Finding the K Shortest (Hyper-)Paths in a Hypergraph

(aka Synthesis Planning)

DM840 - 2022 - Week 45

Department of Mathematics and Computer Science
University of Southern Denmark

November 08, 2022

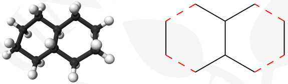
Synthesis Planning

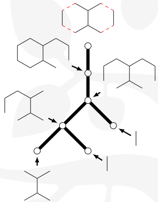
Retrosynthetic method[1]:

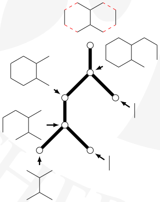
- Bondset
  - Stage 1: Choose bondset
  - **Stage 2: Choose plan for fixing bonds in bondset**
- Types of reactions:
  - **Construction: affixations & cyclizations**
  - Functionalizations

  Plan = sequencing of bond fixations = **unary-binary tree**
- Cost measure for plans
  - External Path Length (EPL)
  - Total Weight of Starting Materials (W)

Decaline







[1] Systematic Synthesis Design. 6. Yield Analysis and Convergency, Hendrickson, James B., 1977

Synthesis Planning

Stage 2: Choose plan for bondset

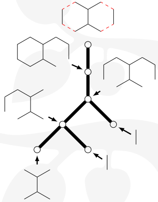
One bondset represents many plans.

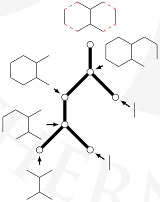
Questions:

- How do we choose the best? Known[2].
- Is one answer enough?

Here: $\boxed{\text{How to compute the } K \text{ best plans}}$

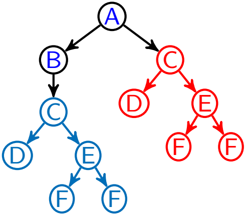
Decaline



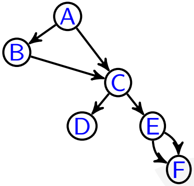


[2] Computational complexity of synthetic chemistry – Basic facts, Smith, Warren D, 1997

Synthesis Plans as Hypergraphs



Unary-Binary tree



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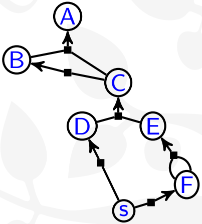


Hypergraph

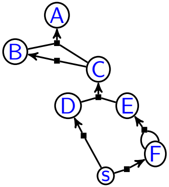
(Actually, the hypergraph above is a hyper**path**)

Hyperpath in Hypergraph[3]

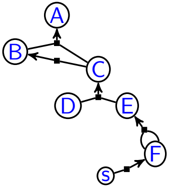
Definition

A *hyperpath* $\pi_{st} = (V_\pi, E_\pi)$ from a source vertex s to a target vertex t is a subhypergraph of H with the following properties: E_π can be ordered in a sequence $\langle e_1, e_2, ..., e_q \rangle$ such that

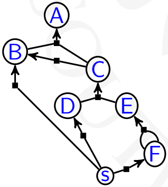
1. $T(e_i) \subseteq \{s\} \cup \{H(e_1), H(e_2), ..., H(e_{i-1})\}$ for all i
2. $t = H(e_q)$
3. Every $v \in V_\pi \setminus \{t\}$ has at least one outgoing hyperarc in E_π
4. Every $v \in V_\pi \setminus \{s\}$ has exactly one ingoing hyperarc in E_π



Hyperpath



Not hyperpath



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[3] variation of Ausiello, G., Franciosa, P., Frigioni, D., TCS (2001)

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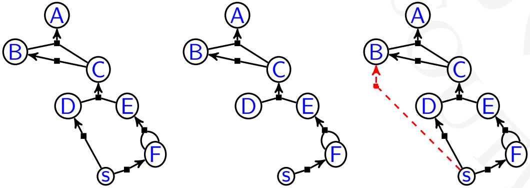

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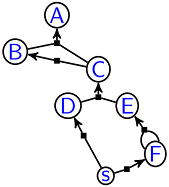
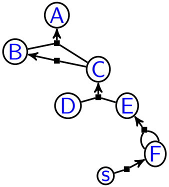
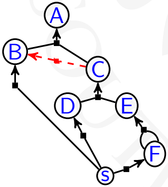


Hyperpath



Not hyperpath



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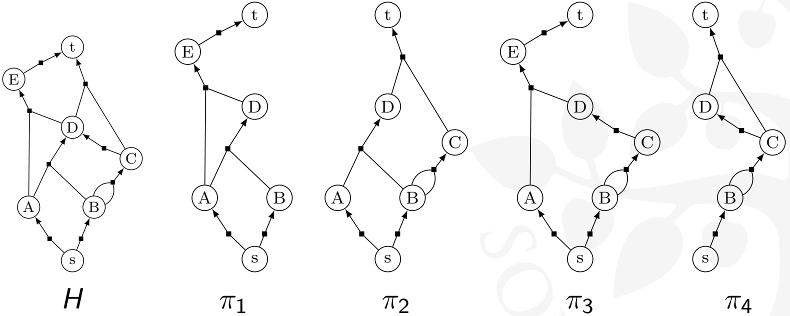
Hyperpaths in Hypergraph, Example



Hypergraph of Synthesis Plans (HoSP)

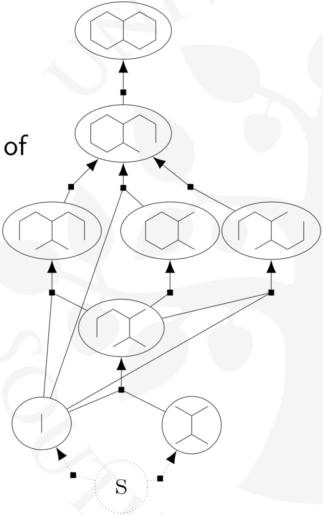
Definition

R: Finite set of reactions

S: Set of starting materials

Let E_R be the representation of R as a set of hyperarcs. Let V_R be the set of vertices appearing in the heads and tails of the hyperarcs in E_R. The *hypergraph of synthesis plans* (*HoSP*) is the hypergraph

$$H = (V_R \cup \{s\}, E_R \cup E_s)$$



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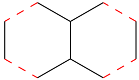
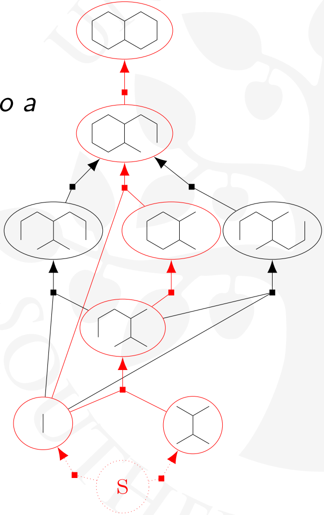


Properties of the HoSP

Lemma

Let H be a HoSP. Then any hyperpath π_{sv} from s to any other vertex v corresponds to a synthesis plan for v.

HoSP is acyclic $\Rightarrow$ topological sorting of vertices exist.



“Cost Functions” – External Path Length

$$\begin{aligned} \pi \quad & st\text{-hyperpath.} \\ \mathcal{S} \quad & \text{set potential starting materials.} \\ i \in \mathcal{S} \cap \pi \quad & \text{starting material of } \pi. \\ P_{it} = (i, e_1, v_1, e_2, v_2, ..., e_{|P_{it}|}, t) \quad & \text{simple } it\text{-path th in } \pi. \end{aligned}$$

$$\mathrm{EPL}_\pi = \sum_i \sum_{P_{it} \in \pi} |P_{it}|, \tag{1}$$

Cost Functions - Total Weight of Starting Materials

π	*st*-hyperpath.
$\mathcal{S}$	set potential starting materials.
$i \in \mathcal{S} \cap \pi$	starting material of π.
$P_{it} = (i, e_1, v_1, e_2, v_2, ..., e_{\lvert P_{it} \rvert}, t)$	simple *it*-path in π.
$r_{v,e}$	retro yield

$$\mathrm{W}_\pi = \sum_i \sum_{P_{it} \in \pi} \prod_{j=0}^{\lvert P_{it} \rvert} r_{v_j, e_{j+1}} \tag{2}$$

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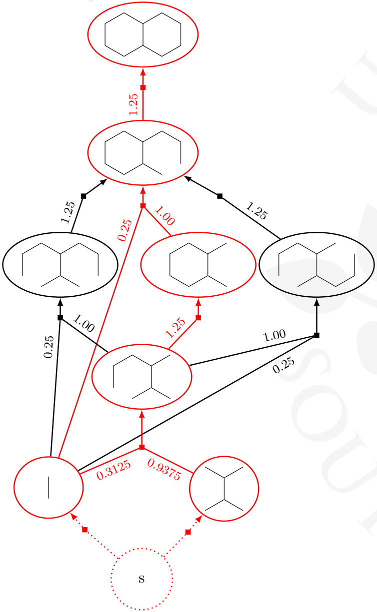
Cost Functions - Total Weight of Starting Materials

$$\begin{array}{rl} \pi & st\text{-hyperpath.} \\ \mathcal{S} & \text{set potential starting materials.} \\ i \in \mathcal{S} \cap \pi & \text{starting material of } \pi. \\ P_{it} = (i, e_1, v_1, e_2, v_2, ..., e_{|P_{it}|}, t) & \text{simple } it\text{-path in } \pi. \\ r_{v,e} & \text{retro yield} \end{array}$$

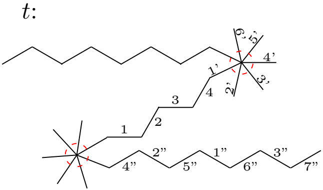
$$W_\pi = \sum_{i} \sum_{P_{it} \in \pi} \prod_{j=0}^{|P_{it}|} r_{v_j, e_{j+1}} \tag{2}$$

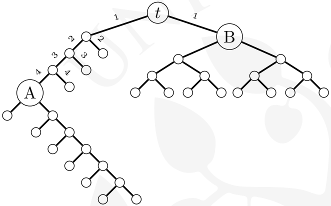
$$W(u) = \begin{cases} 0 & \text{if } u = s \\ 1 & \text{if } u \in \mathcal{S} \\ \sum\limits_{v \in T(p(u))} r_{v,p(u)} W(v) & \text{otherwise} \end{cases} \tag{3}$$

Cost Functions : Example



The Problem with EPL

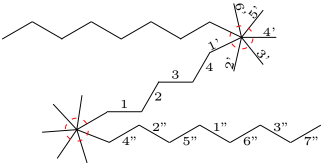


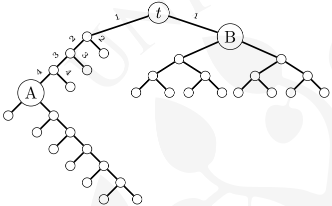


EPL=96

The Problem with EPL

t:





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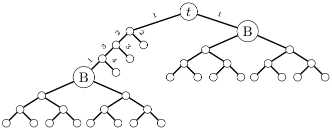
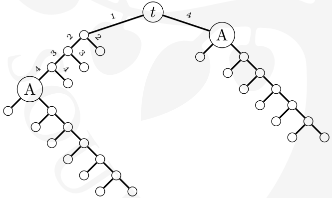


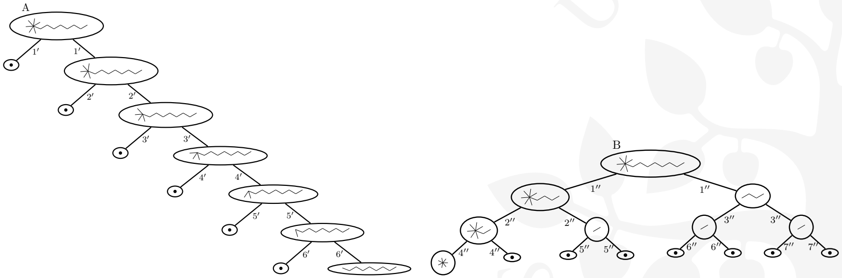
EPL=97



EPL=98

EPL is not *additive*

The Problem with EPL



Yens Algorithm [4]

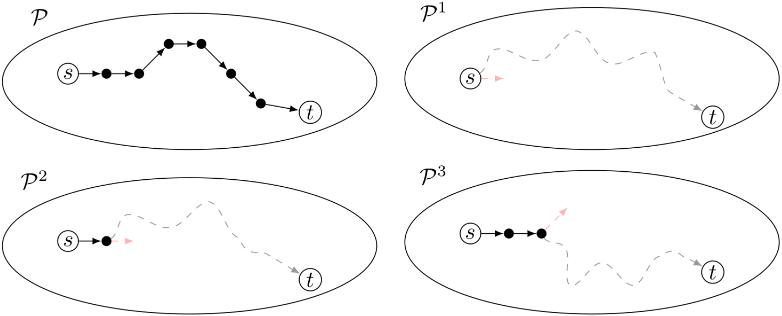
K shortest simple paths from s to t in a (standard) directed graph

[4] Finding the k Shortest Loopless Paths in a Network, Yen, Jin Y, 1971

Yens Algorithm [4]

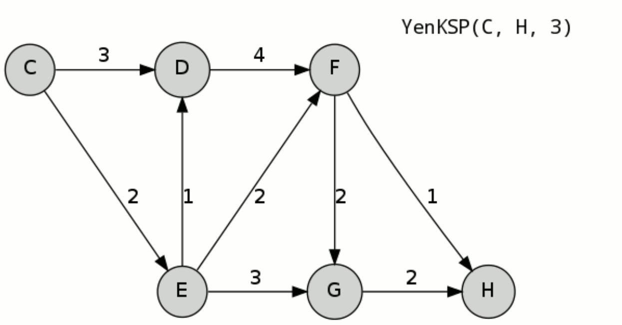
K shortest simple paths from s to t in a (standard) directed graph

$\mathcal{P} = \{P | P \text{ is a path from } s \text{ to } t\}$
$P' \in \mathcal{P}$ is the best.
Partition $\mathcal{P} \setminus P'$ into $|P'|$ disjoint sets, creating $|P'|$ shortest path problems.



[4] Finding the k Shortest Loopless Paths in a Network, Yen, Jin Y, 1971

Yens Algorithm



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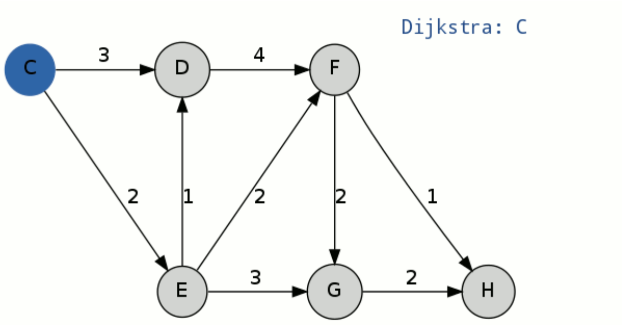

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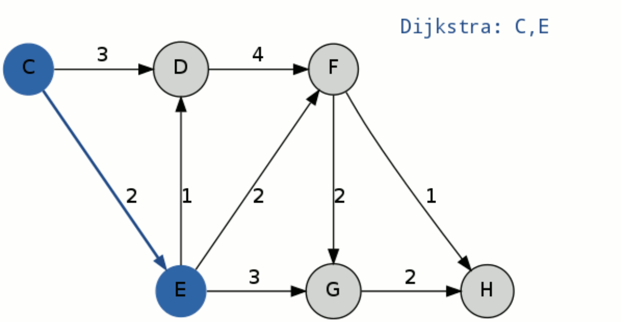

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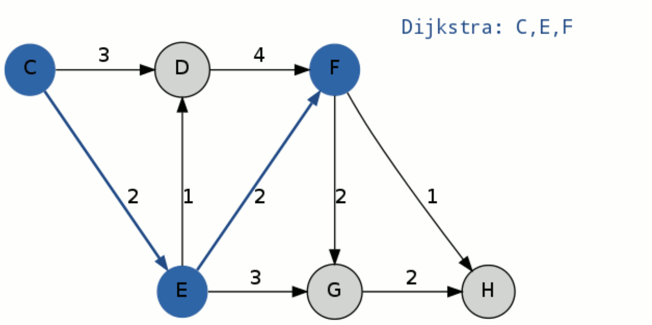

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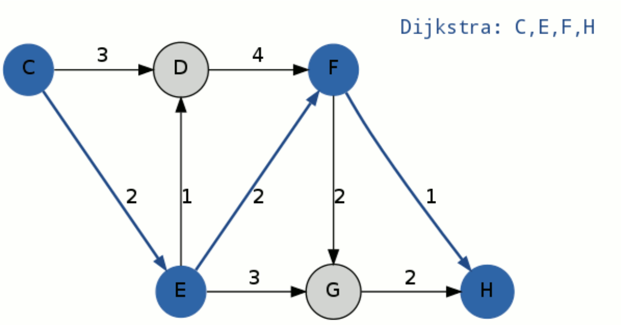

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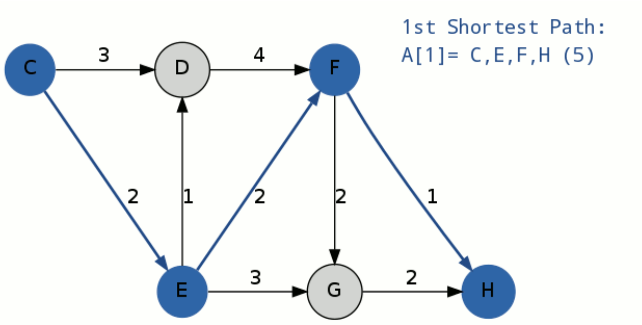

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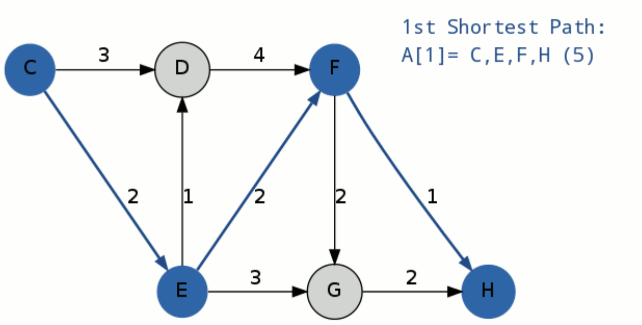

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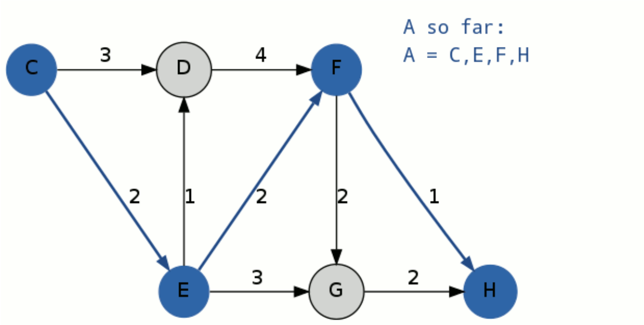

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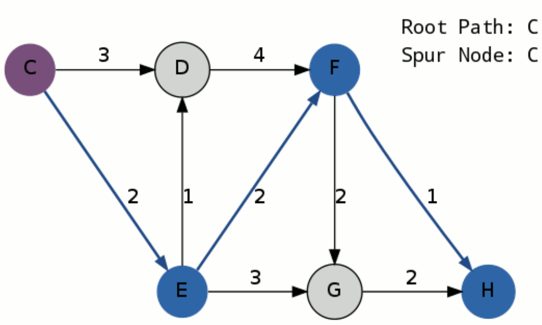

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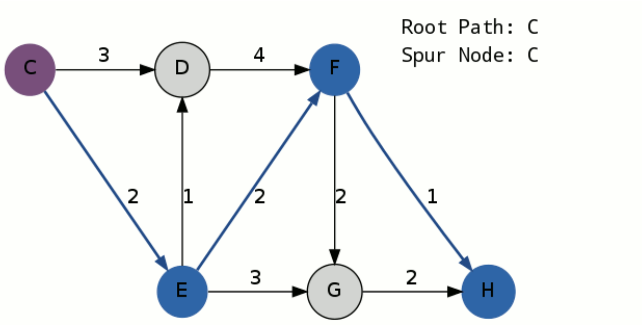

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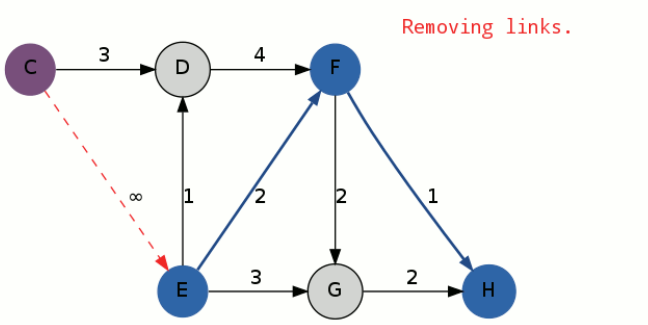

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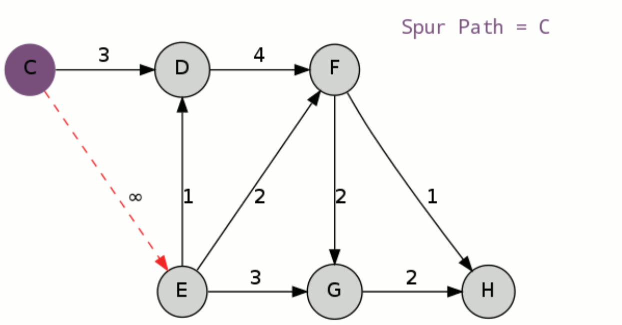

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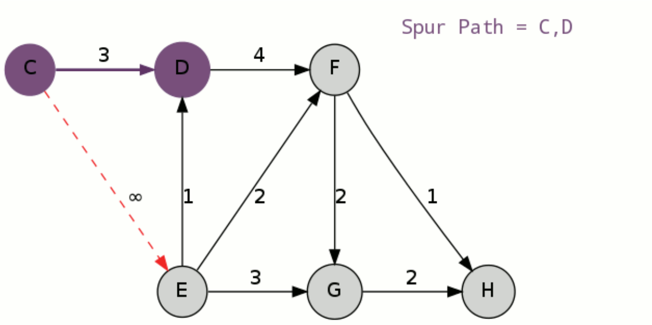

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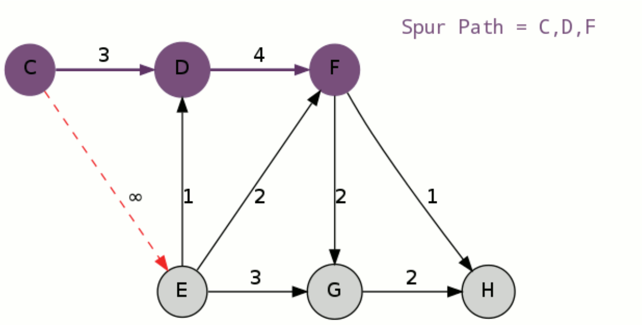

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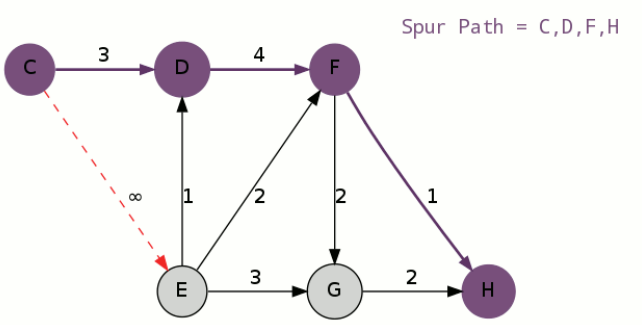

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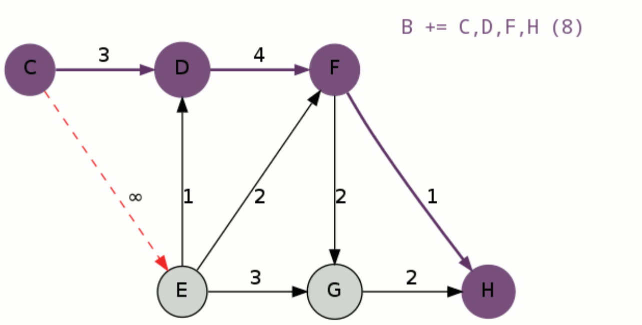

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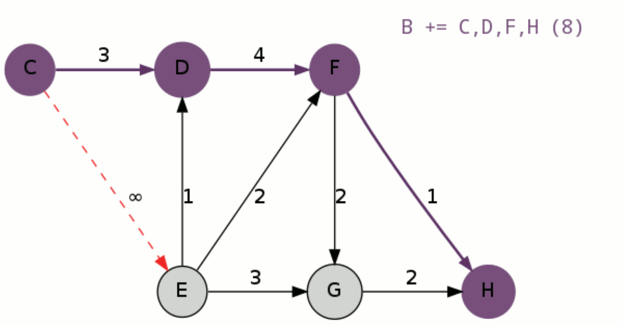

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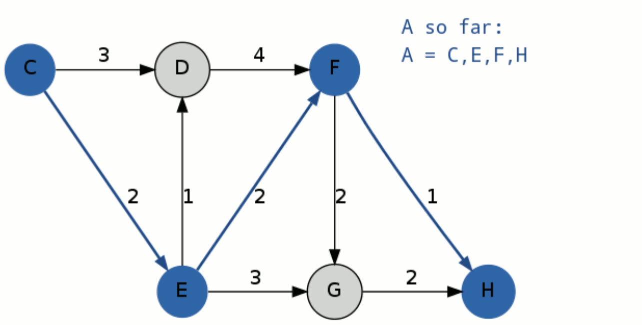

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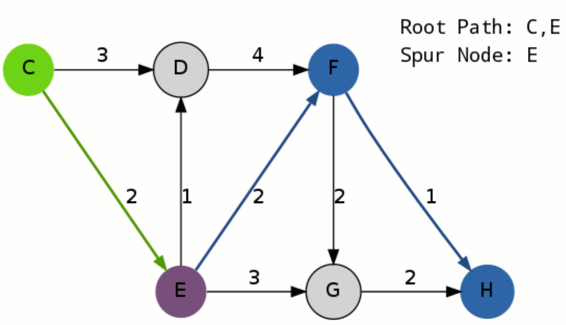

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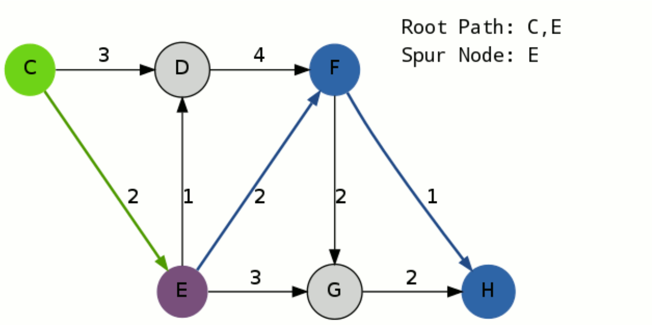

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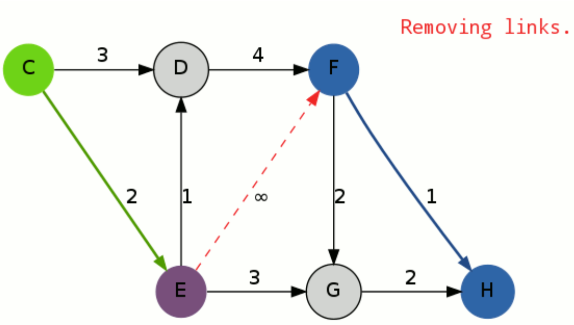

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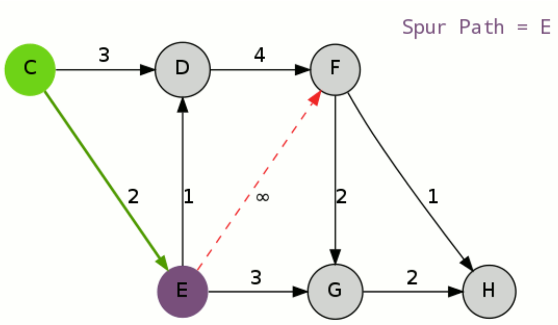

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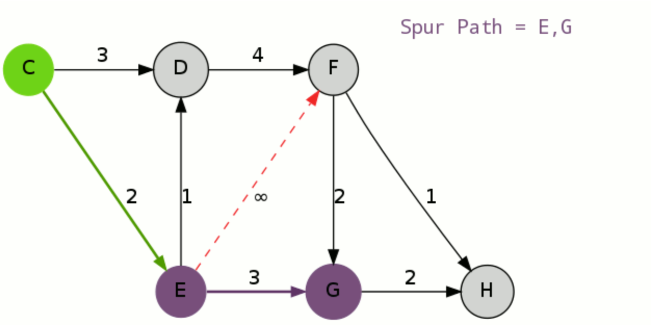

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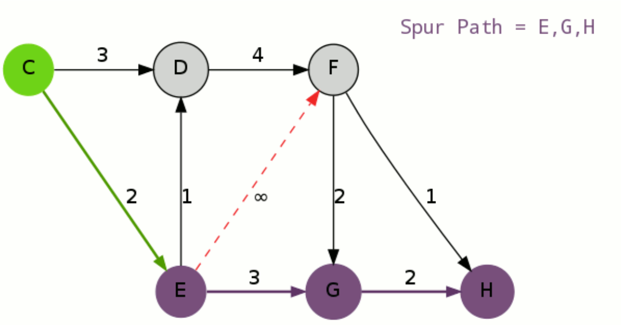

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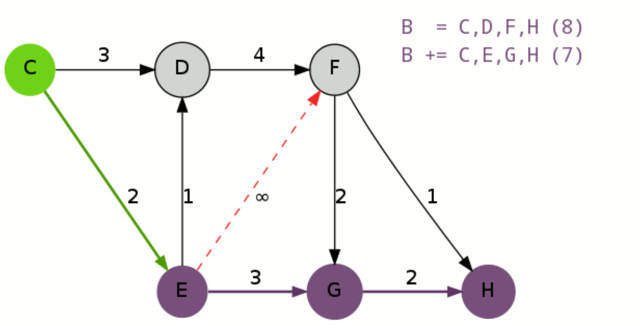

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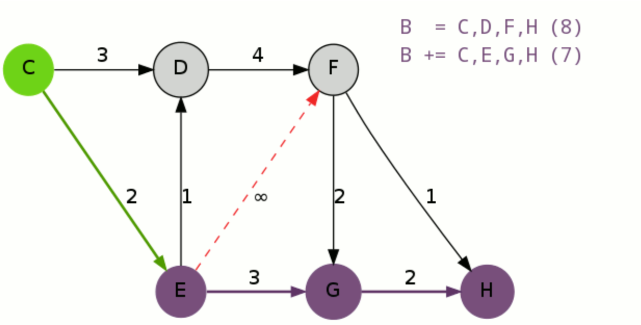

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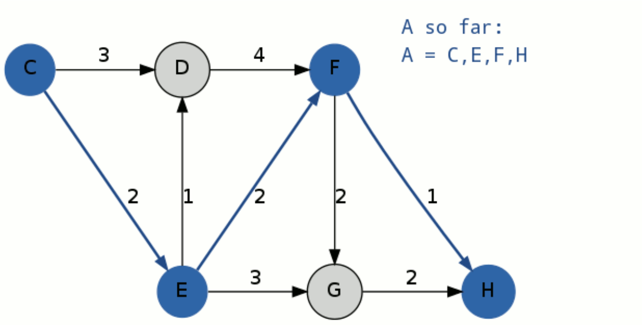

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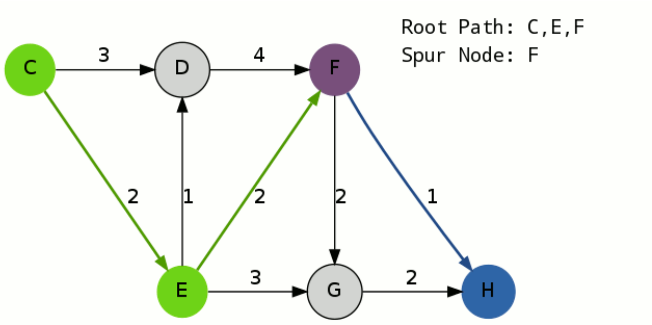
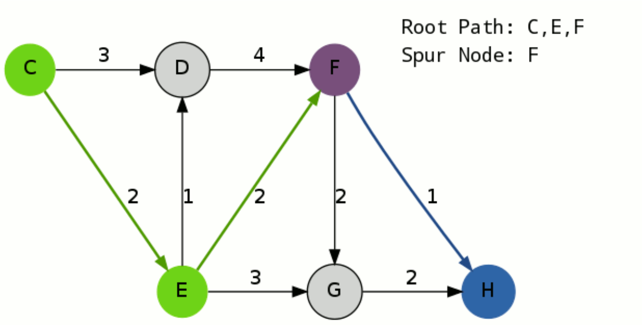


Yens Algorithm



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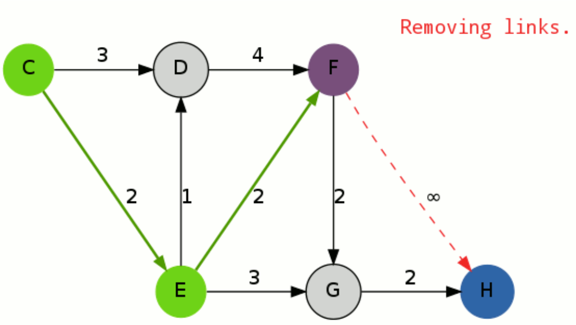

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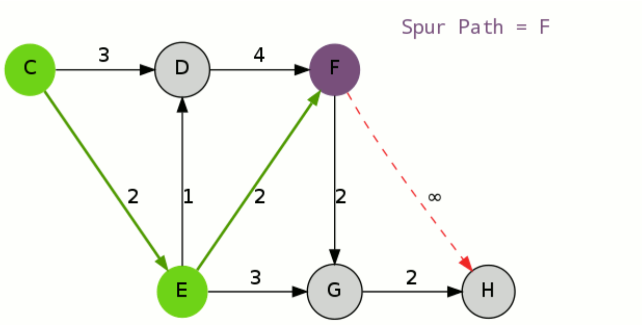

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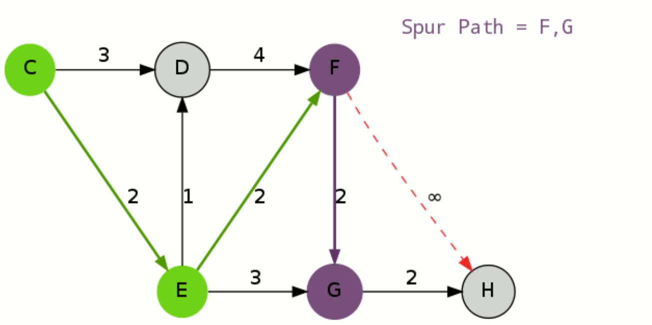

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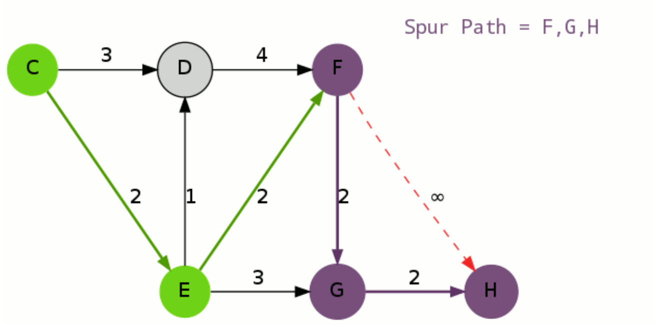

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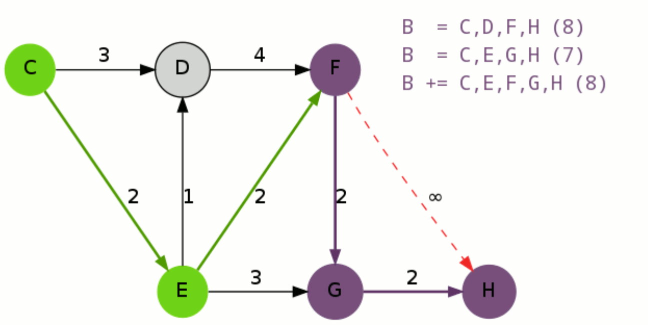

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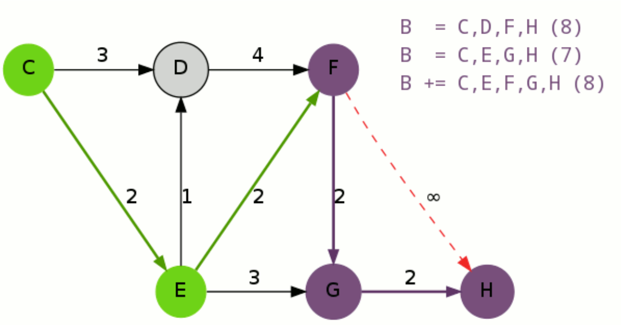

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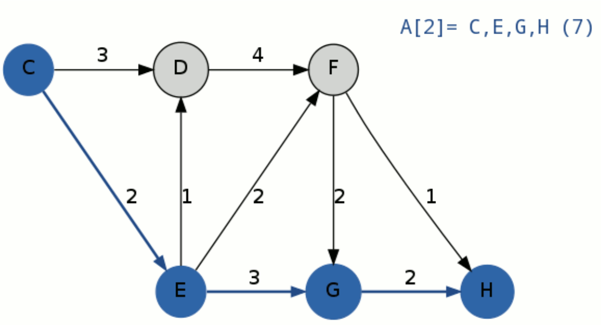

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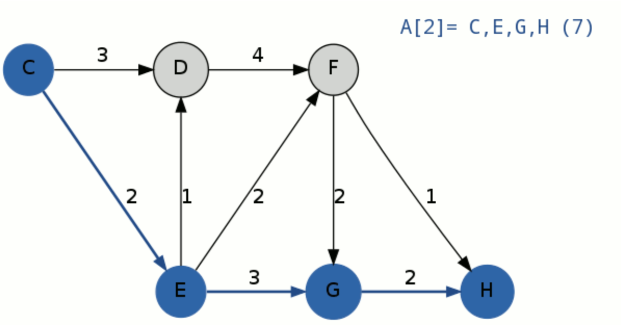

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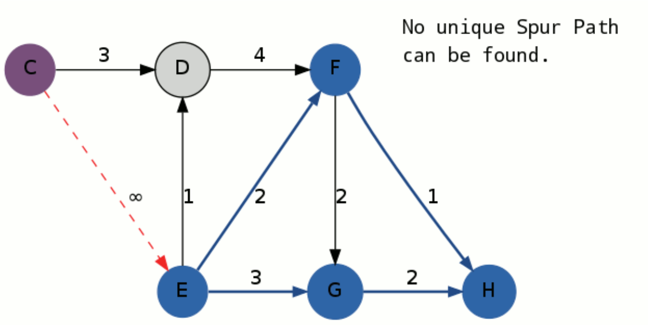

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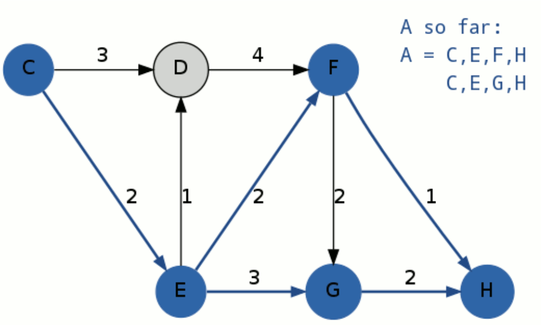

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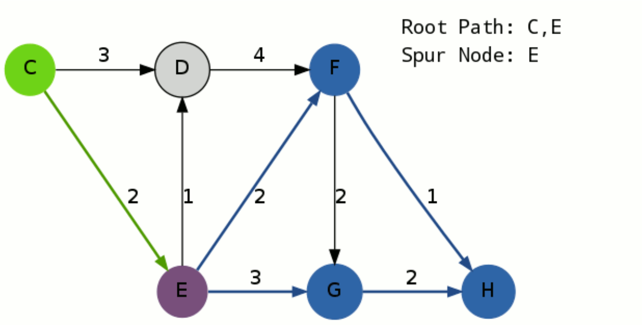

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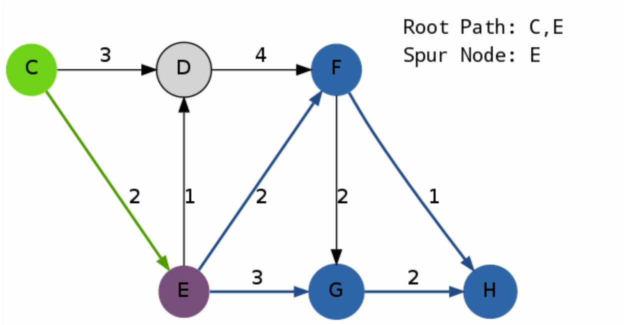

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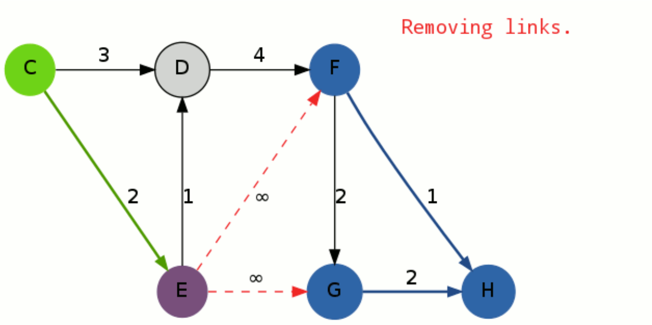

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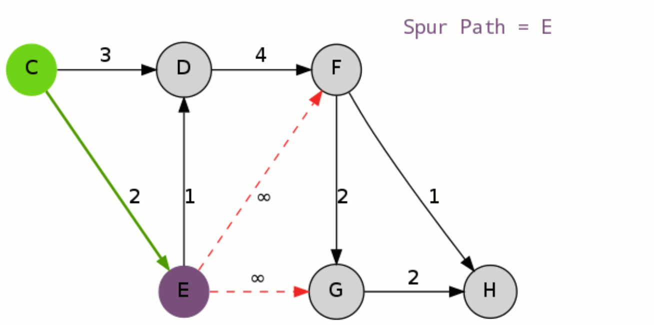

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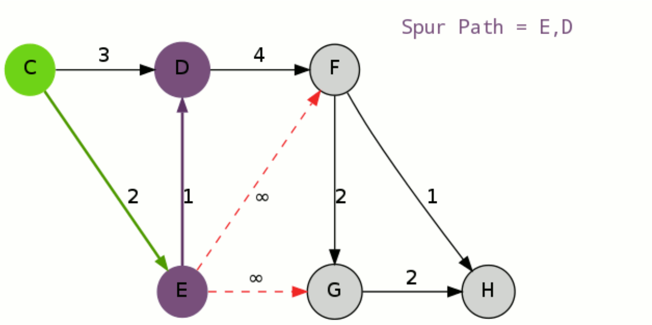

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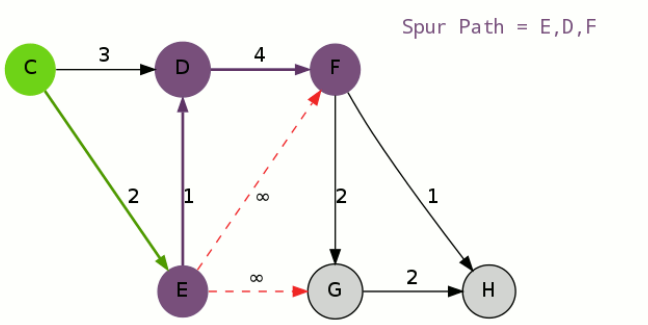

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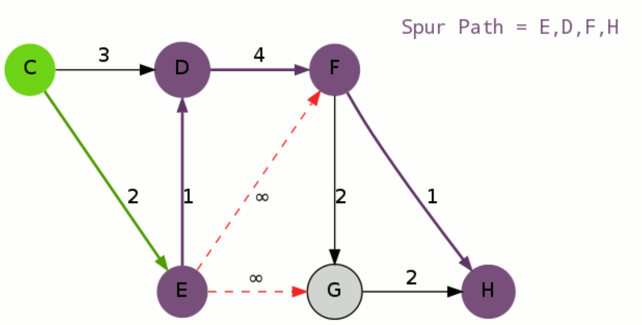

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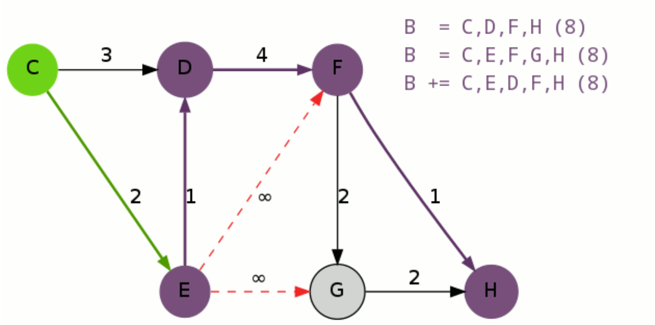

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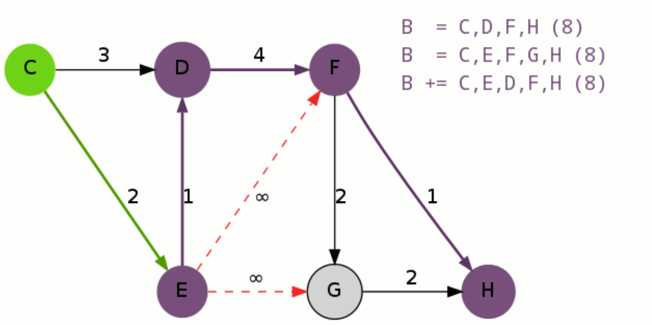

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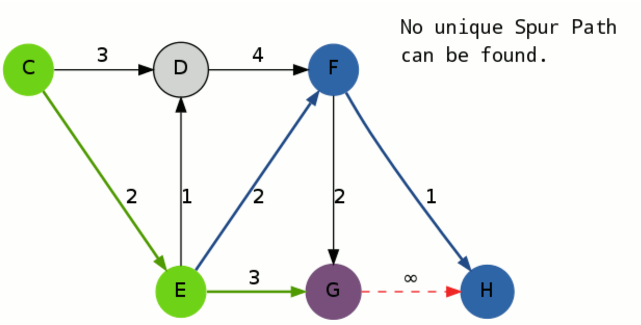

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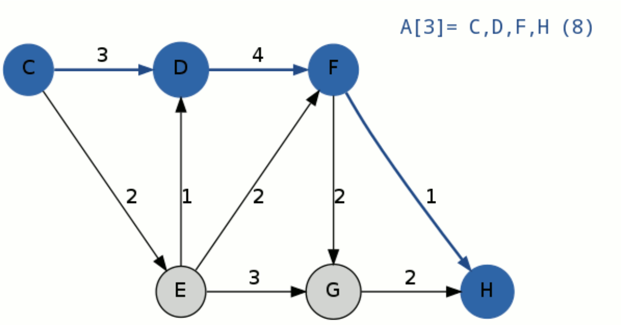

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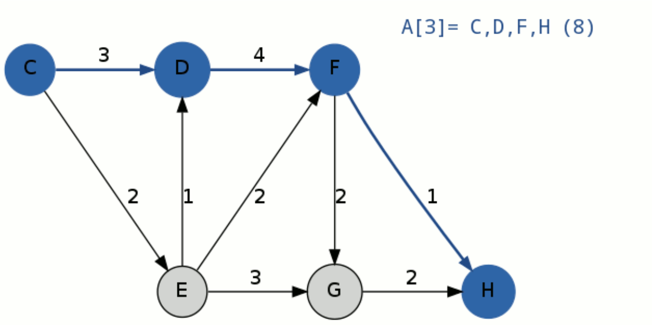

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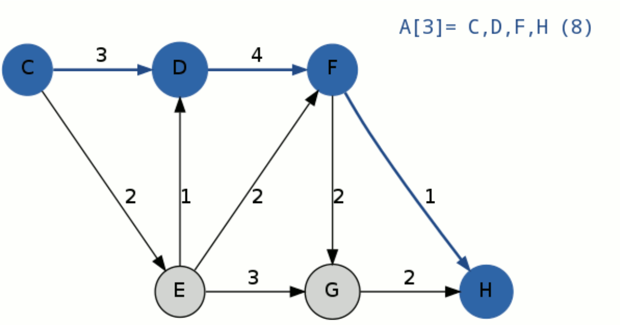

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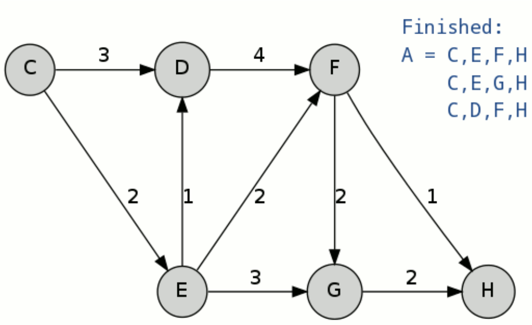

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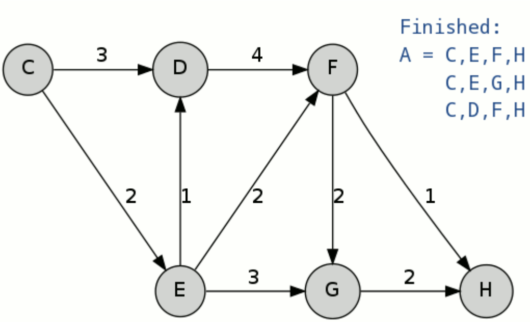

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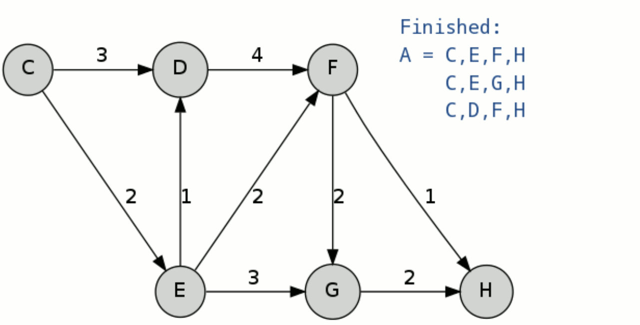


Yens Algorithm

- Implementation details omitted
- Runtime : dominated by $K \cdot |V|$ Dijkstra calls
  (i.e., in any of the K iterations/partitionings max "length of current path" many Dijkstras (i.e., max. $|V|$ many Dijkstras))

K Shortest Hyperpaths Algorithm[5]

Setup:	$H = (V, E)$	directed hypergraph
	$s, t \in V$	s is hyperconnected to t.
	$\mathcal{P} =$	$\{\pi \mid \pi$ is a hyperpath from s to $t\}$
	$\pi_{st} \in \mathcal{P}$	is the shortest.

π_{st} :

$$\left.\begin{array}{l} \text{Topological ordering} \\ \quad (s, u_1, ..., u_{q-2}, u_{q-1}, u_t) \\ \text{Predecessor function} \\ \quad p : V_\pi \to E_\pi \end{array}\right\} \Rightarrow (p(u_1), p(u_2), ..., p(u_{q-1}), p(t))$$

[5] Finding the K shortest hyperpaths, Nielsen et. al., 2005

K Shortest Hyperpaths

Setup:
$$\begin{aligned} \mathcal{P} = &\quad \{\pi | \pi \text{ is a hyperpath from } s \text{ to } t\} \\ \pi_{st} \in \mathcal{P} &\quad \text{is the shortest.} \\ E_{\pi_{st}} = &\quad (p(u_1), p(u_2), ..., p(u_{q-1}), p(t)) \end{aligned}$$

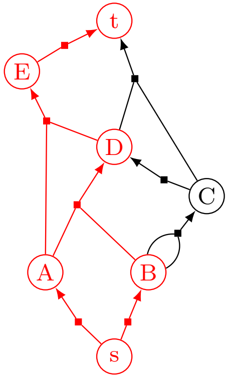
Partition $\mathcal{P} \setminus \{\pi_{st}\}$ to $\mathcal{P}^i, 1 \leq i \leq q$ s.t.:

$$\pi \in \mathcal{P}^i$$
$$\Updownarrow$$
$$E_\pi = (e_1, e_2, ..., e_{m-1}, e_m, p(u_{i+1}), ..., p(u_{q-2}), p(u_{q-1}), p(t)),$$
$$e_m \neq p(u_i)$$

K Shortest Hyperpaths

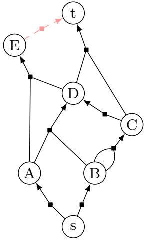
1st iteration

$L = \{(H, \pi_1)\}$

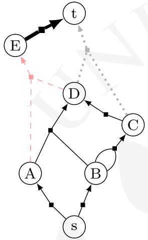


$H : \{\pi_1, \pi_2, \pi_3, \pi_4\}$

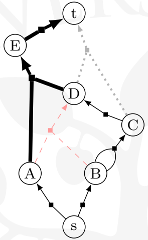
π_1 is red



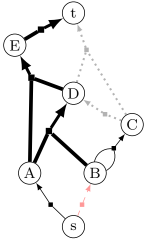
$H^5 : \{\pi_2, \pi_4\}$



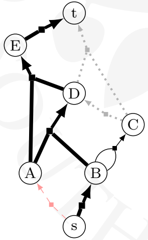
$H^4 : \emptyset$



$H^3 : \{\pi_3\}$



$H^2 : \emptyset$

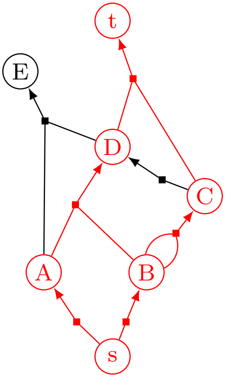


$H^1 : \emptyset$

K Shortest Hyperpaths

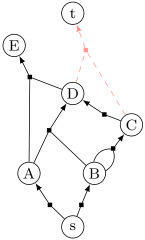
2nd iteration

$L = \{(H^5, \pi_2), (H^3, \pi_3)\}$

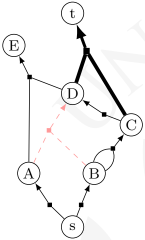


H^5 : $\{\pi_2, \pi_4\}$

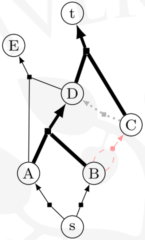
π_2 is red



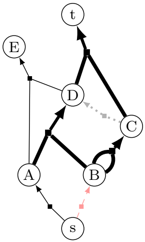
H^{55} : $\emptyset$



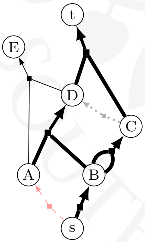
H^{54} : $\{\pi_4\}$



H^{53} : $\emptyset$



H^{52} : $\emptyset$



H^{51} : $\emptyset$

Backwards Branching

$\text{Back-Branch}(\widetilde{H}, \widetilde{\pi})$

$\mathcal{B} = \emptyset$
for $i = 1$ **to** q
    Let $\widetilde{H}^i$ be a new hypergraph
    $\widetilde{H}^i.V = \widetilde{H}.V$
    // Remove hyperarc from of $\widetilde{H}$
    $\widetilde{H}^i.E = \widetilde{H}.E \setminus \{\widetilde{\pi}.p(u_i)\}$
    // Fix back tree
    for $j = i + 1$ **to** q
        $\widetilde{H}^i.\mathrm{BS}(u_j) = \{\widetilde{\pi}.p(u_j)\}$
    $\mathcal{B} = \mathcal{B} \cup \{\widetilde{H}^i\}$
return $\mathcal{B}$

Running time: $O(|V|(|V| + |E|))$

K Shortest Hyperpaths

$\text{K-SYNTHESIS}(H, s, t, K)$

Let L be a new priority queue
$\pi = \text{SHORTESTPATH}(H, s, t)$
$\text{INSERT}\,(L, (H, \pi))$
for $k = 1$ **to** K
    if $L = \emptyset$
        break
    $(H', \pi') = \text{EXTRACT-MIN}(L)$
    output π'
    if $k == K$
        break
    for each H^i in $\text{BACK-BRANCH}(H', \pi')$
        $\pi^i = \text{SHORTESTPATH}(H^i, s, t)$
        if $W(\pi^i) < \infty$
            $\text{INSERT}\,\left(L, (H^i, \pi^i)\right)$

Running time: $O(K|V|(|V| + |E|) + K \lg K)$