An estimator for the tail index of an integrated conditional Pareto-Weibull-type model

Yuri Goegebeur *

Armelle Guillou †

Michael Osmann ‡

Abstract. We introduce a nonparametric regression estimator for a tail heaviness parameter in an integrated conditional Pareto-Weibull-type model. The estimator is based on local log excesses over a high random threshold. Asymptotic properties are derived under proper regularity conditions.

Key words and phrases: Extremes, local estimation, regression, tail index.

1 Introduction

In the recent years, a lot of attention in extreme value theory has been devoted to situations where the variable of interest Y is observed together with a random covariate X. Goegebeur *et al.* (2014) introduced an estimator for the conditional extreme value index $\gamma(x)$ when $\gamma(x) > 0$, while de Wet *et al.* (2015) introduced an estimator for the conditional Weibull-tail coefficient. In both of these cases, a weighted average of the log-excesses over a threshold is used, where the threshold is considered to be non-random. The aim of the present paper is to construct an estimator that can be used for both conditional Weibull-tail distributions and Pareto-type

*Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark (email: yuri.goegebeur@imada.sdu.dk).

†Institut Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg cedex, France (email: armelle.guillou@math.unistra.fr).

‡Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark (email: mosma@imada.sdu.dk).

distributions. To this end, we use a two parameter family of distributions, which contain both the Pareto-type distributions and the Weibull-tail distributions. The estimator is based on a random threshold, as was also done in Stupfler (2013), who introduced an estimator for the conditional extreme value index $\gamma(x)$ with $\gamma(x) \in \mathbb{R}$.

Let $F(y;x) := \mathbb{P}(Y \leq y | X = x)$, the conditional response distribution function, and $\overline{F}(.;x) := 1 - F(.;x)$. Assume

$$\overline{F}(y;x) = \exp\left(-D^{\leftarrow}_{\tau(x)}\left(\ln H\left(y;x\right)\right)\right), \tag{1}$$

where

- $y > y^*(x)$ with $y^*(x) > 0$,
- $D_{\tau(x)}(y) = \int_1^y u^{\tau(x)-1} du$, with $\tau(x) \in [0,1]$,
- H is an increasing function that satisfies $H^{\leftarrow}(t;x) := \inf\{y : H(y;x) \geq t\} = t^{\theta(x)}\ell(t;x)$, where $\theta(x) > 0$, and ℓ is a slowly varying function at infinity, i.e. $\frac{\ell(\lambda y;x)}{\ell(y;x)} \to 1$ as $y \to \infty$ for all $\lambda > 0$.

As noted in Gardes *et al.* (2011), this model includes Weibull-tail distributions with Weibull-tail coefficient $\theta(x)$ if $\tau(x) = 0$, and Pareto-type tails with extreme value index $\theta(x)$ if $\tau(x) = 1$, while $\tau(x) \in (0,1)$ is an intermediate class of distributions. In the following, we let (X_i, Y_i), $i = 1, \ldots, n$, be independent copies of the random vector $(X, Y) \in \mathbb{R}^q \times \mathbb{R}_+$ with $q \geq 1$, where the conditional distribution of Y given $X = x$ satisfies (1). Furthermore, let $x \in \mathbb{R}^q$ be arbitrary and denote by $B(x,h)$, the ball with center x and radius h, i.e. $B(x,h) := \{z \in \mathbb{R}^q : d(x,z) \leq h\}$, with $d(x,z)$ being the distance between x and z. The number of observations in the ball is given by $N_{n,x,h} := \sum_{i=1}^n 1\!\!1_{\{X_i \in B(x,h)\}}$, where $1\!\!1_{\{.\}}$ is the indicator function, and denote by n_x the expected number of observations in $B(x,h)$, i.e. $n_x := n\mathbb{P}\left(X \in B(x,h)\right)$.

Conditional on $N_{n,x,h} = p$, $p \geq 1$, we introduce Z_j, $j = 1, \ldots, p$, as the response variables for which the covariate X_j is in the ball $B(x,h)$, and denote by $Z_{1,p} \leq \ldots \leq Z_{p,p}$ the associated order statistics. In this setting we define our estimator of $\theta(x)$ as

$$\widehat{\theta}(k_x;x) := \frac{1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} [\ln Z_{p-i+1,p} - \ln Z_{p-k_x,p}]$$

with

$$\mu_{\tau(x)}(t) := \int_0^\infty \left(D_{\tau(x)}(u+t) - D_{\tau(x)}(t)\right) \exp(-u) du,$$

and assuming that $k_x \in \{1, \dots, p-1\}$. This estimator is an adaptation of the estimator proposed by Gardes *et al.* (2011) to the regression context. It consists mainly in averaging the log-spacings between the upper order statistics of the response variables for which the covariates are in the ball centered at x.

In the following, we will let $U_h(t;x)$ and $U(t;x)$ be the tail quantile functions corresponding to the conditional distribution function $F_h(y;x) := \mathbb{P}(Y \le y | X \in B(x,h))$ and $F(y;x)$, respectively, i.e. $U_h(.;x) := (1/\overline{F}_h(.;x))^{\leftarrow}$ and $U(.;x) := (1/\overline{F}(.;x))^{\leftarrow}$, where the superscript $\leftarrow$ denotes the generalised inverse as introduced above. In order to control the difference between $U_h(t;x)$ and $U(t;x)$, we define $\omega(u,v,x,h) := \sup_{z\in[u,v]} |\log U_h(z;x) - \log U(z;x)|$, with $u \le v$. The asymptotic properties of $\widehat{\theta}(k_x;x)$ will be examined under the following second order condition.

Assumption $A(\rho(x))$ *There exist* $\rho(x) < 0$ *and* $b(y;x) \to 0$ *for* $y \to \infty$ *such that*

$$\ln \frac{\ell(\lambda y;x)}{\ell(y;x)} = b(y;x) D_{\rho(x)}(\lambda)(1+o(1)),$$

where $o(1)$ *is uniform on* $\lambda \in [1,\infty)$.

Note that this assumption immediately implies that the function $|b(y;x)|$ is regularly varying with index $\rho(x)$.

2 Asymptotic properties

In this section we examine the asymptotic properties of our estimator. We start by establishing the consistency of $\widehat{\theta}(k_x;x)$.

Theorem 1 *Assume that $\overline{F}(.;x)$ satisfies (1) and that $A(\rho(x))$ holds. If $n_x \to \infty$, $k_x \to \infty$ and $\frac{k_x}{n_x} \to 0$ in such a way that for some $\delta > 0$,*

$$\frac{1}{\mu_{\tau(x)}\left(\ln \frac{n_x}{k_x}\right)} \omega \left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right) \longrightarrow 0,$$

then

$$\widehat{\theta}(k_x;x) \xrightarrow{\mathbb{P}} \theta(x).$$

Proof: Let $I_x := \mathbb{N} \cap [(1-n_x^{-1/4})n_x, (1+n_x^{-1/4})n_x]$. According to Lemma 1 in Stupfler (2013), one has that $\mathbb{P}(N_{n,x,h} \in I_x) \to 1$ as $n_x \to \infty$. For any $t > 0$, define the event

$$S(t;x) := \left\{ \left| \widehat{\theta}\left(k_x;x\right) - \theta(x) \right| > t \right\}.$$

Note that after applying the law of total probability one obtains the inequality

$$\mathbb{P}(S(t;x)) \leq \sup_{p \in I_x} \mathbb{P}\left(S(t;x) | N_{n,x,h} = p\right) + \mathbb{P}(N_{n,x,h} \notin I_x).$$

We have thus to show that $\sup_{p \in I_x} \mathbb{P}\left(S(t;x) | N_{n,x,h} = p\right) \to 0$.

To this aim, let $T_i, i = 1, \ldots, p$, be unit Pareto random variables, with $T_{1,p} \leq \ldots \leq T_{p,p}$ the associated order statistics. Given $N_{n,x,h} = p \geq 1$, the distribution of the random vector $(Z_1, \ldots, Z_p)$, is the same as that of the random vector $(U_h\left(T_1;x\right), \ldots, U_h\left(T_p;x\right))$; see Lemma 2 in Stupfler (2013). Thus, denoting

$$\begin{aligned}
\breve{\theta}(k_x;x) &:= \frac{1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \left[\ln U_h\left(T_{p-i+1,p};x\right) - \ln U_h\left(T_{p-k_x,p};x\right)\right], \\
\widetilde{\theta}(k_x;x) &:= \frac{1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \left[\ln U\left(T_{p-i+1,p};x\right) - \ln U\left(T_{p-k_x,p};x\right)\right],
\end{aligned}$$

and

$$R_p(x) := \frac{1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \left[\ln U_h\left(T_{p-i+1,p};x\right) - \ln U_h\left(T_{p-k_x,p};x\right) - \left(\ln U\left(T_{p-i+1,p};x\right) - \ln U\left(T_{p-k_x,p};x\right)\right)\right],$$

we have

$$\mathbb{P}\left(S(t;x)|N_{n,x,h}=p\right) = \mathbb{P}\left(\left|\breve{\theta}(k_x;x) - \theta(x)\right| > t\right) \leq \mathbb{P}\left(\left|\widetilde{\theta}(k_x;x) - \theta(x)\right| > \tfrac{t}{2}\right) + \mathbb{P}\left(\left|R_p(x)\right| > \tfrac{t}{2}\right). \quad (2)$$

The two probabilities on the right-hand side of (2) are now studied separately. Concerning the first one, note that, with $T_i^*(p) := \frac{T_{p-i+1,p}}{T_{p-k_x,p}}$, $i = 1, \ldots, k_x$,

$$\begin{aligned}
\widetilde{\theta}(k_x; x) &= \theta(x) \frac{1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \left[D_{\tau(x)}\left(\ln T_{p-k_x,p} + \ln T_i^*(p)\right) - D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right] \\
&\quad + \frac{1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \ln \frac{\ell\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p} + \ln T_i^*(p)\right)\right); x\right)}{\ell\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right); x\right)} \\
&=: \widetilde{\theta}_1(k_x; x) + \widetilde{\theta}_2(k_x; x).
\end{aligned}$$

For the sequel, it is important to keep in mind that $(T^*_{k_x-i+1}(p), i = 1, \ldots, k_x) \stackrel{D}{=} (T_{1,k_x}, \ldots, T_{k_x,k_x})$, independently of $T_{p-k_x,p}$. Application of a Taylor series expansion to $\widetilde{\theta}_1(k_x; x)$ gives

$$\begin{aligned}
\widetilde{\theta}_1(k_x; x) &= \theta(x) \frac{(\ln T_{p-k_x,p})^{\tau(x)-1}}{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1}} \frac{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i^*(p) \\
&\quad + \frac{\theta(x)}{2} \frac{\tau(x) - 1}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} \frac{1}{k_x} \sum_{i=1}^{k_x} \left(\ln T_{p-k_x,p} + \ln \widetilde{T}_i(p)\right)^{\tau(x)-2} \left(\ln T_i^*(p)\right)^2 \\
&=: \widetilde{\theta}_{11}(k_x; x) + \widetilde{\theta}_{12}(k_x; x)
\end{aligned}$$

where $\ln \widetilde{T}_i(p)$ is a random value between 0 and $\ln T_i^*(p)$. The cases $\tau(x) = 1$ and $\tau(x) \neq 1$ can now be studied separately. If $\tau(x) = 1$, we have that $\widetilde{\theta}_{11}(k_x; x) = \theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i^*(p)$ and $\widetilde{\theta}_{12}(k_x; x) = 0$, and thus for any $t > 0$

$$\begin{aligned}
\sup_{p \in I_x} \mathbb{P}\left(\left|\widetilde{\theta}_1(k_x; x) - \theta(x)\right| > t\right) &= \sup_{p \in I_x} \mathbb{P}\left(\left|\theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i^*(p) - \theta(x)\right| > t\right) \\
&= \sup_{p \in I_x} \mathbb{P}\left(\left|\theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_{k_x-i+1,k_x} - \theta(x)\right| > t\right) \\
&= \mathbb{P}\left(\left|\theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i - \theta(x)\right| > t\right) \\
&\longrightarrow 0,
\end{aligned}$$

by the law of large numbers. Otherwise, if $\tau(x) < 1$, by combining Lemma 6 in Stupfler (2013) with our Lemmas 1 and 3, we deduce that

$$\sup_{p \in I_x} \mathbb{P}\left(\left|\widetilde{\theta}_{11}(k_x; x) - \theta(x)\right| > t\right) \longrightarrow 0,$$

while concerning $\widetilde{\theta}_{12}(k_x;x)$,

$$\left|\widetilde{\theta}_{12}(k_x;x)\right| \leq \frac{\theta(x)}{2}\left(\ln T_{p-k_x,p}\right)^{-1}\frac{\left(\ln T_{p-k_x,p}\right)^{\tau(x)-1}}{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}\frac{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\frac{1}{k_x}\sum_{i=1}^{k_x}\left(\ln T_i^*(p)\right)^2.$$

Using again the law of large numbers combining with the convergence $\sup_{p\in I_x}\mathbb{P}\left(\left(\ln T_{p-k_x,p}\right)^{-1}>t\right)\to 0$ and our Lemma 3, we deduce that

$$\sup_{p\in I_x}\mathbb{P}\left(\left|\widetilde{\theta}_{12}(k_x;x)\right|>t\right)\longrightarrow 0.$$

This leads also for $\tau(x)<1$ to

$$\sup_{p\in I_x}\mathbb{P}\left(\left|\widetilde{\theta}_1(k_x;x)-\theta(x)\right|>t\right)\longrightarrow 0. \tag{3}$$

Concerning now $\widetilde{\theta}_2(k_x;x)$, we have to use assumption $A(\rho(x))$ which ensures that

$$\begin{aligned}\widetilde{\theta}_2(k_x;x) =& \frac{1}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\\ &\cdot\frac{1}{k_x}\sum_{i=1}^{k_x}\ln\frac{\ell\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}+\ln T_i^*(p)\right)-D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right)\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)}{\ell\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)}\\ =& \frac{b\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\\ &\cdot\frac{1}{k_x}\sum_{i=1}^{k_x}D_{\rho(x)}\left(\exp\left(D_{\tau(x)}\left(\ln\left(T_{p-k_x,p}T_i^*(p)\right)\right)-D_{\tau(x)}\left(\ln\left(T_{p-k_x,p}\right)\right)\right)\right)(1+\delta_n)\end{aligned}$$

where $\delta_n\xrightarrow{\mathbb{P}}0$ uniformly in i and p. An application of the mean value theorem, shows that

$$\begin{aligned}&D_{\rho(x)}\left(\exp\left(D_{\tau(x)}\left(\ln\left(T_{p-k_x,p}T_i^*(p)\right)\right)-D_{\tau(x)}\left(\ln\left(T_{p-k_x,p}\right)\right)\right)\right)\\ &=\left[\exp\left(D_{\tau(x)}(\ln\widetilde{T}_i(p)+\ln T_{p-k_x,p})-D_{\tau(x)}(\ln T_{p-k_x,p})\right)\right]^{\rho(x)}\left(\ln\widetilde{T}_i(p)+\ln T_{p-k_x,p}\right)^{\tau(x)-1}\ln T_i^*(p),\end{aligned}$$

where $\ln\widetilde{T}_i(p)$ is a random value between 0 and $\ln T_i^*(p)$. Since

$$\left[\exp\left(D_{\tau(x)}(\ln\widetilde{T}_i(p)+\ln T_{p-k_x,p})-D_{\tau(x)}(\ln T_{p-k_x,p})\right)\right]^{\rho(x)}\leq 1,$$

it follows that

$$\left|\widetilde{\theta}_2(k_x;x)\right|\leq\left|(1+\delta_n)\frac{\left(\ln T_{p-k_x,p}\right)^{\tau(x)-1}}{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}\frac{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}b\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)\frac{1}{k_x}\sum_{i=1}^{k_x}\ln T_i^*(p)\right|.$$

Clearly,

$$\sup_{p\in I_x} \mathbb{P}\left(|(1+\delta_n)-1|>t\right) \longrightarrow 0$$

and

$$\sup_{p\in I_x} \mathbb{P}\left(\left|b\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)\right|>t\right) \longrightarrow 0,$$

(observe that $b(\exp(D_{\tau(x)}(\ln y));x)$ is regularly varying at infinity, and apply Lemma 6 of Stupfler, 2013), from which we deduce that

$$\sup_{p\in I_x} \mathbb{P}\left(\left|\widetilde{\theta}_2(k_x;x)\right|>t\right) \longrightarrow 0$$

according to our Lemma 3. Finally, coming back to $R_p(x)$, we have

$$|R_p(x)| \le \frac{2\omega(T_{p-k_x,p},T_{p,p},x,h)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)} \frac{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}. \tag{4}$$

Since $\omega(u,v,x,h)$ is a decreasing function in u and an increasing function in v, it is clear that for all $t>0$,

$$\left\{\left|\frac{2\omega\left(\frac{n_x}{(1+\delta)k_x},n_x^{1+\delta},x,h\right)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}\right|\le t\right\} \cap \left\{T_{p-k_x,p}\ge\frac{n_x}{(1+\delta)k_x}\right\} \cap \left\{T_{p,p}\le n_x^{1+\delta}\right\} \subseteq \left\{\left|\frac{2\omega(T_{p-k_x,p},T_{p,p},x,h)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}\right|\le t\right\}.$$

By considering the complementary event, we have

$$\left\{\left|\frac{2\omega(T_{p-k_x,p},T_{p,p},x,h)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}\right|>t\right\} \subseteq \left\{\left|\frac{2\omega\left(\frac{n_x}{(1+\delta)k_x},n_x^{1+\delta},x,h\right)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}\right|>t\right\} \cup \left\{T_{p-k_x,p}<\frac{n_x}{(1+\delta)k_x}\right\} \cup \left\{T_{p,p}>n_x^{1+\delta}\right\}.$$

Taking n_x sufficiently large, under the assumption of Theorem 1, we have

$$\sup_{p\in I_x}\mathbb{P}\left(\left|\frac{2\omega(T_{p-k_x,p},T_{p,p},x,h)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}\right|>t\right) \le \sup_{p\in I_x}\mathbb{P}\left(T_{p-k_x,p}<\frac{n_x}{(1+\delta)k_x}\right)+\sup_{p\in I_x}\mathbb{P}\left(T_{p,p}>n_x^{1+\delta}\right)$$
$$\longrightarrow 0,$$

by Lemma 6 in Stupfler (2013) and using the properties of the largest order statistic $T_{p,p}$. This ensures then under our Lemma 2 that

$$\sup_{p\in I_x}\mathbb{P}\left(|R_p(x)|>t\right) \longrightarrow 0.$$

Combining the above results, Theorem 1 follows. ■

Now we establish the asymptotic normality of $\widehat{\theta}(k_x;x)$, when properly normalised.

Theorem 2 *Assume that $\overline{F}(.;x)$ satisfies (1) and that $A(\rho(x))$ holds. If $n_x \to \infty$, $k_x \to \infty$ and $\frac{k_x}{n_x} \to 0$ in such a way that for some $\delta > 0$,*

$$\frac{\sqrt{k_x}}{\mu_{\tau(x)}\left(\ln \frac{n_x}{k_x}\right)} \omega\left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h\right) \longrightarrow 0,$$

and if additionally

$$\sqrt{k_x}\, b\left(\exp\left(D_{\tau(x)}\left(\ln \frac{n_x}{k_x}\right)\right); x\right) \longrightarrow \lambda \in \mathbb{R}$$

and for $\tau(x) < 1$

$$\frac{\sqrt{k_x}}{\ln \frac{n_x}{k_x}} \longrightarrow 0$$

then

$$\sqrt{k_x}\left(\widehat{\theta}(k_x;x) - \theta(x)\right) \xrightarrow{D} \mathcal{N}\left(\frac{\lambda}{1-\rho(x)} \mathbb{1}_{\{\tau(x)=1\}} + \lambda \mathbb{1}_{\{\tau(x)<1\}}, \theta^2(x)\right).$$

Proof: Given $N_{n,x,h} = p \geq 1$, the distribution of $\sqrt{k_x}(\widehat{\theta}(k_x;x) - \theta(x))$ is the same as that of $\sqrt{k_x}(\breve{\theta}(k_x;x) - \theta(x))$. Thus according to Lemma 5 in Stupfler (2013), it is sufficient to prove that the latter has the same distribution as a triangular array of the form

$$D_n + \phi_{np}$$

where $D_n \xrightarrow{D} \mathcal{N}\left(\frac{\lambda}{1-\rho(x)} \mathbb{1}_{\{\tau(x)=1\}} + \lambda \mathbb{1}_{\{\tau(x)<1\}}, \theta^2(x)\right)$ and $\sup_{p \in I_x} \mathbb{P}\left(|\phi_{np}| > t\right) \to 0$ for all $t > 0$, as $n_x \to \infty$. We can use the same decomposition of $\breve{\theta}(k_x;x)$ as in the proof of Theorem 1, that is in terms of $\widetilde{\theta}_{11}(k_x;x)$, $\widetilde{\theta}_{12}(k_x;x)$, $\widetilde{\theta}_2(k_x;x)$ and $R_p(x)$. Expanding further on the term $\widetilde{\theta}_{11}(k_x;x)$ gives

$$\begin{aligned}\widetilde{\theta}_{11}(k_x;x) &\overset{D}{=} \theta(x)\frac{1}{k_x}\sum_{i=1}^{k_x} \ln T_i + \theta(x)\left[\frac{(\ln T_{p-k_x,p})^{\tau(x)-1}}{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1}} \frac{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} - 1\right] \frac{1}{k_x}\sum_{i=1}^{k_x} \ln T_i \\ &=: \widetilde{\theta}_{111}(k_x;x) + \widetilde{\theta}_{112}(k_x;x).\end{aligned}$$

The first term $\widetilde{\theta}_{111}(k_x;x)$ can be dealt with directly with the central limit theorem

$$\sqrt{k_x}\left(\widetilde{\theta}_{111}(k_x;x) - \theta(x)\right) \xrightarrow{D} \mathcal{N}\left(0, \theta^2(x)\right).$$

Note that $\widetilde{\theta}_{112}(k_x;x) = 0$ if $\tau(x) = 1$, so we only need to consider the case $\tau(x) < 1$. For $\widetilde{\theta}_{112}(k_x;x)$, we have thus to show that for all $t > 0$

$$\sup_{p\in I_x} \mathbb{P}\left(\sqrt{k_x}\left|\left(\frac{\ln T_{p-k_x,p}}{\ln p/k_x}\right)^{\tau(x)-1} - 1\right| > t\right) \longrightarrow 0.$$

From the mean value theorem we get

$$\sup_{p\in I_x} \mathbb{P}\left(\sqrt{k_x}\left|\left(\frac{\ln T_{p-k_x,p}}{\ln p/k_x}\right)^{\tau(x)-1} - 1\right| > t\right)$$

$$\leq \sup_{p\in I_x} \mathbb{P}\left(\left(1 - \left|\frac{\ln(\frac{k_x}{p}T_{p-k_x,p})}{\ln(p/k_x)}\right|\right)^{\tau(x)-2} \frac{\sqrt{k_x}}{\ln[(1-n_x^{-1/4})n_x/k_x]}\left|\ln\left(\frac{k_x}{p}T_{p-k_x,p}\right)\right| > t\right).$$

Taylor's theorem gives now

$$\sup_{p\in I_x} \mathbb{P}\left(\left|\ln\left(\frac{k_x}{p}T_{p-k_x,p}\right)\right| > t\right) \leq \sup_{p\in I_x} \mathbb{P}\left(\frac{\left|\frac{k_x}{p}T_{p-k_x,p} - 1\right|}{1 - \left|\frac{k_x}{p}T_{p-k_x,p} - 1\right|} > t\right) = \sup_{p\in I_x} \mathbb{P}\left(\left|\frac{k_x}{p}T_{p-k_x,p} - 1\right| > \frac{t}{1+t}\right),$$

which tends to zero by Lemma 6 in Stupfler (2013), and, with $a > 1$,

$$\sup_{p\in I_x} \mathbb{P}\left(\left|\left(1 - \left|\frac{\ln(\frac{k_x}{p}T_{p-k_x,p})}{\ln(p/k_x)}\right|\right)^{\tau(x)-2} - 1\right| > t\right)$$

$$\begin{aligned}
&\leq \sup_{p\in I_x} \mathbb{P}\left(\left(1 - \left|\frac{\ln T_{p-k_x,p}}{\ln(p/k_x)} - 1\right|\right)^{\tau(x)-3} > a\right) + \sup_{p\in I_x} \mathbb{P}\left(\left|\frac{\ln T_{p-k_x,p}}{\ln(p/k_x)} - 1\right| > \frac{t}{2a}\right) \\
&= \sup_{p\in I_x} \mathbb{P}\left(\left|\frac{\ln T_{p-k_x,p}}{\ln(p/k_x)} - 1\right| > 1 - a^{\frac{1}{\tau(x)-3}}\right) + \sup_{p\in I_x} \mathbb{P}\left(\left|\frac{\ln T_{p-k_x,p}}{\ln(p/k_x)} - 1\right| > \frac{t}{2a}\right) \\
&\to 0.
\end{aligned}$$

Concerning now the term $\widetilde{\theta}_{12}(k_x;x)$ (which only needs to be considered in case $\tau(x) < 1$), remark that

$$\left|\sqrt{k_x}\,\widetilde{\theta}_{12}(k_x;x)\right| \leq \left|\frac{\theta(x)}{2}\,\frac{\sqrt{k_x}}{\ln\frac{n_x}{k_x}}\,\frac{\ln\frac{n_x}{k_x}}{\ln T_{p-k_x,p}}\,\frac{(\ln T_{p-k_x,p})^{\tau(x)-1}}{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}\,\frac{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\,\frac{1}{k_x}\sum_{i=1}^{k_x}(\ln T_i^*(p))^2\right|.$$

Combining again Lemma 6 in Stupfler (2013) with our Lemmas 1 and 3 together with our assumptions, we infer that

$$\sup_{p\in I_x} \mathbb{P}\left(\left|\sqrt{k_x}\,\widetilde{\theta}_{12}(k_x;x)\right| > t\right) \longrightarrow 0.$$

For $\widetilde{\theta}_2(k_x;x)$, we need also to distinguish between the two cases $\tau(x)=1$ and $\tau(x)<1$. We first consider the case $\tau(x)=1$, where we use the fact that $b(.;x)$ is regularly varying at infinity combining with Lemma 6 in Stupfler (2013) and the law of large numbers according to which

$$\sup_{p\in I_x}\mathbb{P}\left(\left|\frac{1}{k_x}\sum_{i=1}^{k_x}\frac{(T_i^*(p))^{\rho(x)}-1}{\rho(x)}-\frac{1}{1-\rho(x)}\right|>t\right)=\mathbb{P}\left(\left|\frac{1}{k_x}\sum_{i=1}^{k_x}\frac{T_i^{\rho(x)}-1}{\rho(x)}-\frac{1}{1-\rho(x)}\right|>t\right)\longrightarrow 0.$$

The convergence

$$\sup_{p\in I_x}\mathbb{P}\left(\left|\sqrt{k_x}\,\widetilde{\theta}_2(k_x;x)-\frac{\lambda}{1-\rho(x)}\right|>t\right)\longrightarrow 0$$

then follows from our assumptions and our Lemma 3. In the case where $\tau(x)<1$, using the same arguments as in the proof of Theorem 1, we have the following decomposition

$$\widetilde{\theta}_2(k_x;x)=:\widetilde{\theta}_{21}(k_x;x)+\widetilde{\theta}_{22}(k_x;x)+\widetilde{\theta}_{23}(k_x;x),$$

where

$$\begin{aligned}
\widetilde{\theta}_{21}(k_x;x) &:= (1+\delta_n)\,b\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)\frac{\left(\ln T_{p-k_x,p}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\frac{1}{k_x}\sum_{i=1}^{k_x}\ln T_i^*(p)\\
\widetilde{\theta}_{22}(k_x;x) &:= (1+\delta_n)\frac{b\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\frac{1}{k_x}\sum_{i=1}^{k_x}\ln T_i^*(p)\\
&\cdot e^{\rho(x)\left[D_{\tau(x)}(\ln\widetilde{T}_i(p)+\ln T_{p-k_x,p})-D_{\tau(x)}(\ln T_{p-k_x,p})\right]}\left\{\left(\ln T_{p-k_x,p}+\ln\widetilde{T}_i(p)\right)^{\tau(x)-1}-\left(\ln T_{p-k_x,p}\right)^{\tau(x)-1}\right\}\\
\widetilde{\theta}_{23}(k_x;x) &:= (1+\delta_n)\,b\left(\exp\left(D_{\tau(x)}\left(\ln T_{p-k_x,p}\right)\right);x\right)\frac{\left(\ln T_{p-k_x,p}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}\\
&\cdot\frac{1}{k_x}\sum_{i=1}^{k_x}\ln T_i^*(p)\left\{e^{\rho(x)\left[D_{\tau(x)}(\ln\widetilde{T}_i(p)+\ln T_{p-k_x,p})-D_{\tau(x)}(\ln T_{p-k_x,p})\right]}-1\right\}.
\end{aligned}$$

Using the regularly varying property of $b(.;x)$, the law of large numbers, our Lemmas 1-3 and our assumptions, combining with the mean value theorem for $\widetilde{\theta}_{22}(k_x;x)$ and $\widetilde{\theta}_{23}(k_x;x)$, we deduce that

$$\begin{aligned}
\sup_{p\in I_x}\mathbb{P}\left(\left|\sqrt{k_x}\,\widetilde{\theta}_{21}(k_x;x)-\lambda\right|>t\right) &\longrightarrow 0,\\
\sup_{p\in I_x}\mathbb{P}\left(\left|\sqrt{k_x}\,\widetilde{\theta}_{22}(k_x;x)\right|>t\right) &\longrightarrow 0,\\
\sup_{p\in I_x}\mathbb{P}\left(\left|\sqrt{k_x}\,\widetilde{\theta}_{23}(k_x;x)\right|>t\right) &\longrightarrow 0.
\end{aligned}$$

For what concerns the remainder term $R_p(x)$, using the same arguments as in the proof of Theorem 1, we get for all $t > 0$, that

$$\left\{ \left| \sqrt{k_x} \frac{2\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{\mu_{\tau(x)}\left(\ln \frac{n_x}{k_x}\right)} \right| > t \right\} \subseteq \left\{ \left| \sqrt{k_x} \frac{2\omega\left(\frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h\right)}{\mu_{\tau(x)}\left(\ln \frac{n_x}{k_x}\right)} \right| > t \right\} \cup \left\{ T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right\}$$
$$\cup \left\{ T_{p,p} > n_x^{1+\delta} \right\}.$$

Taking now n_x sufficiently large, this implies by assumption that

$$\sup_{p\in I_x} \mathbb{P}\left(\left| \sqrt{k_x} \frac{2\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{\mu_{\tau(x)}\left(\ln \frac{n_x}{k_x}\right)} \right| > t \right) \le \sup_{p\in I_x} \mathbb{P}\left(T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right) + \sup_{p\in I_x} \mathbb{P}\left(T_{p,p} > n_x^{1+\delta} \right)$$
$$\longrightarrow 0.$$

This convergence combined with (4) and Lemma 2 ensures that

$$\sup_{p\in I_x} \mathbb{P}\left(\left| \sqrt{k_x} R_p(x) \right| > t \right) \longrightarrow 0.$$

Combining all these convergences yield our Theorem 2. ∎

Appendix

In this section we introduce some lemmas which are useful for establishing the main results.

Lemma 1 *Assume that $n_x \to \infty$, $k_x \to \infty$ such that $\frac{k_x}{n_x} \to 0$. If $\tau(x) < 1$, then there exist a constant $C > 0$, such that*

$$\sup_{p\in I_x} \left| \frac{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} - 1 \right| \le C \left(\ln \frac{n_x}{k_x} \right)^{-1}.$$

Proof: First note that we have $\mu_{\tau(x)}(y) = y^{\tau(x)-1} + \widetilde{R}(y)$, with

$$\widetilde{R}(y) := \frac{\tau(x)-1}{2} y^{\tau(x)-2} \int_0^\infty (1+\xi)^{\tau(x)-2} u^2 e^{-u} du,$$

where ξ is a value between 0 and $\frac{u}{y}$. Hence $|\widetilde{R}(y)| \le y^{\tau(x)-2}$. Consequently

$$\left| \frac{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln \frac{p}{k_x}\right)} - 1 \right| = \left| \frac{\widetilde{R}\left(\ln \frac{p}{k_x}\right)}{\left(\ln \frac{p}{k_x}\right)^{\tau(x)-1} + \widetilde{R}\left(\ln \frac{p}{k_x}\right)} \right| \le \left(\ln \frac{p}{k_x} \right)^{-1} \left(1 + O\left(\left(\ln \frac{p}{k_x} \right)^{-1} \right) \right)^{-1}.$$

Since

$$\sup_{p\in I_x}\left(\ln\frac{p}{k_x}\right)^{-1}\le\left(\ln\frac{n_x\left(1-n_x^{-\frac{1}{4}}\right)}{k_x}\right)^{-1},$$

the result easily follows. ∎

Lemma 2 *Assume that $n_x\to\infty$, $k_x\to\infty$ such that $\frac{k_x}{n_x}\to 0$. Then*

$$\frac{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}\to 1$$

uniformly in $p\in I_x$.

Proof: We start by rewriting the term $\frac{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}-1$ as

$$\frac{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}-1=\left(\frac{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}-1\right)\frac{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}+\frac{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}-1.$$

According to Lemma 2 in Gardes *et al.* (2011), $\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)\sim\left(\ln\frac{n_x}{k_x}\right)^{\tau(x)-1}$. Thus, using a Taylor series expansion combining with the fact that uniformly in $p\in I_x$, $\ln\frac{p}{n_x}\to 0$, we have

$$\left|\frac{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}{\mu_{\tau(x)}\left(\ln\frac{n_x}{k_x}\right)}-1\right|\sim\left|\left(1+\frac{\ln\frac{p}{n_x}}{\ln\frac{n_x}{k_x}}\right)^{\tau(x)-1}-1\right|\longrightarrow 0 \tag{5}$$

uniformly in $p\in I_x$. Moreover, from the proof of Lemma 1, we know that

$$\left|\frac{\mu_{\tau(x)}\left(\ln\frac{p}{k_x}\right)}{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}-1\right|=\left|\frac{\widetilde{R}\left(\ln\frac{p}{k_x}\right)}{\left(\ln\frac{p}{k_x}\right)^{\tau(x)-1}}\right|\le\left(\ln\frac{p}{k_x}\right)^{-1}\longrightarrow 0 \tag{6}$$

uniformly in $p\in I_x$. Combining (5) and (6), our Lemma 2 follows.

∎

Lemma 3 *Assume that I_n is some index set, and, for $p\in I_n$ let $(X_n(p))_n$ and $(Y_n(p))_n$ be sequences of random variables. If for all $\varepsilon>0$ and some $x,y\in\mathbb{R}$,*

$$\sup_{p\in I_n}\mathbb{P}\left(|X_n(p)-x|>\varepsilon\right)\longrightarrow 0$$

and

$$\sup_{p\in I_n} \mathbb{P}\left(|Y_n(p) - y| > \varepsilon\right) \longrightarrow 0$$

as $n \to \infty$*, then*

$$\sup_{p\in I_n} \mathbb{P}\left(|X_n(p)Y_n(p) - xy| > \varepsilon\right) \longrightarrow 0$$

as $n \to \infty$*.*

Proof: Note that for all $p \in I_n$,

$$\{|X_n(p)Y_n(p) - xy| > \varepsilon\} \subseteq \{|(X_n(p) - x)| > 1\} \cup \left\{|(Y_n(p) - y)| > \frac{\varepsilon}{3}\right\} \cup \left\{|y\,(X_n(p) - x)| > \frac{\varepsilon}{3}\right\} \cup \left\{|x\,(Y_n(p) - y)| > \frac{\varepsilon}{3}\right\}.$$

Lemma 3 then follows using the subadditivity property of a probability measure. ■

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