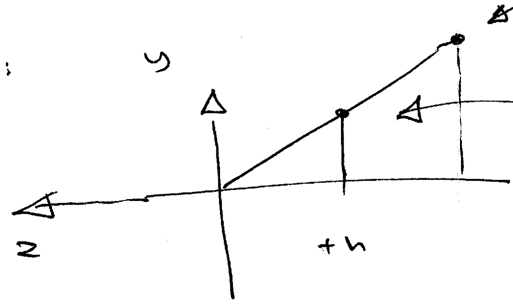
n	= near
f	= far
t	= top
b	= bottom
l	= left
r	= right

Proj.: $\vec{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \cdot \frac{n}{z} \\ y \cdot \frac{n}{z} \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}$

Def.:

for $\boxed{w = z/n}$

For y:



(z, y)

$\left(z \cdot \frac{n}{z},\ y \cdot \frac{n}{z}\right) = \left(n,\ y \cdot \frac{n}{z}\right)$

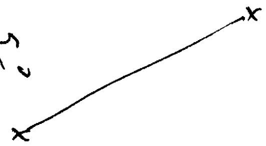
Similar for x

Proposition $\vec{f}$ maps line segments to line segments

Proof: Given line segment $\vec{r}(s) = \vec{r}_0 + s(\vec{r}_1 - \vec{r}_0)$

$s \in [0; 1]$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \vec{r}_0 \qquad \vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$



Wlog $z_1 \geq z_0$

$$\vec{f}(\vec{r}(s)) = \begin{pmatrix} \frac{r(s)_x}{r(s)_z} \cdot n \\ \frac{r(s)_y}{r(s)_z} \cdot n \end{pmatrix} = \begin{pmatrix} \frac{x_0 + s(x_1 - x_0)}{z_0 + s(z_1 - z_0)} \cdot n \\ \frac{y_0 + s(y_1 - y_0)}{z_0 + s(z_1 - z_0)} \cdot n \end{pmatrix}$$

Def.:

$$\boxed{\begin{aligned} w_0 &= z_0 / n \\ w_1 &= z_1 / n \end{aligned}}$$

$$= \begin{pmatrix} \frac{x_0 + s(x_1 - x_0)}{w_0 + s(w_1 - w_0)} \\ \frac{y_0 + s(y_1 - y_0)}{w_0 + s(w_1 + w_0)} \end{pmatrix}$$

$$\underset{\text{claim}}{=} \begin{pmatrix} \frac{x_0}{w_0} + \tilde{f}(s)\left(\frac{x_1}{w_1} - \frac{x_0}{w_0}\right) \\ \frac{y_0}{w_0} + \tilde{f}(s)\left(\frac{y_1}{w_1} - \frac{y_0}{w_0}\right) \end{pmatrix}$$

$$= \vec{f}(\vec{r}_0) + \tilde{f}(s)\left(\vec{f}(\vec{r}_1) - \vec{f}(\vec{r}_0)\right) \qquad (*)$$

For $\boxed{\tilde{f}(s) = \frac{w_1 \cdot s}{w_0 + (w_1 - w_0) \cdot s}}$.

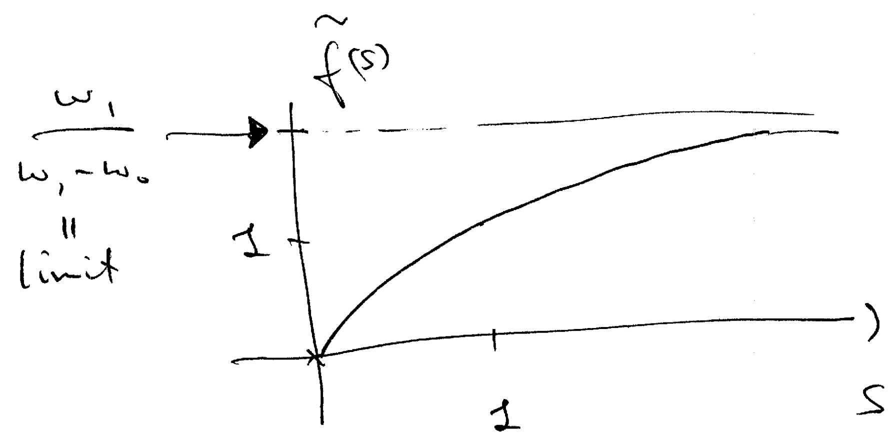
$\tilde{f}$ is increasing

$$\left(\tilde{f}'(s) = \frac{w_1 (w_0 + (w_1 - w_0)\cdot s) - w_1 \cdot s \cdot (w_1 - w_0)}{(w_0 + (w_1 - w_0) s)^2} = \frac{w_1 w_0}{(w_0 + (w_1 - w_0)\cdot s)^2} > 0 \right)$$

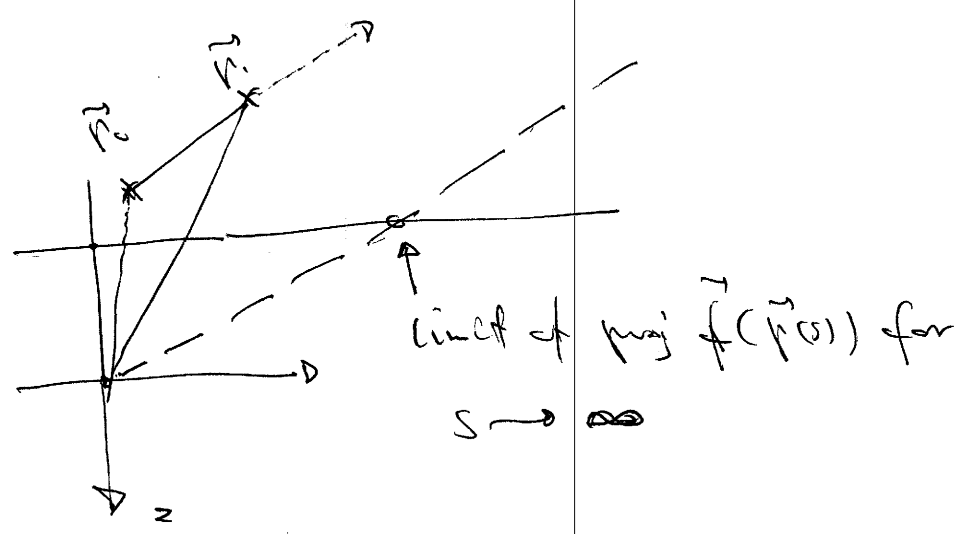
$[$ as $w_1, w_0 > 0]$

$f(0) = 0$

$f(1) = 1$



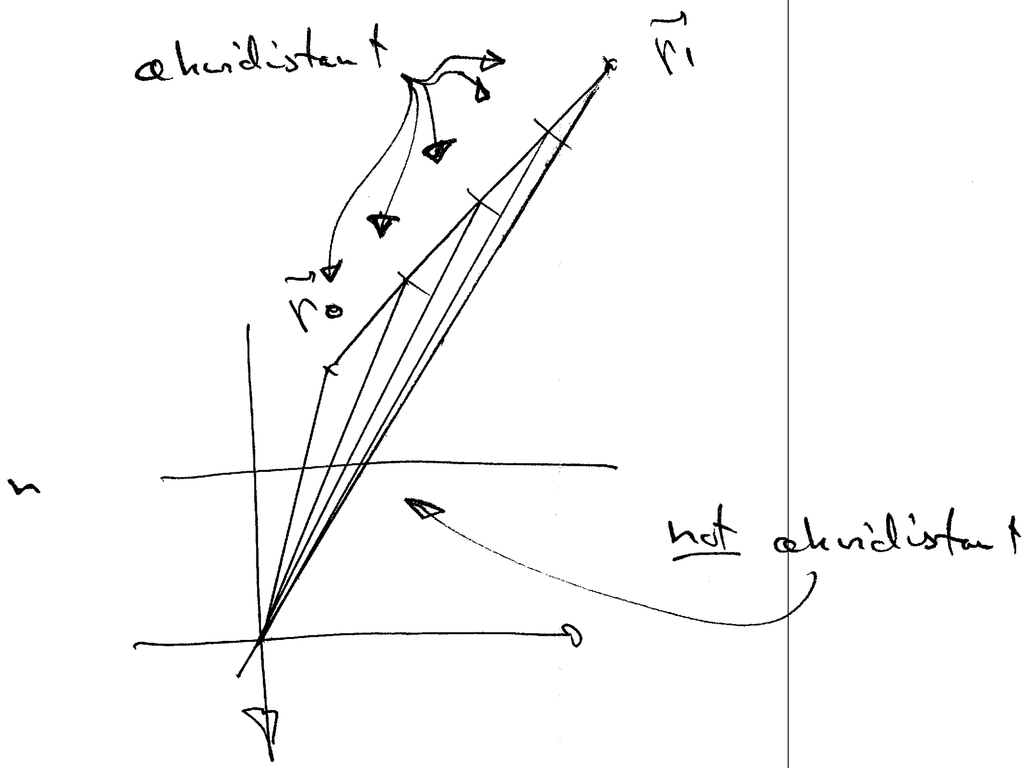
Cf :



So ⊛ is a line segment.

But <u>not traversed ~~at~~ same speed</u> as ~~the~~ original line segment.

Clear this should be the case:



$\tilde{f}(s)$ gives speed correspondances. [world → screen].

Inverse: $t = \tilde{f}(s) = \dfrac{w_1 \cdot s}{w_0 + (w_1 - w_0)\cdot s}$

$$\Updownarrow$$

$$t w_0 + t(w_1 - w_0)s = w_1 s$$

$$\Updownarrow$$

$$t w_0 = s(w_1 - t(w_1 - w_0))$$

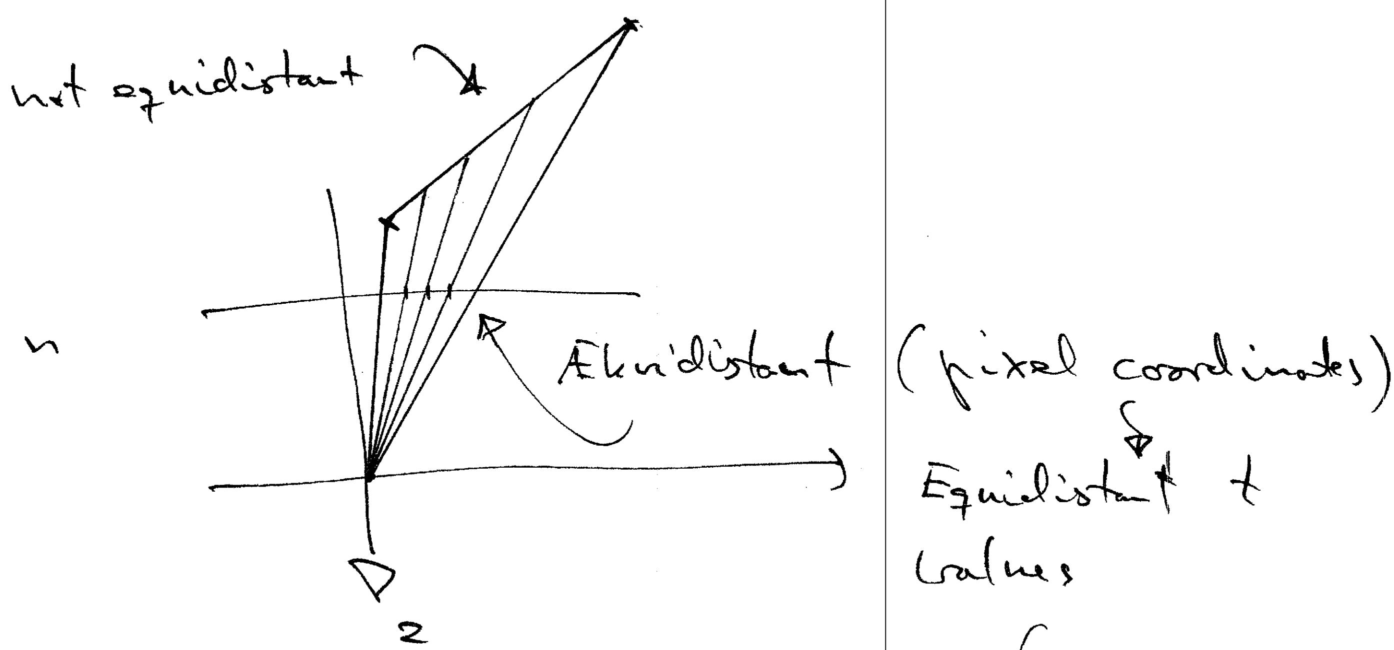
$$\Updownarrow$$

$$\frac{t w_0}{w_1 - t(w_1 - w_0)} = s$$

def. $\quad \| \quad \tilde{g}(t)$

$\tilde{g}(t)$ gives speed correspondance in the other direction [screen → world]

We need $\tilde{g}(t)$ in order to do perspectively correct interpolation [of colors, texture coordinates, ...]



Correct (not eq. dist.) $\tilde{g}(t)$ values used for interpolation.

Above: line segments (single parameter)

For Δ's (barycentric coordinates, two parameters) similar calc. can be done [see hand out].

Proof of claim

For x: Let $\square$ designate $w_0 + (w_1 - w_0) \cdot s$

Then: $$\frac{x_0 + s(x_1 - x_0)}{\square} = \frac{x_0(1-s) + s\,x_1}{\square}$$

$$= \frac{x_0}{w_0} \cdot \frac{w_0(1-s)}{\square} + \frac{s\,x_1}{\square}$$

$$= \frac{x_0}{w_0} \cdot \frac{w_0 + (w_1 - w_0)s - w_1 s}{\square} + \frac{s\,x_1}{\square}$$

$$= \frac{x_0}{w_0} \cdot \frac{\square - w_1 s}{\square} + \frac{s\,x_1}{\square}$$

$$= \frac{x_0}{w_0} + \underbrace{\frac{w_1 \cdot s}{\square}}_{\tilde{f}(s)} \left(\frac{x_1}{w_1} - \frac{x_0}{w_0} \right)$$

$\square$.