Testing for periodicities in functional time series

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Summary

Problem and data example

Periodicities in time series

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- We investigate multivariate and functional time series.
- We do not assume that the period of the periodic component is known.

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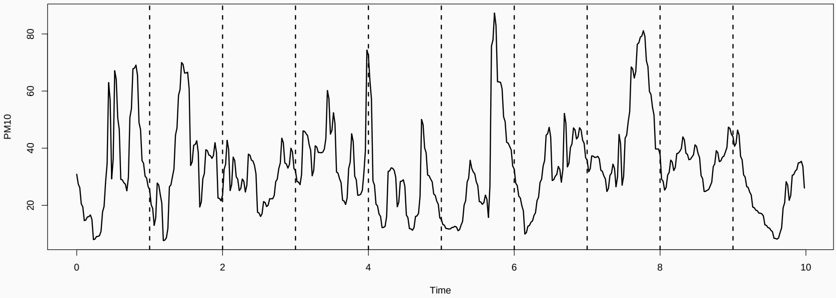
- Data set consists of 175 observation days in the winter season of 2010-2011 (October - March).
- The amount of PM10 is recorded every 30 min (48 observations per day).
- The measurement unit is $\mu g/m^3$.

PM10 data



PM10 time series

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PM10 time series

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- The overall process might not be stationary but consecutive curves might constitute a stationary functional time series.

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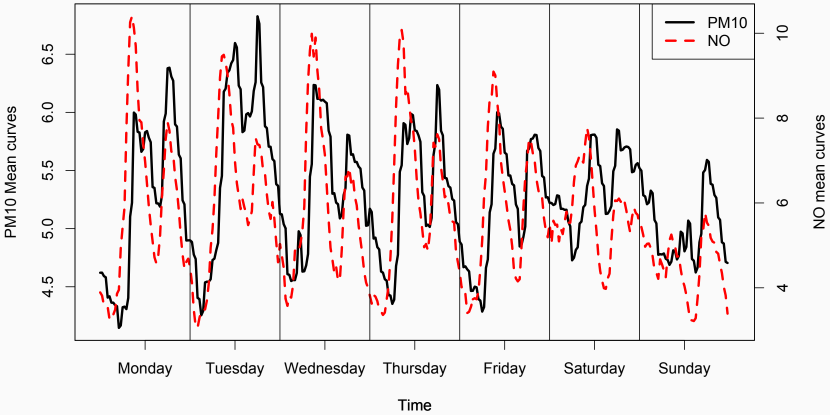
- We expect that some periodic component is present in the data.
- There are some natural periodicities of the periodic component (for example, weekly, monthly, yearly).

Mean curves



PM10
NO
PM10 Mean curves
NO mean curves
4.5
5.0
5.5
6.0
6.5
4
6
8
10
Monday
Tuesday
Wednesday
Thursday
Friday
Saturday
Sunday
Time

Known period

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Known period

- Hörmann, Kokoszka, and Nisol (2018) consider testing for periodicity in functional time series when the period is known.
- The same PM10 time series is considered and the presence of a weekly periodic component is investigated.
- They show that no weekly periodic component can be detected if only daily averages are considered and the functional structure is disregarded.
- Their test based on a fully functional ANOVA test for dependent data indicates that there is a weekly periodic pattern present in the PM10 time series.

Hidden periodicities

- We focus on developing a test when we do not know the period of a periodic component.

Hidden periodicities

- We focus on developing a test when we do not know the period of a periodic component.
- We also investigate if there are any other periodic components present in the PM10 time series.

Model

We consider time series $\{Y_t\}_{t\in\mathbb{Z}}$ with values in a separable Hilbert space $\mathbb{H}$ given by

$$Y_t = \mu + s_t + X_t,$$

where

i) $\mu \in \mathbb{H}$;

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i) $\mu \in \mathbb{H}$;

ii) $\{s_t\}_{t\in\mathbb{Z}}$ is a deterministic sequence with values in $\mathbb{H}$ such that

$$s_t = s_{t+d} \quad \text{and} \quad \sum_{t=1}^{d} s_t = 0$$

for all $t \in \mathbb{Z}$ with some $d > 1$;

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iii) $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary sequence of zero mean random elements with values in $\mathbb{H}$.

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Problem

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$$H_0 : Y_t = \mu + X_t \quad \text{versus} \quad H_1 : Y_t = \mu + s_t + X_t.$$

- We need a test statistic that captures the presence of any periodic component.

Test statistic

Frequency domain approach

- Our test is based on the frequency domain approach to the analysis of functional time series.

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- Our test is based on the frequency domain approach to the analysis of functional time series.
- The frequency domain approach to the analysis of functional time series has been gaining attention in recent years (see, for example, Panaretos and Tavakoli (2013), Hörmann, Kidziński and Hallin (2015), Zhang (2016), Ch. and Rice (2020) among others).

DFT and periodogram

Definition

The DFT of $\{X_t\}_{1 \le t \le n}$ is defined by

$$\mathcal{X}_n(\omega) = n^{-1/2} \sum_{t=1}^{n} X_t e^{-it\omega}$$

with $i = \sqrt{-1}$ for $\omega \in [-\pi, \pi]$ and $n \ge 1$.

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Definition

The periodogram of $\{X_t\}_{1\leq t\leq n}$ is defined by

$$I_n(\omega) = \mathcal{X}_n(\omega) \otimes \mathcal{X}_n(\omega) = \langle \cdot, \mathcal{X}_n(\omega)\rangle \mathcal{X}_n(\omega)$$

for $\omega \in [-\pi, \pi]$ and $n \geq 1$.

Maximum of periodogram

The test statistic is given by

$$M_n = \max_{1 \le j \le q} |||I_n(\omega_j)|||_2 = \max_{1 \le j \le q} \|\mathcal{X}_n(\omega_j)\|^2$$

for $n > 2$, where $q = \lfloor (n-1)/2 \rfloor$,

i) $\omega_j = 2\pi j/n$ are the Fourier frequencies with $1 \le j \le q$;

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i) $\omega_j = 2\pi j/n$ are the Fourier frequencies with $1 \leq j \leq q$;

ii) $|||\cdot|||_2$ is the Hilbert-Schmidt norm and $\|\cdot\|$ is the norm induced by the inner product of $\mathbb{H}$.

Simple simulated example

$\{Y_t\}_{t\geq 1}$ is defined by

$$Y_t = 0.5\cos((2\pi/7)t)\omega + W_t$$

for $t \geq 1$, where

i) $\omega(\tau) = 1$ for $\tau \in [0, 1]$;

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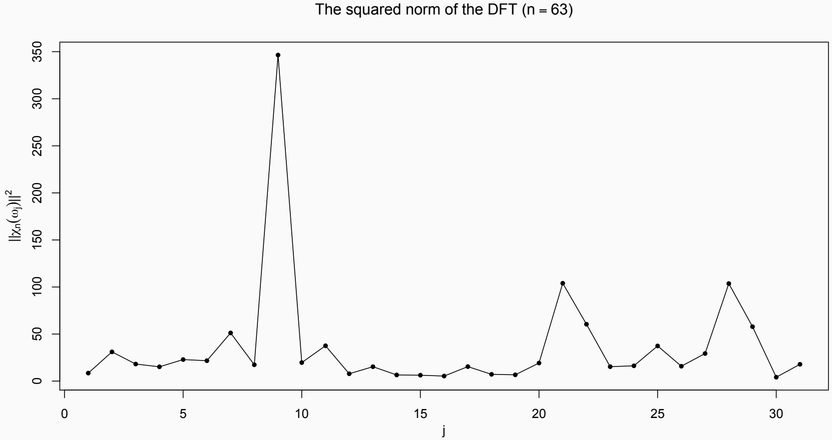
The period of $\{Y_t\}_{t\geq 1}$ is $d = 7$.

Simple simulated example (cont.)



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Maximum of periodogram

- Small values of the maximum of the periodogram indicate that there is no periodic component.
- Large values of the maximum of the periodogram indicate that there is a periodic component.
- We need to establish the asymptotic distribution of the maximum of the periodogram if we want to use it to test for hidden periodicities.

Theorem

Suppose that $\{X_t\}_{t\geq 1}$ *are iid random variables such that* $\mathrm{E}\, X_1 = 0$, $\mathrm{E}\, |X_1|^2 = 1$ *and* $\mathrm{E}\, |X_1|^r < \infty$ *with* $r > 2$*. Then*

$$\max_{1\leq j\leq q} |\mathcal{X}_n(\omega_j)|^2 - \log q \xrightarrow{d} \mathcal{G} \quad as \quad n \to \infty,$$

where $q = \lfloor (n-1)/2 \rfloor$ *and* $\mathcal{G}$ *is the standard Gumbel distribution with the CDF given by* $F(x) = \exp\{-\exp\{-x\}\}$ *for* $x \in \mathbb{R}$*.*

Davis and Mikosch (1999)

Maximum of periodogram

- Does a similar result hold when $\mathbb{H} = \mathbb{R}^d$?

Maximum of periodogram

- Does a similar result hold when $\mathbb{H} = \mathbb{R}^d$?
- Does a similar result hold when $\mathbb{H}$ is an infinite dimensional separable Hilbert space?

Asymptotic results

Intuition

- Assume for the moment that the $\{X_t\}_{1 \leq t \leq n}$ are iid Gaussian random vectors such that $\mathrm{E}\, X_1 = 0$ and $\mathrm{E}[X_1 X_1'] = I_d$, where I_d is the identity matrix.

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- Then we have that

$$\max_{1\leq j\leq q} \|\mathcal{X}_n(\omega_j)\|^2 = \max_{1\leq j\leq q}\left\{\sum_{k=1}^{d} E_{kj}\right\},$$

where E_{kj} are iid $\mathrm{Exp}(1)$ for $1 \leq k \leq d$ and $1 \leq j \leq q$.

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- Then we have that

$$\max_{1 \le j \le q} \|\mathcal{X}_n(\omega_j)\|^2 = \max_{1 \le j \le q} \left\{ \sum_{k=1}^{d} E_{kj} \right\},$$

where E_{kj} are iid $\mathrm{Exp}(1)$ for $1 \le k \le d$ and $1 \le j \le q$.
- Hence, we have a maximum of q iid $\mathrm{Erlang}(d, 1)$ random variables (a special case of the gamma distribution).

Theorem

Suppose that $\{X_t\}_{t\geq 1}$ *are iid vectors in* $\mathbb{R}^d$ *such that* $\mathrm{E}\, X_1 = 0$, $\mathrm{E}[X_1 X_1'] = I_d$ *and* $E\|X_1\|^r < \infty$ *for some* $r > 2$, *where* I_d *is the identity matrix. Then*

$$\max_{1\leq j\leq q} \|\mathcal{X}_n(\omega_j)\|^2 - c_n \xrightarrow{d} \mathcal{G} \quad as \quad n \to \infty,$$

where $q = \lfloor (n-1)/2 \rfloor$,

$$c_n = \log q + (d-1)\log\log q - \log(d-1)!$$

for $n > 3$ *and* $\mathcal{G}$ *is the standard Gumbel distribution with the CDF given by* $F(x) = \exp\{-\exp\{-x\}\}$ *for* $x \in \mathbb{R}$.

Assumptions and notation for the general case

We first investigate the situation when $\{X_t\}_{t\geq 1}$ are iid zero mean random elements with values in $\mathbb{H}$.

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$\{v_k\}_{k\geq 1}$ and $\{\lambda_k\}_{k\geq 1}$ are the eigenvectors and eigenvalues of the covariance operator $\mathrm{E}[X_1 \otimes X_1] = \mathrm{E}[\langle \cdot, X_1 \rangle X_1]$.

Projection onto a finite dimensional subspace

Since $\{v_k\}_{k\geq 1}$ is an ONB of $\mathbb{H}$, we have that

$$X_t = \sum_{k=1}^{\infty} \langle X_t, v_k \rangle v_k$$

for $t \geq 1$.

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We denote

$$X_t^d = \sum_{k=1}^{d} \langle X_t, v_k \rangle v_k, \quad \mathcal{X}_n^d(\omega) = n^{-1/2} \sum_{t=1}^{n} X_t^d e^{-it\omega}$$

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for $t \geq 1$ and $\omega \in [-\pi, \pi]$.

We also denote

$$M_n^d = \max_{1\leq j\leq q} \|\mathcal{X}_n^d(\omega_j)\|^2$$

for $n \geq 1$.

The idea of the proof

We have that

$$(M_n - b_n)/\lambda_1 = \underbrace{(M_n - M_n^{d_n})/\lambda_1}_{A_1} + \underbrace{(M_n^{d_n} - b_n^{d_n})/\lambda_1}_{A_2} + \underbrace{(b_n^{d_n} - b_n)/\lambda_1}_{A_3},$$

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where $d_n \to \infty$ as $n \to \infty$,

$$b_n^{d_n} = \lambda_1 \log q - \lambda_1 \sum_{j=2}^{d_n} \log(1 - \lambda_j/\lambda_1)$$

and $b_n = \lim_{n\to\infty} b_n^{d_n}$.

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We start by noting that

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- A_1 goes to 0 in probability if $d_n \to \infty$ fast enough as $n \to \infty$.
- A_3 goes to 0 as $n \to \infty$ as long as $d_n \to \infty$ as $n \to \infty$.

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- A_1 goes to 0 in probability if $d_n \to \infty$ fast enough as $n \to \infty$.
- A_3 goes to 0 as $n \to \infty$ as long as $d_n \to \infty$ as $n \to \infty$.
- The challenge is A_2. We need a sequence $\{d_n\}_{n \geq 1}$ that grows slowly enough, but at the same time the intersection of the sequences $\{d_n\}_{n \geq 1}$ in A_1 and A_2 cannot be empty.

Gaussian approximation

Suppose that

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- $X_1, \ldots, X_n$ are independent zero mean random vectors in $\mathbb{R}^p$;
- $Y_1, \ldots, Y_n$ are independent zero mean Gaussian random vectors in $\mathbb{R}^p$ such that $\mathrm{E}[Y_i Y_i'] = \mathrm{E}[X_i X_i']$.

Gaussian approximation

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- $Y_1, \dots, Y_n$ are independent zero mean Gaussian random vectors in $\mathbb{R}^p$ such that $\mathrm{E}[Y_i Y_i'] = \mathrm{E}[X_i X_i']$.

Consider the quantity

$$\rho_n(\mathcal{A}) = \sup_{A \in \mathcal{A}} |P(n^{-1/2}(X_1 + \ldots + X_n) \in A) - P(n^{-1/2}(Y_1 + \ldots + Y_n) \in A)|,$$

where $\mathcal{A}$ is a class of Borel sets in $\mathbb{R}^p$.

Classical results

To make $\rho_n(\mathcal{A})$ to be $o(1)$ as $n \to \infty$, we at least need

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- $p = o(n^{1/3})$ as $n \to \infty$ when $\mathcal{A}$ is the class of Euclidean balls;
- $p = o(n^{2/7})$ as $n \to \infty$ when $\mathcal{A}$ is the class of all Borel measurable convex sets.

s-sparsely convex sets

- Chernozhukov, Chetverikov, Kato (2017) define the class $\mathcal{A}^{sp}(s)$ of s-sparsely convex sets.

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- Chernozhukov, Chetverikov, Kato (2017) only consider the situation when s is fixed but in our problem $s = 2d_n \to \infty$ as $n \to \infty$.

Gaussian approximation bound

We obtain that

$$\rho_n(\mathcal{A}^{sp}(2d_n)) \leq C \cdot \frac{d_n^4 \log^{7/6}(d_n n^2)}{\lambda_{d_n}^{1/2} n^{1/6}},$$

where C is a universal constant.

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ii) $\lambda_k > \lambda_{k+1}$ for $k \geq 1$ and $\{k\lambda_k\}_{k\geq 1}$ is eventually monotonic;

iii) there exists $\{d_n\}_{n\geq 1}$ such that $d_n^4/\lambda_{d_n}^{1/2} = o(n^{1/6}/\log^{7/6} n)$ and $d_n = O(n^{\gamma_0})$ as $n \to \infty$ with

$$\gamma_0 < \min\Big\{\min_{k\geq 2}\Big\{\frac{1}{k}\Big(\frac{\lambda_1}{\lambda_k} - 1\Big)\Big\}, 1\Big\};$$

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iv) there exists $\{\ell_k\}_{k\geq 1}$ such that $\ell_k > 0$ for $k \geq 1$, $\sum_{k=1}^{\infty} \ell_k = 1$,

$$\sum_{k=1}^{\infty} \ell_k^{-r/2}\, \mathrm{E}\,|\langle X_1, v_k\rangle|^r < \infty \quad \text{and} \quad \sum_{k>d_n} (\lambda_k/\ell_k)^{r/2} = o(1/n)$$

as $n \to \infty$.

Theorem

Suppose that the assumptions from the previous slide hold. Then

$$\lambda_1^{-1}(M_n - b_n) \xrightarrow{d} \mathcal{G}$$

as $n \to \infty$, *where*

$$b_n = \lambda_1 \log q - \lambda_1 \sum_{j=2}^{\infty} \log(1 - \lambda_j/\lambda_1)$$

with $q = \lfloor (n-1)/2 \rfloor$ *and* $\mathcal{G}$ *is the standard Gumbel distribution with the CDF given by* $F(x) = \exp\{-\exp\{-x\}\}$ *for* $x \in \mathbb{R}$.

Two examples

We write $\alpha_n = \Theta(\beta_n)$ as $n \to \infty$ if there exist $k > 0$, $K > 0$ and $N \geq 1$ such that

$$k\beta_n \leq \alpha_n \leq K\beta_n$$

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The conditions of the main theorem are satisfied in the following two cases:

a) $\lambda_k = \Theta(\rho^k)$ as $k \to \infty$ with $0 < \rho < 1$ (exponential decay);

Two examples

We write $\alpha_n = \Theta(\beta_n)$ as $n \to \infty$ if there exist $k > 0$, $K > 0$ and $N \geq 1$ such that

$$k\beta_n \leq \alpha_n \leq K\beta_n$$

for all $n \geq N$.

The conditions of the main theorem are satisfied in the following two cases:

a) $\lambda_k = \Theta(\rho^k)$ as $k \to \infty$ with $0 < \rho < 1$ (exponential decay);
b) $\lambda_k = \Theta(k^{-\nu})$ as $k \to \infty$ with $\nu > 1$ (polynomial decay).

Linear processes

Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ is a linear process given by

$$X_t = \sum_{k=-\infty}^{\infty} a_k(\varepsilon_{t-k})$$

for each $t \in \mathbb{Z}$, where

- $\{a_k\}_{k\in\mathbb{Z}} \subset \mathcal{L}(\mathbb{H})$ such that $\sum_{k=-\infty}^{\infty} \|\|a_k\|\|_{op} < \infty$;

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- $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are iid zero mean random elements with values in $\mathbb{H}$.

Notation for linear processes

- We denote the DFT of $\varepsilon_1, \ldots, \varepsilon_n$ by

$$\mathcal{E}(\omega) = n^{-1/2} \sum_{t=1}^{n} \varepsilon_t e^{-it\omega}$$

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- We also use the impulse-response operator $A(\omega)$ defined by

$$A(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{-it\omega}$$

  for $\omega \in [-\pi, \pi]$.

Lemma for linear processes

Lemma

Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ *is a linear process such that*

a) $\sum_{k\neq 0}\log(|k|)|||a_k|||_{op} < \infty$;

b) $A^{-1}(\omega)$ *exists for each* $\omega \in [-\pi, \pi]$;

c) $\sup_{\omega\in[0,\pi]} |||A^{-1}(\omega)|||_{op} < \infty$.

Then

$$\max_{1\leq j\leq q} \|A_n^{-1}(\omega_j)\mathcal{X}_n(\omega_j)\|^2 - \max_{1\leq j\leq q} \|\mathcal{E}_n(\omega_j)\|^2 = o_P(1) \quad as \quad n \to \infty.$$

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This is a generalization of the result by Walker (1965).

FAR(1)

- Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ is an FAR(1) model given by

$$X_t = \rho(X_{t-1}) + \varepsilon_t = \sum_{j=0}^{\infty} \rho^j(\varepsilon_{t-j})$$

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  for $t \in \mathbb{Z}$ with $\rho \in \mathcal{L}(\mathbb{H})$ such that $|||\rho^{n_0}||| < 1$ with some $n_0 \geq 1$.
- Then $A^{-1}(\omega)$ exists for each $\omega \in [-\pi, \pi]$ and $\sup_{\omega\in[0,\pi]} |||A^{-1}(\omega)|||_{op} < \infty$.

Corollary

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Suppose that

- $\{X_t\}_{t\in\mathbb{Z}}$ is a linear process and the assumptions of the auxiliary lemma are satisfied;
- $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ satisfies the assumptions of the main theorem.

Then

$$\lambda_1^{-1}\Big(\max_{1\leq j\leq q} \|A^{-1}(\omega_j)\mathcal{X}_n(\omega_j)\|^2 - b_n\Big) \xrightarrow{d} \mathcal{G} \quad \text{as} \quad n\to\infty.$$

The eigenvalue λ_1 and those in the definition of b_n are the eigenvalues of the covariance operator $\mathsf{E}[\varepsilon_1 \otimes \varepsilon_1]$.

Assumption for the FAR(1)

Assumption 2

i) $\hat{\rho}$ is an estimator of ρ such that $|||\hat{\rho} - \rho|||_{op} = o_p\left(a_n^{-1}\right)$ as $n \to \infty$, where $\log n \leq a_n \leq \sqrt{n}$;

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iv) $\mu = 0$.

Test statistic

Theorem

Suppose that $\{\hat{\lambda}_j\}_{j\geq 1}$ *are the eigenvalues of* $(n-1)^{-1}\sum_{k=2}^{n}\hat{\varepsilon}_k\otimes\hat{\varepsilon}_k$, *where*

$$\hat{\varepsilon}_k = X_k - \hat{\rho}\,(X_{k-1}), \quad k = 2,\ldots,n.$$

Under H_0 *and Assumption 2, we have that*

$$T_n = \hat{\lambda}_1^{-1}\max_{1\leq j\leq q}\|(I - e^{-i\omega_j}\hat{\rho}\,)\mathcal{Y}_n(\omega_j)\|^2 - \log q + \sum_{j=2}^{a_n}\log(1-\hat{\lambda}_j/\hat{\lambda}_1) \xrightarrow{d} \mathcal{G}$$

as $n\to\infty$.

Empirical study

Generating functional time series

- The basic building block are the PM10 curves.

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- The estimator of ψ is a PCA based estimator defined in Bosq (2000).

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This construction assures that we get a functional time series which is stationary and behaves similarly as the original PM10 data.

Periodic component

- The periodic component in the simulation study is given by

$$s_t(u) = a\cos(2\pi t/d),$$

where $u \in [0, 1]$ and $d - 2$ is a Poisson distributed random variable P_λ with $\lambda = 5$ or $\lambda = 15$.

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- For a we investigate the values $a = 0, 1, 2$, where $a = 0$ corresponds to H_0.

Empirical rejection rates

		$a = 0$			$a = 1$			$a = 2$		
	α	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
$\lambda = 5$	$n = 100$	0.066	0.029	0.004	0.861	0.799	0.670	1.000	0.999	0.993
	$n = 200$	0.082	0.038	0.006	0.989	0.983	0.970	1.000	1.000	1.000
	$n = 500$	0.093	0.054	0.011	1.000	1.000	0.999	1.000	1.000	1.000
$\lambda = 15$	$n = 100$	0.082	0.041	0.005	0.249	0.165	0.071	0.818	0.758	0.606
	$n = 200$	0.071	0.035	0.006	0.569	0.471	0.293	0.985	0.973	0.922
	$n = 500$	0.096	0.045	0.007	0.990	0.978	0.942	1.000	1.000	1.000

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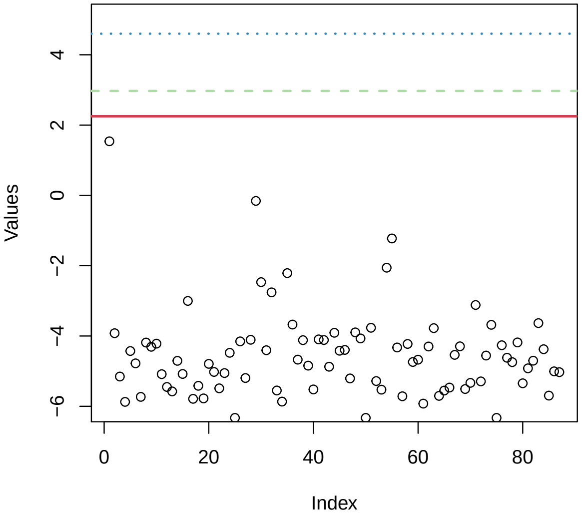
- Since the test we propose is not requiring knowledge of the period d, it is expected to have smaller power.
- We plot the values of the test statistic

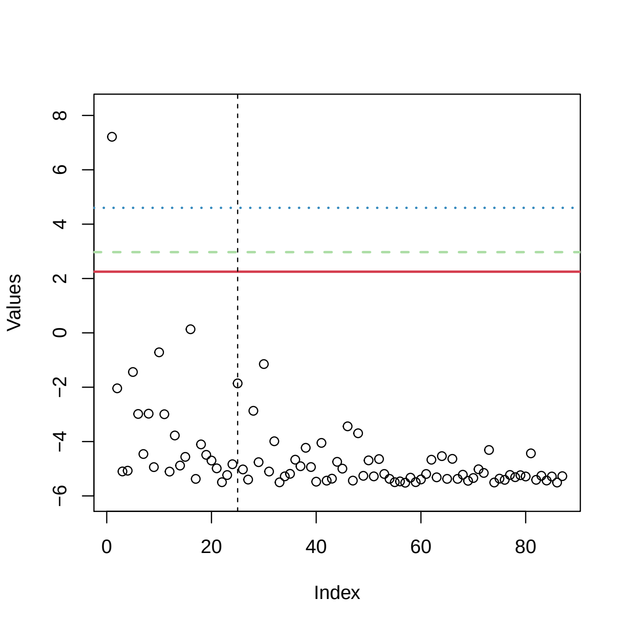
$$T_n(j) := \|(I - e^{-i\omega_j}\hat{\rho})\mathcal{Y}_n(\omega_j)\|^2 - \log q + \sum_{j=2}^{a_n} \log(1 - \hat{\lambda}_j/\hat{\lambda}_1)$$

for $j = 1, \dots, q = 87$.

PM10 time series the weekday effect removed



PM10 time series



Summary

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`https://www.stat.ucdavis.edu/~vaidas/`