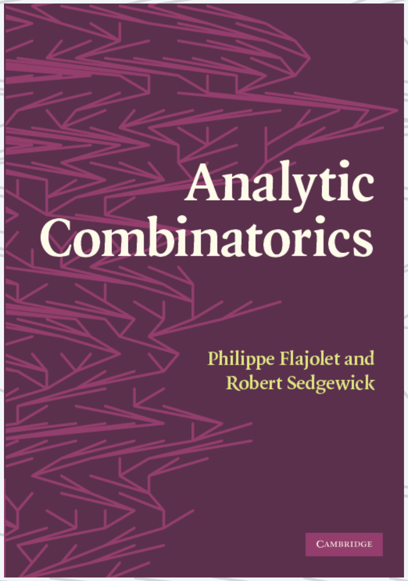
ANALYTIC COMBINATORICS

PART TWO



http://ac.cs.princeton.edu

4. Complex Analysis, Rational and Meromorphic Asymptotics

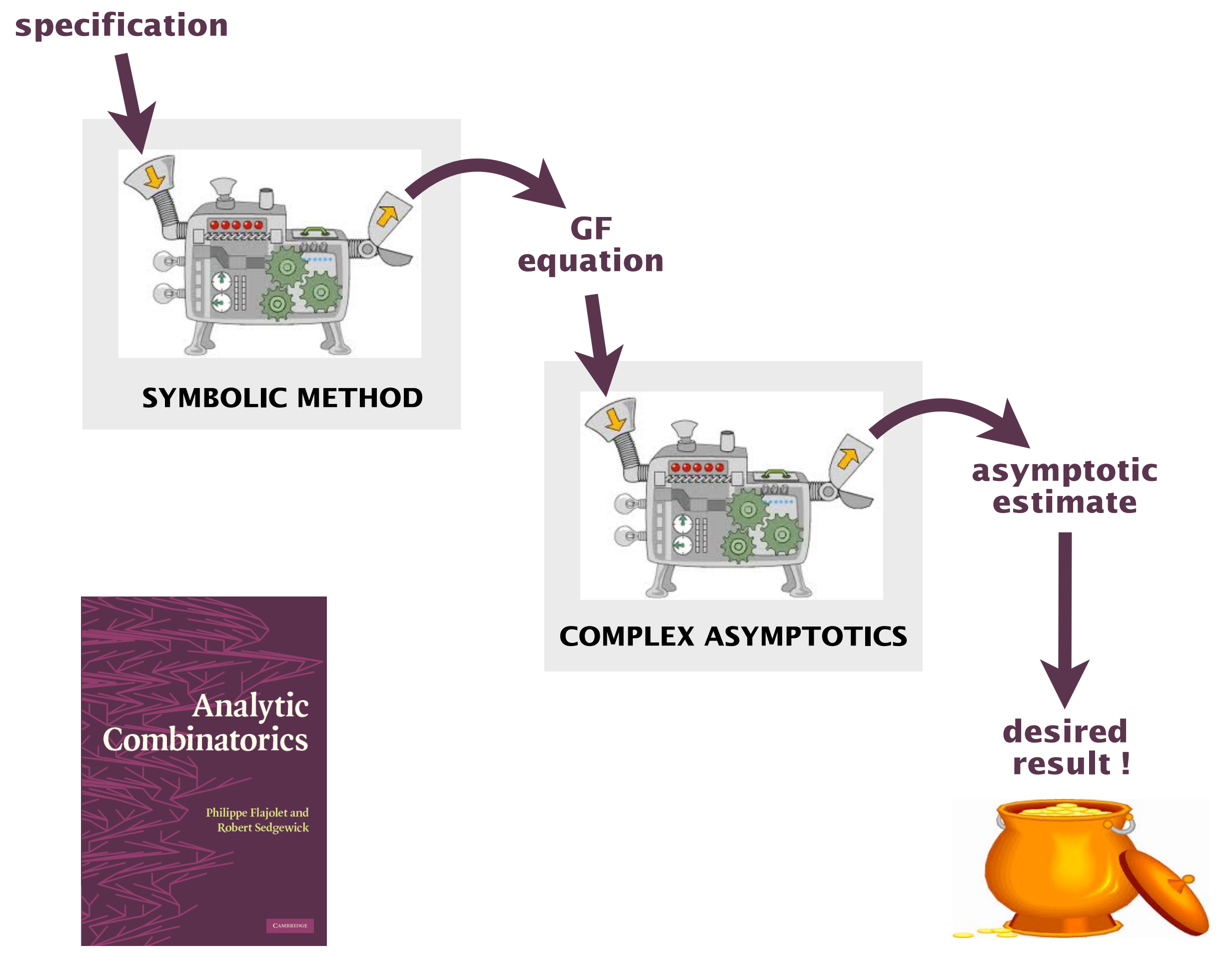
Analytic combinatorics overview

A. SYMBOLIC METHOD

1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



specification

SYMBOLIC METHOD

GF equation

COMPLEX ASYMPTOTICS

asymptotic estimate

desired result !

Analytic combinatorics overview

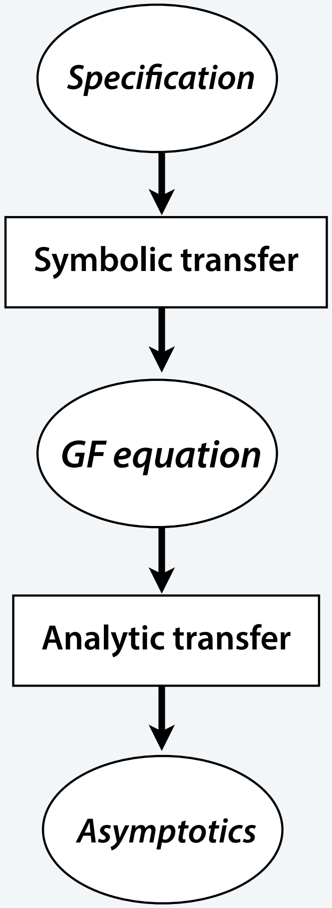
To analyze properties of a large combinatorial structure:

1. Use the symbolic method (lectures 1 and 2).
   - Define a *class* of combinatorial objects.
   - Define a notion of *size* (and associated GF)
   - Use standard constructions to *specify* the structure.
   - Use a *symbolic* transfer theorem.

Result: A direct derivation of a GF equation.

2. Use complex asymptotics (starting with this lecture).
   - Start with GF equation.
   - Use an *analytic* transfer theorem.

Result: Asymptotic estimates of the desired properties.



Ex. Derangements

$$\mathbf{D} = SET(CYC_{>1}(\mathbf{Z}))$$

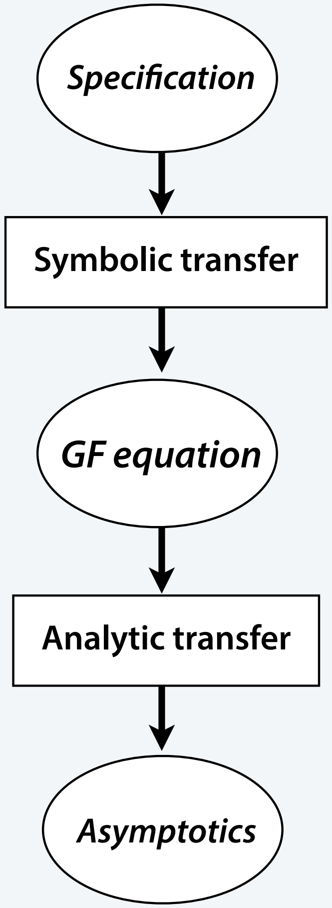
$$\Big\downarrow$$

$$D(z) = \frac{e^{-z}}{1-z}$$

$$\Big\downarrow$$

$$D_N \sim \frac{1}{e}$$

A shift in point of view



Specification

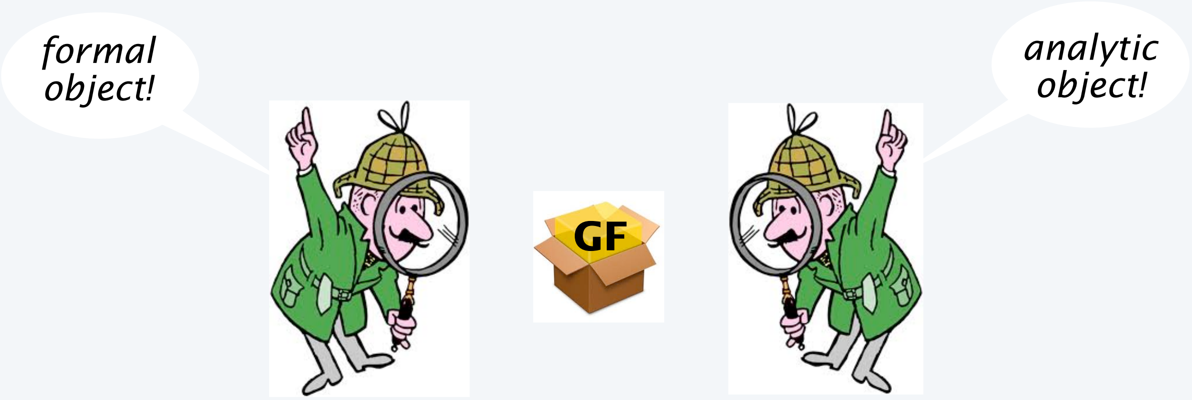
Symbolic transfer

GF equation

Analytic transfer

Asymptotics

generating functions are treated as *formal* objects

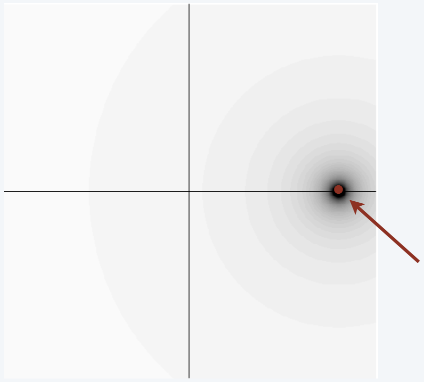


generating functions are treated as *analytic* objects

GFs as analytic objects (complex)

Q. What happens when we assign *complex* values to a GF?

$$f(z) = \frac{e^{-z}}{1-z}$$



stay tuned for
interpretation
of plot

singularity

A. We can use a *series representation* (in a certain domain) that allows us to extract coefficients.

Same useful concepts:

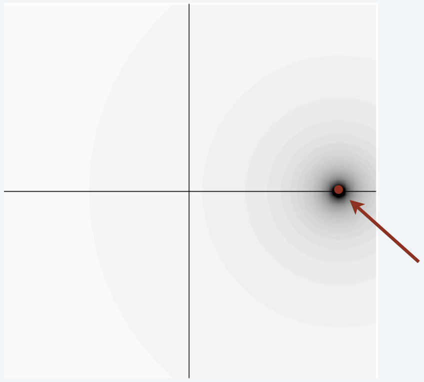
Differentiation:	Compute derivative term-by-term where series is valid.
Singularities:	Points at which series ceases to be valid.
Continuation:	Use functional representation even where series may diverge.

GFs as analytic objects (complex)

Q. What happens when we assign *complex* values to a GF?

$$f(z) = \frac{e^{-z}}{1 - z}$$

stay tuned for
interpretation
of plot →



A. A surprise!

Singularities provide *full information* on growth of GF coefficients!

“Singularities provide a royal road to coefficient asymptotics.”

General form of coefficients of combinatorial GFs

subexponential factor

$$[z^N]F(z) = A^N\theta(N)$$

exponential growth factor

First principle of coefficient asymptotics

The *location* of a function’s singularities dictates the *exponential growth* of its coefficients.

Second principle of coefficient asymptotics

The *nature* of a function’s singularities dictates the *subexponential factor* of the growth.

Examples (preview):

	GF	GF type	singularities		exponential growth	subexp. factor
			location	nature		
strings with no 00	$B_2(z) = \frac{1 - z^2}{1 - 2z - z^3}$	*rational*	$1/\phi, 1/\hat{\phi}$	*pole*	ϕ^N	$\frac{1}{\sqrt{5}}$
derangements	$D(z) = \frac{e^{-z}}{1 - z}$	*meromorphic*	1	*pole*	1^N	e^{-1}
Catalan trees	$G(z) = \frac{1 + \sqrt{1 - 4z}}{2}$	*analytic*	$1/4$	*square root*	4^N	$\frac{1}{4\sqrt{\pi N^3}}$

Theory of complex functions

Quintessential example of the power of abstraction.

Start by defining i to be the square root of -1 so that $i^2 = -1$

Continue by exploring natural definitions of basic operations

- Addition
- Multiplication
- Division
- Exponentiation
- Functions
- Differentiation
- Integration

$1 + i$

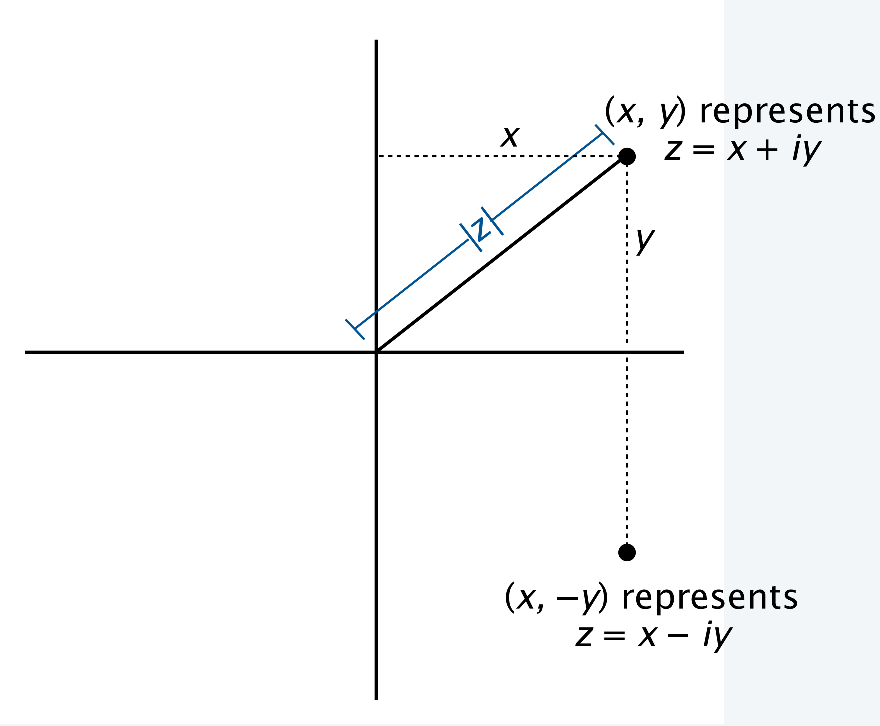
Standard conventions

$z = x + iy$

real part	$\Re z \equiv x$
imaginary part	$\Im z \equiv y$
absolute value	$\|z\| \equiv \sqrt{x^2 + y^2}$
conjugate	$\bar{z} = x - iy$

Quick exercise: $z\bar{z} = |z|^2$

Correspondence with points in the plane



Analytic functions

Definition. A function $f(z)$ defined in Ω is *analytic* at a point z_0 in Ω iff for z in an open disc in Ω centered at z_0 it is representable by a power-series expansion $\displaystyle f(z) = \sum_{N \geq 0} c_N (z - z_0)^N$

Examples:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ldots$$ is analytic for $|z| < 1$.

$$e^z \equiv 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$ is analytic for $|z| < \infty$.

Aside (continued): plotting complex functions

is also an easy (and instructive!) programming exercise.

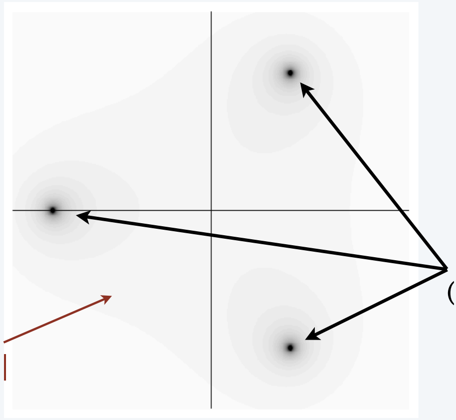
```
public class Plot2Dez
{
  public static void show(ComplexFunction f, int sz)
  {
   StdDraw.setCanvasSize(sz, sz);
   StdDraw.setXscale(0, sz);
   StdDraw.setYscale(0, sz);
   double scale = 2.5;
   for (int i = 0; i < sz; i++)
      for (int j = 0; j < sz; j++)
      {
         double x = ((1.0*i)/sz - .5)*scale;
         double y = ((1.0*j)/sz - .5)*scale;
         Complex z = new Complex(x, y);
         double val = f.eval(z).abs()*10;
         int t;
         if  (val <   0) t = 255;
         else if (val > 255) t = 0;
         else t = 255 - (int) val;
         Color c = new Color(t, t, t);
         StdDraw.setPenColor(c);
         StdDraw.pixel(i, j);
      }
   Color c = new Color(0, 0, 0);
   StdDraw.setPenColor(c);
   StdDraw.line(sz/2, 0, sz/2, sz);
   StdDraw.line(0, sz/2, sz, sz/2);
   StdDraw.show();
  }
}
```

arbitrary factor
to emphasize growth

```
public class Example implements ComplexFunction
{
    public Complex eval(Complex z)
    {  // {1 \over 1+z^3}
        Complex one = new Complex(1, 0);
        Complex d = one.plus(z.times(z.times(z)));
        return d.reciprocal();
    }
   public static void main(String[] args)
   {  Plot2D.show(new Example(), 512);  }
}
```

our convention:
plots are in the 2.5 by 2.5 square
centered at the origin

$$\frac{1}{1+z^3}$$

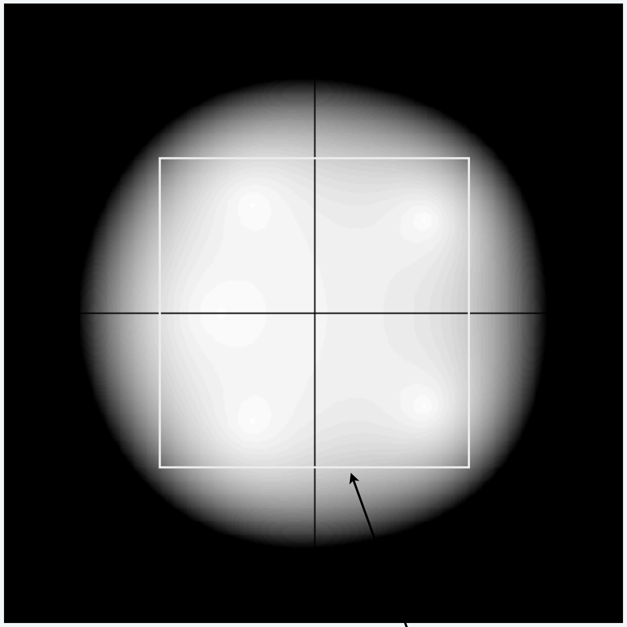


singularities
(where $|f| \to \infty$)

darkness of pixel at (x, y)
is proportional to $|f(x + iy)|$

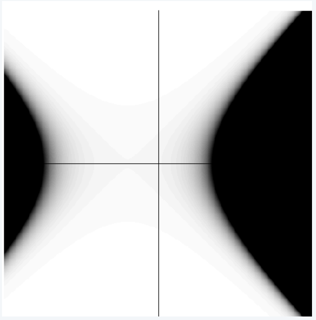
Entire functions (analytic everywhere)

$1 + z + z^5$



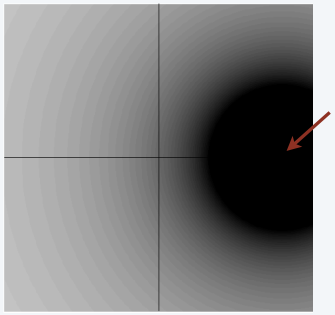
our convention:
highlight the 2.5 by 2.5 square
centered at the origin
when plotting a bigger square

$e^{z+z^2/2}$

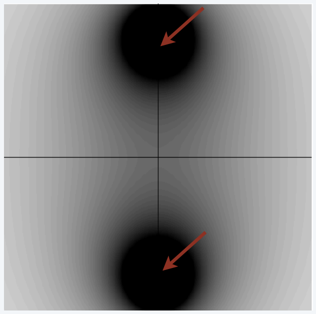


Plots of various rational functions

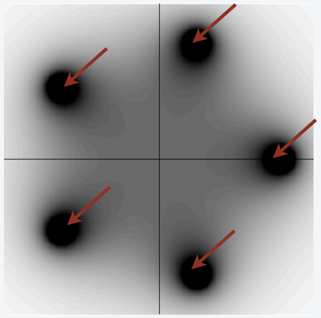
$$\frac{1}{1-z}$$



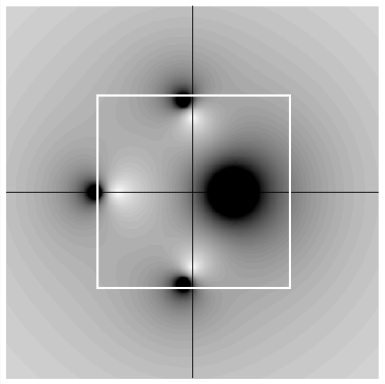
$$\frac{1}{1+z^2}$$



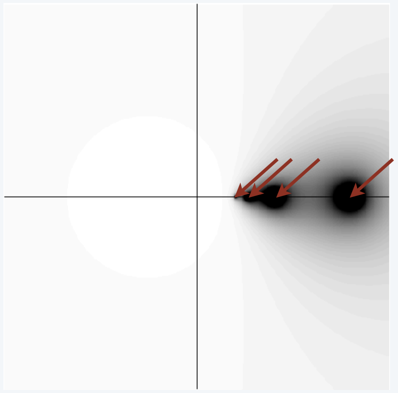
$$\frac{1}{1-z^5}$$



$$\frac{1+z+z^2+z^3}{1-z-z^2-z^3-z^4}$$



$$\frac{z^4}{(1-z)(1-2z)(1-3z)(1-4z)}$$



Meromorphic functions

are complex functions that can be expressed as the ratio of two *analytic functions*.

Note: All rational functions are meromorphic.

$$D(z) = \boxed{\frac{e^{-z}}{1-z}} \qquad G(z) = \frac{1+\sqrt{1-4z}}{2} \qquad R(z) = \boxed{\frac{1}{2-e^z}} \qquad B_P(z) = \boxed{\frac{1+z+z^2+\ldots+z^{P-1}}{1-z-z^2-\ldots-z^P}}$$

$$S_r(z) = \boxed{\frac{z^r}{(1-z)(1-2z)\ldots(1-rz)}} \qquad C(z) = \frac{1}{1-z}\ln\frac{1}{1-z} \qquad I(z) = e^{z+z^2/2}$$

Approach:

- Use *contour integration* to expand into terms for which coefficient extraction is easy.
- Focus on the largest term to approximate.

[Same approach as for rationals, resulting in a more general transfer theorem.]

Meromorphic functions

Definition. A function $h(z)$ defined in Ω is *meromorphic* at z_0 in Ω iff for z in a neighborhood of z_0 with $z \neq z_0$ it can be represented as $f(z)/g(z)$, where $f(z)$ and $g(z)$ are analytic at z_0.

Useful facts:

- A function $h(z)$ that is meromorphic at z_0 admits an expansion of the form

$$h(z) = \frac{h_{-M}}{(z-z_0)^M} + \ldots + \frac{h_{-2}}{(z-z_0)^2} + \frac{h_{-1}}{(z-z_0)} + h_0 + h_1(z-z_0) + h_2(z-z_0)^2 + \ldots$$

and is said to have a pole of order M at z_0.

Proof sketch: If z_0 is a zero of $g(z)$ then $g(z) = (z - z_0)^M G(z)$.
Expand the analytic function $f(z)/G(z)$ at z_0.

- The coefficient h_{-1} is called the residue of $h(z)$ at z_0, written $\operatorname*{Res}_{z=z_0} h(z)$.
- If $h(z)$ has a pole of order M at z_0, the function $(z - z_0)^M h(z)$ is analytic at z_0.

A function is meromorphic in Ω iff it is analytic in Ω *except for a set of isolated singularities*, its *poles*.

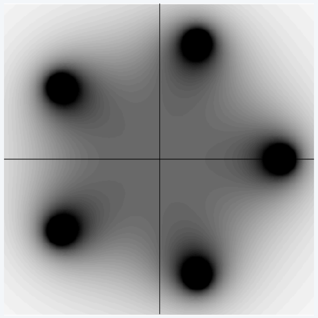
Meromorphic functions

Definition. A function $h(z)$ defined in Ω is *meromorphic* at z_0 in Ω iff for z in a neighborhood of z_0 with $z \neq z_0$ it can be represented as $f(z)/g(z)$, where $f(z)$ and $g(z)$ are analytic at z_0.

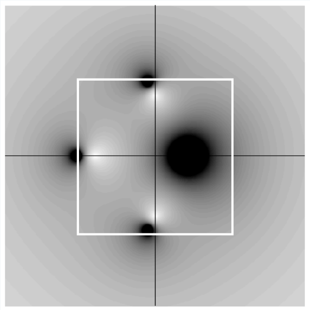
function	region of meromorphicity
$1 + z + z^2$	everywhere
$\frac{1}{z}$	everywhere but $z = 0$
$D(z) = \frac{e^{-z}}{1 - z}$	everywhere but $z = 1$
$\frac{1}{1 + z^2}$	everywhere but $z = \pm i$
$S_r(z) = \frac{z^r}{(1 - z)(1 - 2z) \ldots (1 - rz)}$	everywhere but $z = 1, 1/2, 1/3, \ldots$
$R(z) = \frac{1}{2 - e^z}$	everywhere but $z = \ln 2 \pm 2\pi k i$

Plots of various meromorphic functions

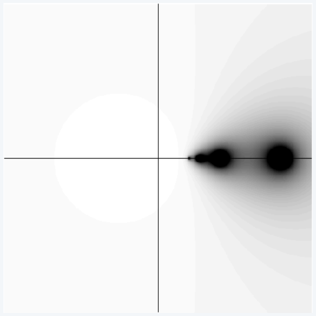
$$\frac{1}{1 - z^5}$$



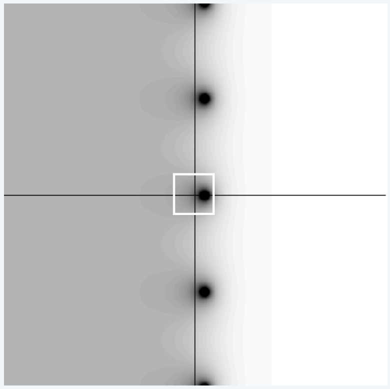
$$\frac{1 + z + z^2 + z^3}{1 - z - z^2 - z^3 - z^4}$$



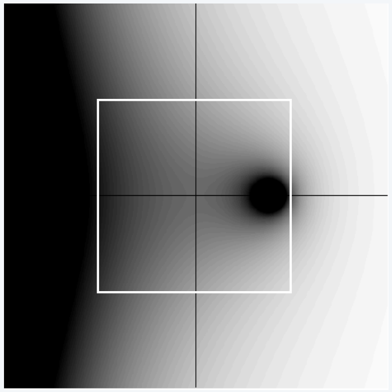
$$\frac{z^4}{(1 - z)(1 - 2z)(1 - 3z)(1 - 4z)}$$



$$\frac{1}{2 - e^z}$$



$$\frac{e^{-z}}{1 - z}$$



AC transfer theorem for meromorphic GFs (leading term)

Theorem. Suppose that $h(z) = f(z)/g(z)$ is meromorphic in $|z| \le R$ and analytic both at $z = 0$ and at all points $|z| = R$. If α is a unique closest pole to the origin of $h(z)$ in R, then α is real and $\boxed{[z^N]\frac{f(z)}{g(z)} \sim c\beta^N N^{M-1}}$ where M is the order of α, $\boxed{c = (-1)^M \frac{Mf(\alpha)}{\alpha^M g^{(M)}(\alpha)}}$ and β = 1/α.

Proof sketch for $M = 1$:

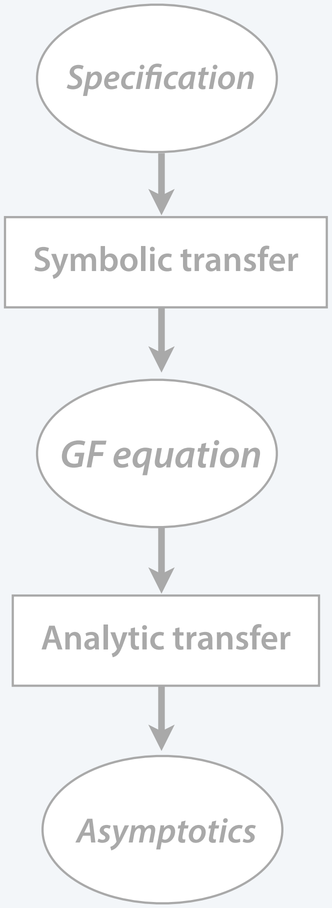
- Series expansion (valid near α): $h(z) = \frac{h_{-1}}{\alpha - z} + h_0 + h_1(\alpha - z) + h_2(\alpha - z)^2 + \ldots$ ← elementary from Pringsheim's and coefficient extraction theorems
- One way to calculate constant: $h_{-1} = \lim_{z\to\alpha}(\alpha - z)h(z)$
- Approximation at α: $h(z) \sim \frac{h_{-1}}{\alpha - z} = \frac{1}{\alpha}\frac{h_{-1}}{1 - z/\alpha} = \frac{h_{-1}}{\alpha}\sum_{N\ge 0}\frac{z^N}{\alpha^N}$

See next slide for calculation of c and $M > 1$.

Notes:

- Error is *exponentially small* (and next term may involve periodicities due to complex roots).
- Result is the same as for rational functions.

Bottom line



Specification

Symbolic transfer

GF equation

Analytic transfer

Asymptotics

Analytic transfer for meromorphic GFs: $f(z)/g(z) \sim c\,\beta^N$

- Compute the dominant pole α (smallest real with $g(z) = 0$).
- (Check that no others have the same magnitude.)
- Compute the residue $h_{-1} = -f(\alpha)/g'(\alpha)$.
- Constant c is $h_{-1}\,/\alpha$.
- Exponential growth factor β is $1/\alpha$

Not order 1 if $g'(\alpha) = 0$.
Adjust to (slightly) more
complicated order M case.

AC transfer for meromorphic GFs

Analytic transfer for meromorphic GFs: $f(z)/g(z) \sim c\,\beta^N$

- Compute the dominant pole α (smallest real with $g(z) = 0$).
- (Check that no others have the same magnitude.)
- Compute the residue $h_{-1} = -f(\alpha)/g'(\alpha)$.
- Constant c is $h_{-1}\ /\alpha$.
- Exponential growth factor β is 1/α

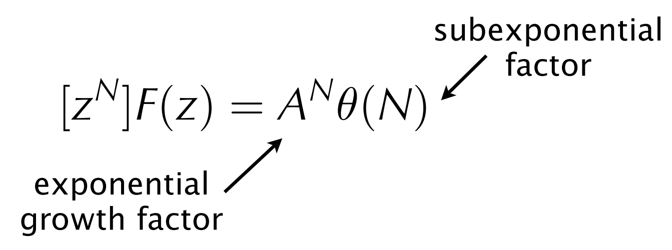
Examples.

$h(z) = f(z)/g(z)$	α	h_{-1}	$[z^N]h(z)$
$\dfrac{z}{1-z-z^2}$	$\hat{\phi} = \dfrac{1}{\phi}$	$\dfrac{\hat{\phi}}{(1+2\hat{\phi})} = \dfrac{\hat{\phi}}{\sqrt{5}}$	$\sim \dfrac{1}{\sqrt{5}}\phi^N$
$\dfrac{e^{-z}}{1-z}$	1	$\dfrac{1}{e}$	$\dfrac{1}{e}$
$\dfrac{e^{-z-z^2/2-z^3/3}}{1-z}$	1	$\dfrac{1}{e^{H_3}}$	$\dfrac{1}{e^{H_3}}$

$$\hat{\phi} = \frac{\sqrt{5}-1}{2}$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

General form of coefficients of combinatorial GFs (revisited)



$$[z^N]F(z) = A^N\theta(N)$$

subexponential factor

exponential growth factor

First principle of coefficient asymptotics

The *location* of a function’s singularities dictates the *exponential growth* of its coefficients.

Second principle of coefficient asymptotics

The *nature* of a function’s singularities dictates the *subexponential factor* of the growth.

When F(z) is a meromorphic function $f(z)/g(z)$

- If the smallest real root of $g(z)$ is α then the exponential growth factor is $1/\alpha$.
- If α is a pole of order M, then the subexponential factor is cN^{M-1}.