Master Thesis for the Cand.Scient degree in Mathematics at the University of Copenhagen.

Generalized Murray- von Neumann Dimension & L^2-Homology for finite von Neumann Algebras

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by David Kyed
m99dk@math.ku.dk

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Supervisor: Ryszard Nest

Introduction

In the present thesis, we give an introduction to the generalized Murray von Neumann dimension developed by W. Lück and apply the results to describe the theory of L^2-homology and L^2-Betti numbers for von Neumann algebras, developed by A. Connes and D. Shlyakhtenko.
The contents of the thesis is summarized in the following paragraph.

Summary: The thesis contains three chapters, where the major part of the theory is contained in the first and the third chapter. The second chapter serves as an interlude, connecting the first and the third chapter.

In the first chapter, we introduce the generalized Murray von Neumann dimension, as defined by W. Lück in [Lüc98]. This notion of dimension is defined on the category of modules over a finite von Neumann algebra $\mathscr{M}$[1], and extends the classical notion of Murray- von Neumann dimension of finitely generated projective modules. Following Lück, [Lüc97] and [Lüc98], we prove that this generalized dimension inherits many of the properties of the classical Murray- von Neumann dimension. The most important of these properties, can be found in Theorem 1.4.7.
The reader interested in applications of this dimension theory, other than L^2-Betti numbers for von Neumann algebras, may find these in [Lüc02]. Lastly, we discuss the so-called induction functor associated with a pair of von Neumann algebras (one contained in the other), and proof that this functor preserves the extended dimension. (see e.g. Theorem 1.5.1)

In the second chapter, we first recapitulate some basic facts on Hochschild homology which will be needed in order to describe the theory of L^2-homology for von Neumann algebras. Next we give a brief introduction to group von Neumann algebras and define the notion of group homology (and in particular L^2-homology) for a discrete group G. We end the chapter with a discussion of the induction-functor in the context of group von Neumann algebras.

The third chapter contains the primary material of this thesis; namely a presentation of the theory of L^2-Homology and L^2-Betti numbers for finite von Neumann algebras, as defined by A. Connes and D. Shlyakhtenko in [CS03]. The L^2-homology of a finite von Neumann algebra $\mathscr{M}$, is defined as the Hocschild homology $H_*(\mathscr{M}, \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$ and because of the choice of coefficients, these groups all become left modules over the von Neumann algebra tensor product $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$.
Using the generalized Murray- von Neumann dimension, the Hochschild modules therefore all have a dimension over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$; and the L^2-Betti numbers of $\mathscr{M}$ is defined as this sequence of dimensions. Next we turn towards the development of computational results and, in particular, we prove the so-called compression formula, (Theorem 3.2.8) which gives a relation between the L^2- Betti numbers of a finite factor $\mathscr{M}$ and the L^2-Betti numbers of the corner-algebra $p\mathscr{M}p$ associated with a projection $p \in \mathscr{M}$. Finally, we give a more detailed description of the zero'th and first L^2-Betti number and compute these in some special cases.

[1] See e.g. Definition 1.1.10

About this text: Reading this project requires basic knowledge of homological algebra. The relevant theory of homological algebra can, for instance, be found in [CE], [Wei] and/or [Fox]. Moreover, some knowledge about operator algebras — and especially von Neumann algebras — is required; in particular in relation to the contents of Chapter 3. We refer to [KR1] and [KR2] for background information on operator algebras.
On the last page, a list of notation can be found. It does not contain all notation used in the text, but shall be considered as a supplement for the text, containing short descriptions of parts of the notation which appear without further explanation in the main text. When a result (not developed by the author) is presented, we will give a reference to the (if possible) original source. This will be done in a bracket placed just before the statement, in the Lemma, Proposition or Theorem in question, is given. However, if the results are classical, we will sometimes give a general reference in the beginning of the relevant section.
Despite the fact that the thesis has only one author, most of the text is written in plural form. So, when the word "we" appear (as it already has), it is to be interpreted as the reader and author in conjunction.

Throughout the text, unless explicitly stated otherwise, the following assumptions will be made:

- All vector spaces are assumed to be over the complex numbers.

- All algebras are assumed to be associative.

- The symbol $\otimes$ will be used to denote algebraic tensor products (of modules, algebras, etc.) while the symbol $\bar{\otimes}$ will be used to denote completed tensor products. (of Hilbert spaces, von Neumann algebras, etc.)

Finally, I would like to thank Ryszard Nest and Erik Christensen, for patiently listening to all my questions and helping me find answers to them.

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David Kyed.

Contents

Chapter 1

The generalized Murray- von Neumann Dimension

In this chapter we introduce the notion of Murray- von Neumann dimension of a finitely generated projective module over a finite von Neumann algebra $\mathscr{M}$. Subsequently, we generalize it to cover all modules over $\mathscr{M}$ and prove that many of the attractive properties carry over to this generalized dimension function.
We start out, by presenting some basic results of module theory, which will be needed in the sequel.

1.1 Module theory

Let R be a unital $*$-ring of characteristic zero and let $\mathrm{Mod}(R)$ denote the category of left R-modules, with morphisms being the R-linear maps. We denote by 1 the unit in R and we will often also use the symbol 1 to denote the identity homomorphism on a given object in $\mathrm{Mod}(R)$. Unless otherwise stated, throughout this section all modules are left modules and the term "ideal" will always mean left ideal.

Lemma 1.1.1. *If P is a finitely generated projective R-module, then P is isomorphic to the image of an idempotent R-homomorphism $p : R^n \to R^n$ for some $n \in \mathbb{N}$.*

Proof. Since P is finitely generated and projective, P is isomorphic to a direct summand P' in R^n for some $n \in \mathbb{N}$. That is, $R^n = P' \oplus Q'$ for a suitable submodule Q' in R^n.
We now define $p : R^n \to R^n$ by

$$p(x, y) = (x, 0) \quad \text{for } x \in P', y \in Q'.$$

Clearly p has the desired properties. □

Remark 1.1.2. *Any morphism $f : R^n \to R^m$ is given by right-multiplication with a (unique) matrix $A \in M_{n,m}(R)$. To see this, we consider the natural basis $e_1, \dots, e_n$ for R^n, where*

$$e_i := (0, \dots, 0, 1, 0, \dots, 0), \qquad \text{(the 1 in the } i\text{'th position)}$$

and define the k'th row in A to be the image $f(e_k)$. Then $f(e_k) = e_k A$ for all $k \in \{1, \dots, n\}$ and since $(e_1, \dots, e_n)$ is a basis, the two maps must agree everywhere.
Clearly the relations $f(e_k) = e_k A$, $k = 1, \dots, n$, determines A completely.

Proposition 1.1.3. *If M is an R-module and $p : M \to M$ an idempotent R-homomorphism, then $M = pM \oplus (1 - p)M$.*

Proof. Using that p is idempotent, it is straightforward to check that

$$M \ni x \longmapsto (px, (1-p)x) \in pM \oplus (1-p)M,$$

is an isomorphism. □

Lemma 1.1.1 has an inverse.

Lemma 1.1.4. *If P is a projective R-module and $q : P \to P$ is an idempotent R-homomorphism, then qP is a projective R-module.*

Proof. By Proposition 1.1.3, we have $P = qP \oplus (1-q)P$ and since P is projective there exists a free R-module F, which contains P as a direct summand. Then F also contains qP as a direct summand and we conclude that qP is projective. □

Definition 1.1.5. *The ring R is said to be* semi-hereditary *if all finitely generated (left) ideals in R are projective as R-modules. More generally, a projective R-module M is said to be* semi-hereditary, *if all finitely generated R-submodules of M are projective.*

Proposition 1.1.6. [CE] *The following statements are equivalent*

(i) The ring R is semi-hereditary.

(ii) Every projective module over R is semi-hereditary.

Proof.

(ii)⇒ (i): Since R is free (and hence projective) as an R-module and since R is semi hereditary as an R-module if, and only if, it is semi hereditary as a ring, the implication follows.

(i)⇒ (ii): Since every projective module is isomorphic to a submodule in a free module and since projectivity is preserved under finite direct sums, it suffices to prove the following claim:

Each finitely generated submodule of a free module is the direct sum of a finite number of modules, each of which is isomorphic to a finitely generated ideal in R.

Proof of claim: Assume M to be a finitely generated submodule in the free module F and let $(x_i)_{i\in I}$ be a basis for F.
Since M is finitely generated, we can choose a minimal $n \in \mathbb{N}$ and $i_1, \dots, i_n \in I$ such that $M \subseteq \mathrm{span}_R\{x_{i_1}, \dots, x_{i_n}\} =: F'$.
We proceed by induction on the number n.
If $n = 1$ we have $F' = Rx_{i_1} \simeq R$ (since $\{x_i | i \in I\}$ is a basis) and hence the submodule M is isomorphic to a finitely generated R-submodule of R; that is, to a finitely generated ideal in R, which is projective by assumption.
Assume now, inductively, that the claim is proven for some fixed $n \in \mathbb{N}$ and consider the case $n + 1$. That is, assume that

$$F' = \mathrm{span}_R\{x_{i_1}, \dots, x_{i_n}, x_{i_{n+1}}\} = \mathrm{span}_R\{x_{i_1}, \dots, x_{i_n}\} \oplus Rx_{i_{n+1}}.$$

Then the R-linear map $f : \mathrm{span}_R\{x_{i_1}, \dots, x_{i_n}\} \oplus Rx_{i_{n+1}} \longrightarrow R$ given by

$$(v, rx_{i_{n+1}}) \longmapsto r \in R,$$

maps M onto a finitely generated ideal J in R, which by assumption is projective. We then have a short-exact sequence

$$0 \longrightarrow \ker(f) \cap M \xrightarrow{\subseteq} M \xrightarrow{f} J \longrightarrow 0 \; ,$$

which splits (since J is projective) so that $M \simeq (\ker(f) \cap M) \oplus J$. Since $\ker(f) \cap M$ is a finitely generated (because M is finitely generated) submodule in $\mathrm{span}_R\{x_{i_1}, \ldots, x_{i_n}\}$, the induction hypothesis applies. □

Our interest in semi-hereditary rings comes from fact that finite von Neumann algebras are semi-hereditary, as we shall see later. (Corollary 1.3.21) At this point, we need some more algebraic constructions.

Definition 1.1.7. *For an R-module M, we define* the dual module $M^* := \mathrm{Hom}_R(M, R)$*, with R-module structure given by*

$$(rf)(x) = f(x)r^*,$$

for $r \in R, f \in \mathrm{Hom}_R(M, R)$ and $x \in M$.

At first glance, the action of R on the dual module may look strange, but this definition will be justified later. (see e.g. Corollary 1.4.8 and Definition 1.3.5)

Definition 1.1.8. *Let M be an R-module and let K be an R-submodule of M. We then define the* (algebraic) closure *of K in M as*

$$\overline{K}^M := \{x \in M \mid \forall f \in \mathrm{Hom}_R(M, R) : (K \subseteq \ker(f)) \Rightarrow (x \in \ker(f))\}.$$

When there is no possibility of confusion, about which module the closure is performed relative to, we will sometimes denote the closure of K by $\overline{K}$ to simplify notation and sometimes by $\overline{K}^{\mathrm{alg}}$ to distinguish it from topological closures. The submodule K is said to be closed in M *when $\overline{K}^M = K$. We also define*

- $TM := \{x \in M \mid \forall f \in \mathrm{Hom}_R(M, R) : f(x) = 0\} = \overline{\{0\}}^M$.
- $PM := M/TM$.

The algebraic closure of modules has the following properties.

Proposition 1.1.9. *Let L, M and N be modules over R. Then the following holds.*

(i) If $L \subseteq M \subseteq N$ then $\overline{L}^N \subseteq \overline{M}^N$.

(ii) If $f : M \to N$ is a homomorphism and N is projective, then $\ker(f)$ is closed in M.

Proof.
(i) Let $x \in \overline{L}^N$ be given and consider any $\varphi \in \mathrm{Hom}_R(N, R)$ that vanishes on M. Then φ especially vanishes on $L \subseteq M$ and since $x \in \overline{L}^N$ we get $\varphi(x) = 0$. Hence $x \in \overline{M}^N$.

(ii) Assume first that N is free and choose a basis $(x_i)_{i\in I}$. Let $x \in \overline{\ker(f)}^M$ be given. We want to show that $f(x) = 0$. Since $(x_i)_{i\in I}$ is a basis, every element $y \in N$ has a unique expansion $\sum_{i\in I} r_i(y)x_i$,[1] and hence we get a family of homomorphisms $\varphi_i : N \to R$ by setting

$$\varphi_i(y) = \varphi_i(\sum_{j\in I} r_j(y)x_j) := r_i(y).$$

[1]With only finitely many $r_i(y)$'s non-zero

Note, that this family separates points in N. That is, if $\varphi_i(y) = 0$ for all $i \in I$ then $y = 0$. For any $i \in I$, we have $\varphi_i \circ f \in \mathrm{Hom}_R(M, R)$ and vanishing on $\ker(f)$ and hence $\varphi_i(f(x)) = 0$. Thus $f(x) = 0$. This proves (ii) in the special case when N is free.
Assume now, that N is projective and consider an $x \in \overline{\ker(f)}^M$. Assume moreover that $f(x) \neq 0$. Since N is projective, it is a direct summand in a free module F, which therefore can be written as $F = N \oplus Q$ for some module Q. By the argument above, there exists a $\varphi \in \mathrm{Hom}_R(F, R)$ such that $\varphi(f(x), 0) \neq 0$. But then

$$M \ni \xi \longmapsto \varphi(f(\xi), 0) \in R,$$

is a homomorphism, which obviously vanishes on $\ker(f)$ and is nonzero on x — contradicting the choice of x.
Hence $f(x) = 0$, and $\overline{\ker(f)}^M = \ker(f)$. □

1.1.1 The generalized Murray- von Neumann dimension

In this section we define the Murray- von Neumann dimension of a finitely generated projective module over a finite von Neumann algebra. Following Lück ([Lüc97],[Lüc98]), we generalize the notion of Murray- von Neumann dimension to the category of all modules over a finite von Neumann algebra. We shall restrict our selves to the class of finite von Neumann algebras, who possesses a normal, faithful, tracial state. Hence the following definition.

Definition 1.1.10. *Throughout the text (unless explicitly stated otherwise) the term* finite von Neumann algebra*, will mean a von Neumann algebra $\mathscr{M}$ with the following properties:*

- *The unit 1 is a finite projection in $\mathscr{M}$.*
- *$\mathscr{M}$ possesses a normal, faithful, tracial state.*

Actually the first condition in Definition 1.1.10 is redundant, which can be seen in the following way. If $\mathscr{M}$ is any von Neumann algebra with a normal, faithful, tracial state and $p \in \mathscr{M}$ is a projection equivalent to 1, then there exists a partial isometry $V \in \mathscr{M}$ with $V^*V = 1$ and $VV^* = p$. Hence

$$\tau(1 - p) = \tau(V^*V - VV^*) = \tau(V^*V) - \tau(VV^*) = 0,$$

and since τ is faithful we get $p = 1$.

Let $\mathscr{M}$ be finite von Neumann algebra with unit 1 and let τ be a fixed normal, faithful, tracial state on $\mathscr{M}$. The trace τ gives rise to a tracial functional τ_n on $M_n(\mathscr{M})$, by setting

$$\tau_n(\{a_{ij}\}_{i,j=1}^n) := \sum_{i=1}^n \tau(a_{ii}).$$

By computing τ_n on a matrix-product of the form A^*A, one easily checks that τ_n is positive and faithful. (But not a state, since it takes the value n on the unit matrix.) Given a finitely generated projective $\mathscr{M}$-module P, we know by Lemma 1.1.1 and Remark 1.1.2 that P is (isomorphic to) $\mathscr{M}^n A$ for some idempotent matrix $A \in M_n(\mathscr{M})$. We now define the *Murray- von Neumann dimension* of P as

$$\dim_{\mathscr{M}}(P) := \tau_n(A) \in [0, \infty[$$

Of course we have to check that this is well-defined

Lemma 1.1.11. *The Murray- von Neumann dimension of P is independent of the choice of idempotent matrix.*

For the proof, some notation will be convenient: For a matrix $B \in M_{n,m}(\mathscr{M})$ we denote by R_B the $\mathscr{M}$-homomorphism from $\mathscr{M}^n$ to $\mathscr{M}^m$, multiplying from the right with B.

Proof. Let A, A' be idempotents in $M_n(\mathscr{M})$ and $M_m(\mathscr{M})$ respectively and assume that $\varphi : \mathscr{M}^n A \to \mathscr{M}^m A'$ is an isomorphism of $\mathscr{M}$-modules. By extending the matrices A and A' by zeros, (this does not effect their traces) we may assume that $m = n$.
Define $\tilde{\varphi} : \mathscr{M}^n \to \mathscr{M}^n$ by

$$\mathscr{M}^n A \oplus \mathscr{M}^n(1-A) \ni (x,y) \longmapsto (\varphi(x), 0) \in \mathscr{M}^n A' \oplus \mathscr{M}^n(1-A')$$

and $\hat{\varphi} : \mathscr{M}^n \to \mathscr{M}^n$ by

$$\mathscr{M}^n A' \oplus \mathscr{M}^n(1-A') \ni (x,y) \longmapsto (\varphi^{-1}(x), 0) \in \mathscr{M}^n A \oplus \mathscr{M}^n(1-A)$$

Then both $\tilde{\varphi}$ and $\hat{\varphi}$ are $\mathscr{M}$-linear maps and hence of the form $R_{\tilde{X}}$ and $R_{\hat{X}}$ for some matrices $\tilde{X}, \hat{X} \in M_n(\mathscr{M})$. Using that A and A' are idempotent, a direct computation shows that

$$R_{A'} = R_{\hat{X}A\tilde{X}} \quad \text{and} \quad R_A = R_{\tilde{X}\hat{X}A}$$

Since τ_n is a trace, we get

$$\tau_n(A') = \tau_n(\hat{X}A\tilde{X}) = \tau_n(\tilde{X}\hat{X}A)) = \tau_n(A),$$

as desired. □

We now want to extend the Murray- von Neumann dimension to arbitrary $\mathscr{M}$-modules. This is done in the following way.

Definition 1.1.12. *Let M be any $\mathscr{M}$-module. We then define*

$$\dim'_{\mathscr{M}}(M) := \sup\{\dim_{\mathscr{M}}(P) \mid P \textit{ is a finitely generated projective submodule of } M\} \in [0,\infty].$$

As it stands, it is not completely obvious that $\dim'_{\mathscr{M}}(\cdot)$ actually extends the Murray- von Neumann dimension, but as we will see in Section 1.4 this is the case.
Before we are able to investigate the extended dimension function, a rather large amount of theory is required. First we need some results on finite von Neumann algebras.

1.2 Miscellaneous on finite von Neumann algebras

In this section we collect some general results concerning finite von Neumann algebras.
Throughout the section, $\mathscr{M}$ denotes a finite von Neumann algebra with unit 1, equipped with a fixed normal, faithful, tracial state τ. Let $L^2(\mathscr{M})$ denote the Hilbert space completion of $\mathscr{M}$, in the GNS-construction with respect to τ. That is, the completion of $\mathscr{M}$ with respect to the (norm $\|\cdot\|_2$ induced by the) inner product $(a,b) \mapsto \tau(b^*a) =: \langle a|b\rangle$. Let $\eta : \mathscr{M} \to L^2(\mathscr{M})$ denote the inclusion $\mathscr{M} \subseteq L^2(\mathscr{M})$. Since τ is faithful, the GNS-representation π_τ of $\mathscr{M}$ on $L^2(\mathscr{M})$ is faithful and $\pi_\tau(\mathscr{M})$ is closed in the strong operator topology on $\mathscr{B}(L^2(\mathscr{M}))$ since τ is normal. (see e.g. [KR2] Corollary 7.1.7) Thus, π_τ is a $*$-algebra-isomorphism of von Neumann algebras from $\mathscr{M}$ to $\pi_\tau(\mathscr{M})$ and we will therefore often suppress the reference to π_τ and just consider $\mathscr{M}$ as acting on $L^2(\mathscr{M})$.
We denote by $\mathscr{M}^{\mathrm{op}}$ the opposite algebra of $\mathscr{M}$ and by τ^{op} the trace on $\mathscr{M}^{\mathrm{op}}$ given by

$$\tau^{\mathrm{op}}(m^{\mathrm{op}}) := \tau(m) \quad \text{for } m^{\mathrm{op}} \in \mathscr{M}^{\mathrm{op}}.$$

Everything said above about $(\mathscr{M}, \tau)$ carry over to the pair $(\mathscr{M}^{\mathrm{op}}, \tau^{\mathrm{op}})$ and (as with $\mathscr{M}$) we identify $\mathscr{M}^{\mathrm{op}}$ with its image under the GNS-representation, with respect to τ^{op}, on $L^2(\mathscr{M}^{\mathrm{op}}, \tau^{\mathrm{op}}) =: L^2(\mathscr{M}^{\mathrm{op}})$. The algebraic tensor product $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ is naturally represented on $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$

and we denote by $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ the strong closure of $\mathscr{M}\otimes\mathscr{M}^{\text{op}}$ in $\mathscr{B}(L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\text{op}}))$.
Recall that the $*$-algebra-isomorphism class of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$, does not depend on the representation space $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\text{op}})$, but only on the $*$-algebra-isomorphism classes of $\mathscr{M}$ and $\mathscr{M}^{\text{op}}$. (see e.g. [KR2] Theorem 11.2.10) The normal states τ and τ^{op} on $\mathscr{M}$ and $\mathscr{M}^{\text{op}}$ respectively, gives rise to a normal state $\tau\otimes\tau^{\text{op}}$ on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ (see e.g. [KR2] Proposition 11.2.7), with the property that

$$\tau\otimes\tau^{\text{op}}(m\otimes n^{\text{op}}) = \tau(m)\tau^{\text{op}}(n^{\text{op}}) \quad \text{for all } m\in\mathscr{M}, n^{\text{op}}\in\mathscr{M}^{\text{op}}.$$

Because both τ and τ^{op} are faithful traces, the tensor-state $\tau\otimes\tau^{\text{op}}$ is actually a faithful trace on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$, as we will prove later. (Proposition 1.2.9).
Before doing this, we introduce the so-called *conjugation operator* on $L^2(\mathscr{M})$.

1.2.1 The conjugation operator

In this section we introduce the conjugation operator J on $L^2(\mathscr{M})$ and study the relationship between $\mathscr{M}'$ and $\mathscr{M}$. The presented results are classical and can be found, for instance, in [Dix] and/or [KR2].

Lemma 1.2.1. *The map $J:\eta(\mathscr{M})\to\eta(\mathscr{M})$ given by $\eta(a)\mapsto\eta(a^*)$ extends to an anti-linear isometry $J: L^2(\mathscr{M})\to L^2(\mathscr{M})$ with the property that $J^2=1$.*

Proof. Obviously J is anti-linear on $\eta(\mathscr{M})$ and since τ is a trace it follows that

$$\|\eta(a)\|_2^2 = \tau(a^*a) = \tau(aa^*) = \|\eta(a^*)\|_2^2 = \|J(a)\|_2^2.$$

Hence $J:\eta(\mathscr{M})\to\eta(\mathscr{M})$ is isometric and extends therefore to an anti-linear isometry (also denoted by J) on $L^2(\mathscr{M})$. Since $a^{**}=a$ for all $a\in\mathscr{M}$, it is clear that $J^2=1$. □

As indicated, we shall refer to J as the *conjugation operator* on $L^2(\mathscr{M})$.

Lemma 1.2.2. *The conjugation operator J on $L^2(\mathscr{M})$ has the following properties:*

1. *For all $x,y\in L^2(\mathscr{M})$ we have $\langle Jx\,|\,Jy\rangle = \langle y\,|\,x\rangle$.*

2. *We have $J\mathscr{M}J\subseteq\mathscr{M}'$, where the commutant is taken relative to $\mathscr{B}(L^2(\mathscr{M}))$.*

Proof. By continuity, we only need to check the relation in *1.* on the dense subspace $\eta(\mathscr{M})$. For $x=\eta(a)$ and $y=\eta(b)$, we have

$$\langle Jx\,|\,Jy\rangle = \langle\eta(a^*)\,|\,\eta(b^*)\rangle = \tau(ba^*) = \tau(a^*b) = \langle\eta(b)\,|\,\eta(a)\rangle = \langle y\,|\,x\rangle.$$

To proof the second claim we fix arbitrary $a,b,c\in\mathscr{M}$. Then

$$\begin{aligned} c(JbJ)(\eta(a)) &= c(Jb)(\eta(a^*)) = cJ(\eta(ba^*)) = \eta(cab^*) \\ (JbJ)c(\eta(a)) &= JbJ(\eta(ca)) = J(\eta(ba^*c^*)) = \eta(cab^*). \end{aligned}$$

Since $\eta(\mathscr{M})$ is dense in $L^2(\mathscr{M})$, this shows that JbJ commutes with every $c\in\mathscr{M}$ and since b was arbitrary we now have $J\mathscr{M}J\subseteq\mathscr{M}'$.

□

Consider an arbitrary $a\in\mathscr{M}$. On the dense subspace $\eta(\mathscr{M})$, this operator acts as $\eta(x)\mapsto\eta(ax)$. Associated with a, there is another natural linear operator on $\eta(\mathscr{M})$, namely

$$\eta(x)\overset{R_a}{\longmapsto}\eta(xa).$$

We now want to see that R_a is bounded with respect to the norm on $L^2(\mathscr{M})$. Since $J\eta(1) = \eta(1)$ we have

$$R_a(\eta(x)) = \eta(xa) = x(Ja^*J)\eta(1) = (Ja^*J)x\eta(1) = Ja^*J(\eta(x)),$$

where the third equality follows from Lemma 1.2.2. From this it follows that $R_a = Ja^*J|_{\eta(\mathscr{M})}$ and hence that R_a is bounded. It therefore extends to a bounded operator, also denoted R_a, on $L^2(\mathscr{M})$. Note, that $a \mapsto R_a$ is linear and that $R_{a^*} = R_a^*$.
Our next aim is to show that $\mathscr{M}' = \{R_a | a \in \mathscr{M}\}$. We already proved the inclusion "$\supseteq$" in Lemma 1.2.2, since for each $a \in \mathscr{M}$ we have $R_a = Ja^*J$. From this equation it also follows that $\{R_a | a \in \mathscr{M}\}$ is weakly closed in $\mathscr{B}(L^2(\mathscr{M}))$. To prove the opposite inclusion, some more work is required. In the following we will use the symbol L_a to denote the continuous extension, of the map $\eta(m) \mapsto \eta(am)$, to $L^2(\mathscr{M})$. This is the operator we until now have identified with a and this identification will be reactivated after the proof of Corollary 1.2.7. Hopefully, this notation will help clarify the arguments below.

Definition 1.2.3. *For any $x \in L^2(\mathscr{M})$ we define $L_x^0 : \eta(\mathscr{M}) \to L^2(\mathscr{M})$ by setting $L_x^0(\eta(m)) := R_m x$ and $R_x^0 : \eta(\mathscr{M}) \to L^2(\mathscr{M})$ by setting $R_x^0(\eta(m)) := L_m x$.*

Note, that both L_x^0 and R_x^0 are densely defined unbounded (in general) operators and that $L_x^0 = L_x|_{\eta(\mathscr{M})}$ and $R_x^0 = R_x|_{\eta(\mathscr{M})}$, whenever $x \in \eta(\mathscr{M})$.
Also note, that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{M}$ such that $\|\eta(x_n) - x\|_2 \underset{n\to\infty}{\longrightarrow} 0$, then for any $m \in \mathscr{M}$ we have

$$L_x^0(\eta(m)) = R_m(x) = \lim_n R_m(\eta(x_n)) = \lim_n(\eta(x_n m)) = \lim_n L_{x_n}(\eta(m)).$$

A similar formula holds for R_x^0, by an analogous computation.
The reason for the uppercase zero is that we now show that L_x and R_x are closable operators and we wish to reserve the symbols L_x and R_x to denote their closures. To see that L_x is closable, we need to check that the closure of its graph

$$\overline{\{(\eta(m), L_x^0\eta(m)) | m \in \mathscr{M}\}} \subseteq L^2(\mathscr{M}) \oplus L^2(\mathscr{M}),$$

is the graph of some (unbounded) operator. For this it suffices to see, that if $(\eta(a_n))_{n\in\mathbb{N}} \subseteq \eta(\mathscr{M})$ is null-sequence and $(L_x^0\eta(a_n))_{n\in\mathbb{N}}$ converges to z, then z must be zero. To see this, we consider any $m \in \mathscr{M}$. Then

$$\begin{aligned} |\langle z \,|\, \eta(m)\rangle| &= \lim_n |\langle L_x^0\eta(a_n) \,|\, \eta(m)\rangle| \\ &= \lim_n |\langle R_{a_n} x \,|\, \eta(m)\rangle| \\ &= \lim_n |\langle x \,|\, R_{a_n^*}\eta(m)\rangle| \\ &= \lim_n |\langle x \,|\, L_m(\eta(a_n^*))\rangle| \\ &\leq \lim_n \|x\|_2 \|L_m\| \|\eta(a_n^*)\|_2 \qquad (\|L_m\| \text{ being the operator-norm}) \\ &= \lim_n \|x\|_2 \|L_m\| \|\eta(a_n)\|_2 \longrightarrow 0. \end{aligned}$$

Thus, z is orthogonal to the dense subset $\eta(\mathscr{M})$ and is therefore zero.
In the same way we see that R_x^0 is closable and we denote by L_x and R_x the closure of L_x^0 and R_x^0 respectively.

Lemma 1.2.4. *If $T \in \{R_a | a \in \mathscr{M}\}'$ and $x \in L^2(\mathscr{M})$ is a vector such that L_x is bounded, then also L_{Tx} is bounded and $TL_x = L_{(Tx)}$. In particular $T = L_{T(\eta(1))}$.*

Proof. For $m \in \mathscr{M}$ we have

$$TL_x(\eta(m)) = TR_m x = R_m Tx = L_{(Tx)}^0\eta(m),$$

and since $L^0_{(Tx)}$ agrees with the bounded operator TL_x on the dense subspace $\eta(\mathscr{M})$, we conclude that L_{Tx} is bounded and equal to TL_x. Since L_1 is the identity on $L^2(\mathscr{M})$ we have $T = L_{(T\eta(1))}$. □

Proposition 1.2.5. *The following holds.*

$$\{R_a | a \in \mathscr{M}\}' = \{L_x | L_x \text{ is bounded }\} \qquad \text{and} \qquad \{L_a | a \in \mathscr{M}\}' = \{R_x | R_x \text{ is bounded }\}$$

Proof. We show the first equality. By the above lemma, every $T \in \{R_a | a \in \mathscr{M}\}'$ equals $L_{T(\eta(1))}$ and the inclusion "$\subseteq$" follows. Conversely, if L_x is bounded and $a \in \mathscr{M}$, then for any $m \in \mathscr{M}$ we get

$$L_x R_a(\eta(m)) = L_x(\eta(ma)) = R_{(ma)}x = R_a R_m x = R_a L_x(\eta(m)),$$

and hence $L_x \in \{R_a | a \in \mathscr{M}\}'$.
The second equality is proven similarly, using the obvious variant of Lemma 1.2.4 for operators in $\{L_a | a \in \mathscr{M}\}'$. □

Lemma 1.2.6. *If $x, y \in L^2(\mathscr{M})$ such that L_x and R_y are both bounded, then L_x and R_y commutes.*

Proof. Since both L_x and R_y are bounded, it suffices to check that they commute on the dense subspace $\eta(\mathscr{M})$. Choose a sequence $(y_n)_{n\in\mathbb{N}} \in \mathscr{M}$ such that $\|y - \eta(y_n)\|_2 \underset{n\to\infty}{\longrightarrow} 0$. For any $m \in \mathscr{M}$ we have

$$\begin{aligned} L_x R_y(\eta(m)) &= L_x(\lim_n \eta(m y_n)) \\ &= \lim_n L_x(\eta(m y_n)) \\ &= \lim_n R_{m y_n}(x) \\ &= \lim_n R_{y_n} R_m(x) \\ &= \lim_n R_{y_n} L_x(\eta(m)) \\ &= R_y L_x(\eta(m)), \end{aligned}$$

and the claim follows. □

From this, the desired equality between $\{L_a | a \in \mathscr{M}\}'$ and $\{R_b | b \in \mathscr{M}\}$ follows.

Corollary 1.2.7. *We have*

$$\{L_x | L_x \text{ is bounded }\} = \{L_a | a \in \mathscr{M}\} \quad \text{and} \quad \{R_y | R_y \text{ is bounded }\} = \{R_b | b \in \mathscr{M}\},$$

and hence

$$\{L_a | a \in \mathscr{M}\}' = \{R_b | b \in \mathscr{M}\}.$$

Proof. Lemma 1.2.6 and Proposition 1.2.5 in conjunction with the Double Commutant Theorem, gives

$$\{L_x | L_x \text{ is bounded}\} \subseteq \{R_y | R_y \text{ is bounded}\}' = \{L_a | a \in \mathscr{M}\},$$

and hence $\{L_x | L_x \text{ is bounded}\} = \{L_a | a \in \mathscr{M}\}$. Similarly we get

$$\{R_y | R_y \text{ is bounded}\} = \{R_b | b \in \mathscr{M}\},$$

and (using Proposition 1.2.5) we conclude that

$$\{R_b | b \in \mathscr{M}\} = \{R_y | R_y \text{ is bounded}\} = \{L_a | a \in \mathscr{M}\}'.$$

□

Remark 1.2.8. *By what is proven above, the map $a \mapsto Ja^*J$ is a $*$-preserving, bijective and linear map between $\mathscr{M}$ and $\mathscr{M}'$. But for $a, b \in \mathscr{M}$ we have*

$$J(ab)^*J = Jb^*a^*J = (Jb^*J)(Ja^*J),$$

so that the map is product-reversing. Hence, if we consider the map from $\mathscr{M}^{\mathrm{op}}$ to $\mathscr{M}'$ given by $a^{\mathrm{op}} \mapsto Ja^*J$, we get a $*$-algebra-isomorphism.

With the results above at our disposal, we can now prove the promised properties of the tensor-state $\tau \otimes \tau^{\mathrm{op}}$.

Proposition 1.2.9. *The normal tensor-state $\tau \otimes \tau^{\mathrm{op}} : \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \to \mathbb{C}$ is a faithful trace.*

Proof. We first show that $\tau \otimes \tau^{\mathrm{op}}$ is a trace. Since

$$\tau \otimes \tau^{\mathrm{op}}(x \otimes y^{\mathrm{op}}) = \tau(x)\tau^{\mathrm{op}}(y^{\mathrm{op}}) \text{ for all } x \in \mathscr{M}, y \in \mathscr{M}^{\mathrm{op}},$$

the restriction of $\tau \otimes \tau^{\mathrm{op}}$ to $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ is tracial.
Consider any $x, y \in \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ and choose bounded nets $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ in $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$, converging strongly to x and y respectively. (The Kaplansky density Theorem)
Then, for any $z \in \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$, the net $(x_i z)_{i \in I}$ is bounded and converges weakly to xz and similarly $(zy_j)_{j \in J}$ is bounded and converges weakly to zy. Because $\tau \otimes \tau^{\mathrm{op}}$ is normal, it is weakly continuous on bounded sets (see e.g. [KR2] Prop. 7.4.5), and hence

$$\begin{aligned}
\tau \otimes \tau^{\mathrm{op}}(xy) &= \lim_i \tau \otimes \tau^{\mathrm{op}}(x_i y) \\
&= \lim_i \lim_j \tau \otimes \tau^{\mathrm{op}}(x_i y_j) \\
&= \lim_i \lim_j \tau \otimes \tau^{\mathrm{op}}(y_j x_i) \\
&= \tau \otimes \tau^{\mathrm{op}}(yx).
\end{aligned}$$

We now want to see that $\tau \otimes \tau^{\mathrm{op}}$ is faithful. Set $\xi_0 := \eta_\tau(1)$ and $\xi_0^{\mathrm{op}} := \eta_{\tau^{\mathrm{op}}}(1)$. For $x \otimes y^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ we have

$$\tau \otimes \tau^{\mathrm{op}}(x \otimes y^{\mathrm{op}}) = \langle x\xi_0 | \xi_0 \rangle \langle y^{\mathrm{op}} \xi_0^{\mathrm{op}} | \xi_0^{\mathrm{op}} \rangle = \langle (x \otimes y^{\mathrm{op}})(\xi_0 \otimes \xi_0^{\mathrm{op}}) | \xi_0 \otimes \xi_0^{\mathrm{op}} \rangle.$$

Since both $\tau \otimes \tau$ and $\omega_{\xi_0 \otimes \xi_0^{\mathrm{op}}} : T \mapsto \langle T(\xi_0 \otimes \xi_0^{\mathrm{op}}) | \xi_0 \otimes \xi_0^{\mathrm{op}} \rangle$ are normal states, we conclude that $\tau \otimes \tau^{\mathrm{op}} = \omega_{\xi_0 \otimes \xi_0^{\mathrm{op}}}$.
Assume now that $x \in \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ with $\tau \otimes \tau^{\mathrm{op}}(x^*x) = 0$. Then

$$0 = \omega_{\xi_0 \otimes \xi_0^{\mathrm{op}}}(x^*x) = \langle x(\xi_0 \otimes \xi_0^{\mathrm{op}}) | x(\xi_0 \otimes \xi_0^{\mathrm{op}}) \rangle = \|x(\xi_0 \otimes \xi_0^{\mathrm{op}})\|_2^2. \qquad (*)$$

Let J denote the conjugation operator on $L^2(\mathscr{M})$ and J^{op} the conjugation operator on $L^2(\mathscr{M}^{\mathrm{op}})$. For any $m \in \mathscr{M}$ we have $Jm^*J\xi_0 = R_m \xi_0 = \eta(m)$ and hence we see that

$$X := \{((JmJ) \otimes (J^{\mathrm{op}} n^{\mathrm{op}} J^{\mathrm{op}}))\xi_0 \otimes \xi_0^{\mathrm{op}} | m \in \mathscr{M}, n \in \mathscr{M}^{\mathrm{op}}\} = \eta_\tau(\mathscr{M}) \otimes \eta_{\tau^{\mathrm{op}}}(\mathscr{M}^{\mathrm{op}}).$$

Since $\eta_\tau(\mathscr{M})$ is dense in $L^2(\mathscr{M})$ and $\eta_{\tau^{\mathrm{op}}}(\mathscr{M}^{\mathrm{op}})$ is dense in $L^2(\mathscr{M}^{\mathrm{op}})$, it follows that X is dense in $L^2(\mathscr{M}) \otimes L^2(\mathscr{M}^{\mathrm{op}})$ — and hence in $L^2(\mathscr{M}) \bar{\otimes} L^2(\mathscr{M}^{\mathrm{op}})$.
Each element of the form $(JmJ) \otimes (J^{\mathrm{op}} n^{\mathrm{op}} J^{\mathrm{op}})$ clearly commutes with $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$, (see e.g. Corollary 1.2.7) and hence with the strong closure $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$.
We have now proven, that the vector $\xi_0 \otimes \xi_0^{\mathrm{op}}$ is cyclic for $(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})'$ and therefore separating for $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$. Thus, the formula $(*)$ above implies $x = 0$, and hence $\tau \otimes \tau^{\mathrm{op}}$ is faithful. □

Remark 1.2.10. *In the preceding text, we have consequently used the inclusion-map η to distinguish between $\mathscr{M}$ and $\eta(\mathscr{M})$. To simplify notation, we will often suppress the map η in the future and simply think of $\mathscr{M}$ as a subspace of $L^2(\mathscr{M})$.*

1.2.2 An isomorphism of Hilbert spaces

Because of the great importance of the tensor-product $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ in the theory of L^2-homology for von Neumann algebras, we will devote some time here to investigate an alternative description of the GNS-construction for this tensor product, with respect to $\tau\otimes\tau^{\text{op}}$. This will turn out useful in some future proofs.
For $x, y \in \mathscr{M}$ we consider $x\otimes y^{\text{op}} \in \mathscr{M}\otimes\mathscr{M}^{\text{op}}$ and the linear mapping

$$\mathscr{M} \ni m \xmapsto{\Psi_{x\otimes y^{\text{op}}}} \tau(ym)x = \langle m\,|\,y^*\rangle x \in \mathscr{M}.$$

The following holds.

Lemma 1.2.11. *For each $x\otimes y^{\text{op}} \in \mathscr{M}\otimes\mathscr{M}^{\text{op}}$, the linear operator $\Psi_{x\otimes y^{\text{op}}}$ is bounded with respect to the norm inherited from $L^2(\mathscr{M})$.*

Proof. For any $m\in\mathscr{M}$ we have

$$\begin{aligned}\|\Psi_{x\otimes y^{\text{op}}}m\|_2^2 &:= \langle\tau(ym)x\,|\,\tau(ym)x\rangle\\ &= |\tau(ym)|^2\tau(x^*x)\\ &= \|x\|_2^2|\langle m\,|\,y^*\rangle|^2\\ &\le \|x\|_2^2\|m\|_2^2\|y^*\|_2^2 &&\text{(Cauchy-Schwartz)}\\ &= \|x\|_2^2\|y\|_2^2\|m\|_2^2,\end{aligned}$$

where the last equality follows from the fact that τ is a trace, such that

$$\|y\|_2^2 = \tau(y^*y) = \tau(yy^*) = \|y^*\|_2^2.$$

□

Since $\Psi_{x\otimes y^{\text{op}}}$ is bounded, it extends by continuity to an operator, also denoted $\Psi_{x\otimes y^{\text{op}}}$, on $L^2(\mathscr{M})$. By construction of $\Psi_{x\otimes y^{\text{op}}}$, the map

$$\mathscr{M}\times\mathscr{M}^{\text{op}} \ni (x,y) \longmapsto \Psi_{x\otimes y} \in \mathscr{B}(L^2(\mathscr{M}))$$

is bilinear and we therefore get a linear map

$$\Psi : \mathscr{M}\otimes\mathscr{M}^{\text{op}} \longrightarrow \mathscr{B}(L^2(\mathscr{M})),$$

with the property that $\Psi(x\otimes y^{\text{op}}) = \Psi_{x\otimes y^{\text{op}}}$.

Proposition 1.2.12. *The linear map Ψ given by*

$$\mathscr{M}\otimes\mathscr{M}^{\text{op}} \ni x\otimes y^{\text{op}} \overset{\Psi}{\longmapsto} \Psi_{x\otimes y^{\text{op}}} \in \mathscr{FR}(L^2(\mathscr{M}))$$

is an isometry, when $\mathscr{M}\otimes\mathscr{M}^{\text{op}}$ is endowed with the norm inherited from $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\text{op}}))$ and $\mathscr{FR}(L^2(\mathscr{M}))$ (the finite-rank operators) is endowed with the Hilbert-Schmidt norm $\|\cdot\|_{\mathscr{HS}}$.

Proof. Let $T = \sum_{i=1}^n x_i\otimes y_i^{\text{op}} \in \mathscr{M}\otimes\mathscr{M}^{\text{op}}$ be given and assume, without loss of generality, that the vectors $x_1,\ldots,x_n$ are mutually orthogonal. Then

$$\|\sum_i x_i\otimes y_i^{\text{op}}\|_2^2 = \sum_{i,j}\langle x_i\,|\,x_j\rangle\langle y_i\,|\,y_j\rangle = \sum_i \|x_i\|_2^2\|y_i\|_2^2.$$

Let $(e_\alpha)_{\alpha\in\mathbb{A}}$ be an orthonormal basis for $L^2(\mathscr{M})$.
Computing the Hilbert-Schmidt norm of Ψ_T we get.

$$\begin{aligned}
\|\Psi_T\|^2_{\mathscr{HS}} &:= \|\sum_i \Psi_{x_i\otimes y_i^{\mathrm{op}}}\|^2_{\mathscr{HS}} \\
&:= \sum_\alpha \|\sum_i \Psi_{x_i\otimes y_i^{\mathrm{op}}} e_\alpha\|_2^2 \\
&= \sum_\alpha \|\sum_i \langle e_\alpha | y_i^*\rangle x_i\|_2^2 \\
&= \sum_\alpha \sum_i |\langle e_\alpha | y_i^*\rangle|^2 \|x_i\|_2^2 \qquad \text{(since } x_i \perp x_j.\text{)} \\
&= \sum_i \|x_i\|_2^2 \|y_i^*\|_2^2 \\
&= \sum_i \|x_i\|_2^2 \|y_i\|_2^2,
\end{aligned}$$

and the result follows. □

Corollary 1.2.13. *The map Ψ, from Proposition 1.2.12, extends to an isomorphism of Hilbert spaces from $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$ to $\mathscr{HS}(L^2(\mathscr{M}))$.*

Proof. We first prove that $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ is dense in $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$. By construction, $L^2(\mathscr{M})\otimes L^2(\mathscr{M}^{\mathrm{op}})$ is dense in $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$, so it suffices to see that every element T in $L^2(\mathscr{M})\otimes L^2(\mathscr{M}^{\mathrm{op}})$ is the limit (in $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$-norm) of a sequence from $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$.
By linearity of the inclusion $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}\subseteq L^2(\mathscr{M})\otimes L^2(\mathscr{M}^{\mathrm{op}})$, we may assume that $T = x\otimes y$ for some $x\in L^2(\mathscr{M})$ and $y\in L^2(\mathscr{M}^{\mathrm{op}})$. Then there exists sequences $(x_k)_{k\in\mathbb{N}}\subseteq\mathscr{M}$, $(y_k^{\mathrm{op}})_{k\in\mathbb{N}}\subseteq\mathscr{M}^{\mathrm{op}}$ such that

$$\lim_k \|x_k - x\|_2 = 0 \quad\text{and}\quad \lim_k \|y_k^{\mathrm{op}} - y\|_2 = 0.$$

From this it follows that $x_k\otimes y_k \underset{k\to\infty}{\longrightarrow} x\otimes y$, since

$$\begin{aligned}
\|x\otimes y - x_k\otimes y_k^{\mathrm{op}}\|_2^2 &= \langle x\otimes y | x\otimes y\rangle - \langle x_k\otimes y_k^{\mathrm{op}} | x\otimes y\rangle - \langle x\otimes y | x_k\otimes y_k^{\mathrm{op}}\rangle + \langle x_k\otimes y_k^{\mathrm{op}} | x_k\otimes y_k^{\mathrm{op}}\rangle \\
&= \|x\|_2^2\|y\|_2^2 + \|x_k\|_2^2\|y_k^{\mathrm{op}}\|_2^2 - \langle x_k | x\rangle\langle y_k^{\mathrm{op}} | y\rangle - \langle x | x_k\rangle\langle y | y_k^{\mathrm{op}}\rangle \underset{k\to\infty}{\longrightarrow} 0
\end{aligned}$$

Hence $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ is dense in $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$ and therefore the linear map Ψ extends isometrically to

$$\Psi : L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}}) \longrightarrow \overline{\mathscr{FR}(L^2(\mathscr{M}))}^{\mathscr{HS}}.$$

We now note, that every rank-one operator on $L^2(\mathscr{M})$ has the form $\Psi_{x\otimes y}$ for suitable $x\in L^2(\mathscr{M})$ and $y\in L^2(\mathscr{M}^{\mathrm{op}})$ and by linearity it follows that Ψ maps $L^2(\mathscr{M})\otimes L^2(\mathscr{M}^{\mathrm{op}})$ onto $\mathscr{FR}(L^2(\mathscr{M}))$. Since $\overline{\mathscr{FR}(L^2(\mathscr{M}))}^{\mathscr{HS}} = \mathscr{HS}(L^2(\mathscr{M}))$, the claim follows. □

Proposition 1.2.14. *The Hilbert spaces $L^2(\mathscr{M},\tau)\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}},\tau^{\mathrm{op}})$ and $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}},\tau\otimes\tau^{\mathrm{op}})$ are isomorphic.*

To simplify notation, we denote $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}},\tau\otimes\tau^{\mathrm{op}})$ by $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$.

Proof. We view $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ as a subspace of both $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$ and $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$ in the natural way. By the proof of Corollary 1.2.13, $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ is dense in $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$ and a direct

computation shows that the identity on $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ is an isometry between the two subspaces.
It is therefore sufficient to prove that $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ is dense in $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ with respect to the norm inherited from $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$. Let η denote the inclusion of $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ into $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$.
Every element T in $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \subseteq \mathscr{B}(L^2(\mathscr{M}) \bar{\otimes} L^2(\mathscr{M}^{\mathrm{op}}))$ is the strong operator limit of a net (T_α) from $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$. Thus, for every $\xi \in L^2(\mathscr{M}) \bar{\otimes} L^2(\mathscr{M}^{\mathrm{op}})$ we have that $T_\alpha \xi \longrightarrow T\xi$.
If we choose ξ to be $\eta(1 \otimes 1)$, we get

$$\eta(T_\alpha) = T_\alpha \eta(1 \otimes 1) \longrightarrow T\eta(1 \otimes 1) = \eta(T),$$

and the proof is complete, since $\eta(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$ is dense in $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$ by construction. □

In the light of the above proposition, we often choose to identify $L^2(\mathscr{M}) \bar{\otimes} L^2(\mathscr{M}^{\mathrm{op}})$ with $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$.

Proposition 1.2.15. [CS03] *Let $m \in \mathscr{M}$. The isomorphism $\Psi : L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}) \xrightarrow{\sim} \mathscr{HS}(L^2(\mathscr{M}))$ intertwines the following four, pairwise commuting, actions.*

1. *The $\mathscr{M}$-action $T \mapsto mT$ on $\mathscr{HS}(L^2(\mathscr{M}))$ with the (extension of the) action*
$$\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}} \ni x \otimes y^{\mathrm{op}} \longmapsto mx \otimes y^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}},$$
on $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$.
2. *The $\mathscr{M}^{\mathrm{op}}$-action $T \mapsto Tm$ on $\mathscr{HS}(L^2(\mathscr{M}))$ with the (extension of the) action*
$$\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}} \ni x \otimes y^{\mathrm{op}} \longmapsto x \otimes (ym)^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}},$$
on $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$.
3. *The $\mathscr{M}^{\mathrm{op}}$-action $T \mapsto Jm^*JT$ on $\mathscr{HS}(L^2(\mathscr{M}))$ with the (extension of the) action*
$$\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}} \ni x \otimes y^{\mathrm{op}} \longmapsto xm \otimes y^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$$
on $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$.
4. *The $\mathscr{M}$-action $T \mapsto TJm^*J$ on $\mathscr{HS}(L^2(\mathscr{M}))$ with the (extension of the) action*
$$\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}} \ni x \otimes y^{\mathrm{op}} \longmapsto x \otimes (my)^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$$
on $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$.

Proof. The pairwise commutativity of the actions follows from Corollary 1.2.7. The proofs of *1.*,*2.*,*3.* and *4.* are essentially identical, so we only prove the third here.
Put $h_m(T) := (Jm^*J)T$ for $T \in \mathscr{HS}(L^2(\mathscr{M}))$ and $H_m(x \otimes y^{\mathrm{op}}) := (xm) \otimes y^{\mathrm{op}}$
for $x \otimes y^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ and extend H_m to $L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}})$ by linearity and continuity. We need to show that $\Psi \circ H_m = h_m \circ \Psi$.
That is, commutativity of the following diagram.

$$\begin{array}{ccc} \mathscr{HS}(L^2(\mathscr{M})) & \xrightarrow{h_m} & \mathscr{HS}(L^2(\mathscr{M})) \\ \Psi \Big\uparrow \simeq & & \simeq \Big\uparrow \Psi \\ L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}) & \xrightarrow[H_m]{} & L^2(\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}) \end{array}$$

For $x \otimes y^{\mathrm{op}} \in \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ and $\xi \in L^2(\mathscr{M})$ we get

$$h_m(\Psi(x \otimes y^{\mathrm{op}}))(\xi) = Jm^*J(\Psi(x \otimes y^{\mathrm{op}})\xi) = \langle \xi | y^* \rangle Jm^*Jx = \langle \xi | y^* \rangle xm = \Psi H_m(x \otimes y^{\mathrm{op}})(\xi).$$

By linearity, this implies that $\Psi \circ H_m$ and $h_m \circ \Psi$ agrees on the dense subspace $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ and by continuity the two maps must agree everywhere.

□

1.3 Hilbert modules

In this section we define the category of *Hilbert modules* associated with a fixed (finite) von Neumann algebra $\mathscr{M}$. As it turns out, this category is very "similar" (to be specified later) to the category of finitely generated projective modules over $\mathscr{M}$ and this similarity will be the cornerstone, when proving the essential properties of the extended dimension function.
In the following $\mathscr{M}$ denotes a finite von Neumann algebra, endowed with a fixed, faithful, normal, tracial state τ. We denote by $L^2(\mathscr{M})$ the Hilbert space completion of $\mathscr{M}$ in the GNS-construction with respect to τ and identify $\mathscr{M}$ with its (isomorphic) image under the GNS-representation on $L^2(\mathscr{M})$.
The algebra $\mathscr{M}$ acts diagonally on the direct sum of Hilbert spaces $\oplus_{i=1}^n L^2(\mathscr{M}) =: L^2(\mathscr{M})^n$, as

$$(x_1, \dots, x_n) \longmapsto (ax_1, \dots, ax_n), \qquad (a \in \mathscr{M})$$

and we denote this operator by $\operatorname{diag}(a)$. Note, that $n = \infty$ is allowed and in this case the direct sum is to be interpreted as the usual l^2-sum of Hilbert spaces.
Also note, that $a \mapsto \operatorname{diag}(a)$ is a faithful, normal representation of $\mathscr{M}$, since it is just the n-fold amplification of the (faithful, normal) GNS-representation π_τ of $\mathscr{M}$. Hence $\operatorname{diag}(\mathscr{M})$ is a von Neumann algebra in $\mathscr{B}(L^2(\mathscr{M})^n)$.

Definition 1.3.1. *A* Hilbert module *over $\mathscr{M}$ (or Hilbert $\mathscr{M}$-module) is a pair $(\mathscr{H}, \pi)$, where $\mathscr{H}$ is a Hilbert space and π is a (unital) representation of $\mathscr{M}$ on $\mathscr{H}$, with the following property: There exists a closed* $\operatorname{diag}(\mathscr{M})$*-invariant subspace $\mathscr{H}' \subseteq L^2(\mathscr{M})^n$ and a unitary $U : \mathscr{H} \to \mathscr{H}'$ such that the following diagram commutes for every $a \in \mathscr{M}$.*

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{U} & L^2(\mathscr{M})^n \\ \downarrow{\scriptstyle \pi(a)} & & \downarrow{\scriptstyle \operatorname{diag}(a)} \\ \mathscr{H} & \xrightarrow[U]{} & L^2(\mathscr{M})^n \end{array}$$

The Hilbert $\mathscr{M}$-module $(\mathscr{H}, \pi)$ is said to be finitely generated *if $\mathscr{H}'$ can be chosen as a subspace of $L^2(\mathscr{M})^n$ for finite $n \in \mathbb{N}$.*
By a morphism *between Hilbert modules $(\mathscr{H}, \pi)$ and $(\mathscr{K}, \rho)$ over $\mathscr{M}$ we shall mean a bounded linear operator $f : \mathscr{H} \to \mathscr{K}$ which respects the actions of $\mathscr{M}$. That is,*

$$f(\pi(a)x) = \rho(a)f(x) \text{ for all } a \in \mathscr{M} \text{ and } x \in \mathscr{H}.$$

*We will refer to such an operator as an $\mathscr{M}$-*equivariant *operator and the space of $\mathscr{M}$-equivariant operators from $(\mathscr{H}, \pi)$ to $(\mathscr{K}, \rho)$ is denoted by $\operatorname{Hom}_{\mathscr{M}}^{\mathtt{FGHM}}(\mathscr{H}, \mathscr{K})$ or $\mathscr{B}(\mathscr{H}, \mathscr{K})^{\mathscr{M}}$.*
Two Hilbert modules, $(\mathscr{H}, \pi)$ and $(\mathscr{K}, \rho)$, over $\mathscr{M}$ are said to be isomorphic *if there exists a unitary $\mathscr{M}$-equivariant operator between them.*
The category of finitely generated Hilbert $\mathscr{M}$-modules is denoted $\mathtt{FGHM}(\mathscr{M})$

Note, that if $(\mathscr{H}, \pi)$ is a Hilbert $\mathscr{M}$-module, then π is automatically a normal representation of $\mathscr{M}$ and hence $\pi(\mathscr{M})$ a von Neumann algebra in $\mathscr{B}(\mathscr{H})$. We shall often omit the reference to the representation π and simply speak of $\mathscr{H}$ as a Hilbert $\mathscr{M}$-module. In the following, we shall primarily be interested in finitely generated Hilbert $\mathscr{M}$-modules. In the remark below, some easy facts about these are collected.

Remark 1.3.2. *Let $n \in \mathbb{N}$ and let $\mathscr{K}$ be a closed $\mathscr{M}$-invariant subspace of $L^2(\mathscr{M})^n$ and consider the orthogonal projection $p \in \mathscr{B}(L^2(\mathscr{M})^n)$ onto $\mathscr{K}$. Since $\mathscr{K}$ is $\mathscr{M}$-invariant, the projection p is an $\mathscr{M}$-equivariant operator on $L^2(\mathscr{M})^n$. Thus, every finitely generated Hilbert $\mathscr{M}$-module is isomorphic to $pL^2(\mathscr{M})^n$ for suitable $n \in \mathbb{N}$ and $\mathscr{M}$-equivariant orthogonal projection p.*
Note also, that if $\mathscr{H}_1$ and $\mathscr{H}_2$ are finitely generated Hilbert $\mathscr{M}$-modules, then so is the direct sum of Hilbert spaces $\mathscr{H}_1 \oplus \mathscr{H}_2$, with respect to the diagonal action.

If $f : (\mathscr{H}_1, \langle\cdot|\cdot\rangle_1) \to (\mathscr{H}_2, \langle\cdot|\cdot\rangle_2)$ is a morphism of finitely generated Hilbert $\mathscr{M}$-modules, then so is the adjoint $f^* : (\mathscr{H}_2, \langle\cdot|\cdot\rangle_2) \to (\mathscr{H}_1, \langle\cdot|\cdot\rangle_1)$. To see this, we fix an $a \in \mathscr{M}$ and arbitrary $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$.
Then, if we suppress the $*$-algebra-homomorphisms from $\mathscr{M}$ to $\mathscr{B}(\mathscr{H}_1)$ and $\mathscr{B}(\mathscr{H}_2)$ respectively, we get

$$\begin{aligned}\langle x\,|\,(f^*a - af^*)y\rangle_1 &= \langle x\,|\,(f^*a)y\rangle_1 - \langle x\,|\,(af^*)y\rangle_1\\ &= \langle fx\,|\,ay\rangle_2 - \langle a^*x\,|\,f^*y\rangle_1\\ &= \langle a^*fx\,|\,y\rangle_2 - \langle fa^*x\,|\,y\rangle_2\\ &= \langle (a^*f - fa^*)x\,|\,y\rangle_2\\ &= 0.\end{aligned}$$

Thus, $f^*a = af^*$. In particular, if U is an $\mathscr{M}$-equivariant unitary from $\mathscr{H}_1$ to $\mathscr{H}_2$ then also U^* is $\mathscr{M}$-equivaraint and hence the notion of isomorphism in $\mathtt{FGHM}(\mathscr{M})$ makes sense. (See also Remark 1.3.4 below)
If $(\mathscr{H}, \pi)$ is any Hilbert $\mathscr{M}$-module and $T \in \mathscr{B}(\mathscr{H})$ is a self-adjoint $\mathscr{M}$-equivariant operator, then also $f(T)$ is $\mathscr{M}$-equivariant for any bounded Borel function f on $\sigma(T)$, since the space of $\mathscr{M}$-equivariant operators on $\mathscr{H}$ is equal to the von Neumann algebra $\pi(\mathscr{M})'$.

Example 1.3.3. *By definition, $L^2(\mathscr{M})^n$ is a Hilbert module over $\mathscr{M}$ when endowed with the diagonal action, and finitely generated exactly when n is finite.*
In the following we will often suppress the the diagonal-notation $\operatorname{diag}(a)$, *and simply write $a(x_1, \dots, x_n)$ in stead of* $\operatorname{diag}(a)(x_1, \dots, x_n)$.
By the results of Section 1.2, the map $\mathscr{M} \ni m \longmapsto ma \in \mathscr{M}$ extends to an operator R_a in $\mathscr{B}(L^2(\mathscr{M}), L^2(\mathscr{M}))^{\mathscr{M}}$ for every $a \in \mathscr{M}$ and we proved that the map

$$\mathscr{M}^{\text{op}} \ni a \longmapsto R_a = Ja^*J \in \mathscr{B}(L^2(\mathscr{M}), L^2(\mathscr{M}))^{\mathscr{M}},$$

is a $*$-algebra-isomorphism.
Thus, for $n, m \in \mathbb{N}$ and a matrix $A \in M_{n,m}(\mathscr{M})$ the $\mathscr{M}$-linear map $R_A : \mathscr{M}^n \to \mathscr{M}^m$ given by

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n)A,$$

extends to an operator (also denoted R_A) in $\mathscr{B}(L^2(\mathscr{M})^n, L^2(\mathscr{M})^m)^{\mathscr{M}}$ and every element in $\mathscr{B}(L^2(\mathscr{M})^n, L^2(\mathscr{M})^m)^{\mathscr{M}}$ arises in this way.
As in the case $n = 1$, we see that the map

$$M_n(\mathscr{M})^{\text{op}} \ni A^{\text{op}} \longmapsto R_A \in \mathscr{B}(L^2(\mathscr{M})^n)^{\mathscr{M}},$$

is a $*$-algebra-isomorphism.

Remark 1.3.4. *By definition, the morphisms in $\mathtt{FGHM}(\mathscr{M})$ is the bounded $\mathscr{M}$-equivariant operators and it is therefore natural to define an isomorphism in $\mathtt{FGHM}(\mathscr{M})$ as an invertible $\mathscr{M}$-equivariant operator. We now prove, that this notion of isomorphism is equivalent to the one given in Definition 1.3.1.*
So, let $\mathscr{H}$ and $\mathscr{K}$ be finitely generated Hilbert $\mathscr{M}$-modules and assume $f : \mathscr{H} \to \mathscr{K}$ to be a bijective $\mathscr{M}$-equivariant operator and consider the polar decomposition $f = V|f|$ of f. Then V is an isometry from $\overline{\operatorname{rg}|f|}$ to $\overline{\operatorname{rg}(f)} = \mathscr{K}$. (see e.g. [MV] Proposition 18.18)
Since f is invertible so is f^* (closed range theorem) and hence also f^*f is invertible. Thus

$$0 \notin \{\sqrt{t} | t \in \sigma(f^*f)\} = \sigma(\sqrt{f^*f}) = \sigma(|f|),$$

and hence also $|f|$ is invertible. Thus, V is an isometry from $\mathscr{H}$ to $\mathscr{K}$. To see that V is $\mathscr{M}$-equivaraint, we let $\xi \in \mathscr{H}$ and $a \in \mathscr{M}$ be given. Then, if we suppress the $*$-homomorphisms from $\mathscr{M}$ to $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}(\mathscr{K})$ respectively, we get

$$aV(|f|x) = af(x) = f(ax) = V|f|(ax) = Va(|f|x), \qquad \text{(for any } x \in \mathscr{H})$$

and since $\operatorname{rg}(|f|) = \mathscr{H}$, *we conclude that V is $\mathscr{M}$-equivaraint.*

Having established the basic properties of Hilbert $\mathscr{M}$-modules, we turn our attention towards the category of finitely generated projective modules over $\mathscr{M}$.

Definition 1.3.5. *Let P be a finitely generated projective module over $\mathscr{M}$. By an* inner product *on P we mean a map $\langle\cdot|\cdot\rangle : P \times P \to \mathscr{M}$ such that for $\alpha, \beta \in \mathscr{M}$ and $p, q, r \in P$ we have*

1. $\langle \alpha p + \beta q | r\rangle = \alpha\langle p|r\rangle + \beta\langle q|r\rangle$.
2. $\langle p|q\rangle = \langle q|p\rangle^*$
3. $\langle p|p\rangle \in \mathscr{M}_+$ *and* $\langle p|p\rangle = 0$ *only if* $p = 0$.
4. *The map $P \ni p \mapsto \langle\cdot|p\rangle \in \mathrm{Hom}_{\mathscr{M}}(P, \mathscr{M}) =: P^*$ is an isomorphism of $\mathscr{M}$-modules.*[2]

By a morphism *between finitely generated projective $\mathscr{M}$-modules, P and Q, with inner product we shall simply mean an $\mathscr{M}$-linear map. The set of morphisms from P to Q is denoted $\mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGPIP}}(P, Q)$. We denote by $\mathtt{FGPIP}(\mathscr{M})$, the category of finitely generated projective $\mathscr{M}$-modules with inner product, which in this way becomes a full subcategory of the category of finitely generated projective $\mathscr{M}$-modules. The latter category, will be denoted $\mathtt{FGP}(\mathscr{M})$.*

Example 1.3.6. *As an example of an object in $\mathtt{FGPIP}(\mathscr{M})$, we can consider $\mathscr{M}^n$ (for some $n \in \mathbb{N}$) endowed with the* standard inner product $\langle\cdot|\cdot\rangle_{\mathrm{st}}$ *given by*

$$\langle (a_1, \dots, a_n) | (b_1, \dots, b_n)\rangle_{\mathrm{st}} := \sum_{i=1}^{n} a_i b_i^*.$$

One easily checks, that $\langle\cdot|\cdot\rangle_{\mathrm{st}}$ fulfills the requirements in Definition 1.3.5.
Unless otherwise mentioned, we always view $\mathscr{M}^n$ as an element in $\mathtt{FGPIP}(\mathscr{M})$ with respect to the standard inner product.

In the following remark, we collect some easy facts concerning finitely generated projective $\mathscr{M}$-modules with inner product.

Remark 1.3.7. *Consider a morphism $f : (P_1, \langle\cdot|\cdot\rangle_1) \to (P_2, \langle\cdot|\cdot\rangle_2)$ of finitely generated projective modules with inner product. Just as in the case of Hilbert spaces, we get a (unique)*[3] *adjoint morphism $f^* : (P_2, \langle\cdot|\cdot\rangle_2) \to (P_1, \langle\cdot|\cdot\rangle_1)$ by requiring that*

$$\langle f(x)|y\rangle_2 = \langle x|f^*(y)\rangle_1 \quad \text{for all } x \in P_1 \text{ and } y \in P_2.$$

Note, that if $A \in M_{n,m}(\mathscr{M})$ and we consider the $\mathscr{M}$-linear map $R_A : \mathscr{M}^n \to \mathscr{M}^m$, then $R_A^* = R_{A^*}$ when both $\mathscr{M}^n$ and $\mathscr{M}^m$ are endowed with their standard inner products.

If $(P_1, \langle\cdot|\cdot\rangle_1), (P_2, \langle\cdot|\cdot\rangle_2) \in \mathtt{FGPIP}(\mathscr{M})$, then $\langle\cdot|\cdot\rangle : P_1 \oplus P_2$ given by

$$\langle (p_1, p_2) | (q_1, q_2)\rangle := \langle p_1|q_1\rangle_1 + \langle p_2|q_2\rangle_2,$$

is an inner product on $P_1 \oplus P_2$. Thus $\mathtt{FGPIP}(\mathscr{M})$ is stable under direct sums.
Unless otherwise mentioned, we always view $P_1 \oplus P_2$ as an element in $\mathtt{FGPIP}(\mathscr{M})$ with respect to this inner product.
Note, that for any pair of objects $P, Q \in \mathtt{FGPIP}(\mathscr{M})$ the morphisms $\mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGPIP}}(P, Q)$ has the structure of a complex vector space, where the addition is pointwise and multiplication by scalars is defined by

$$(\lambda f)(x) := (\lambda 1_{\mathscr{M}}) f(x).$$

In this language, the contents of Example 1.3.3 may be reformulated in the following way:
The map from $\mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGPIP}}(\mathscr{M}^n, \mathscr{M}^m)$ to $\mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGHM}}(L^2(\mathscr{M})^n, L^2(\mathscr{M})^m)$ which extends a morphism by continuity, is an isomorphism of vector spaces.

[2] See e.g. Definition 1.1.7

[3] Existence of f^* follows from 4. and uniqueness follows from 3. in Definition 1.3.5.

We adopt the following language from the theory of Hilbert spaces.

Definition 1.3.8. *An endomorphism f in $\mathtt{FGPIP}(\mathscr{M})$ is called* self-adjoint *if $f^* = f$ and an isomorphism u is called unitary if $u^* = u^{-1}$.*
An endomorphism $f : (P, \langle\cdot|\cdot\rangle) \longrightarrow (P, \langle\cdot|\cdot\rangle)$ is called positive, if $\langle f(x)|x\rangle \in \mathscr{M}_+$ for all $x \in P$.

The main goal of this section, is to prove that the categories $\mathtt{FGPIP}(\mathscr{M})$ and $\mathtt{FGHM}(\mathscr{M})$ are actually equivalent categories. (see e.g. Section 1.3.1 below for a definition) To this end, we need to define a potential equivalence between the two categories. This is done in the following way.

Consider an object $(P, \langle\cdot|\cdot\rangle)$ in $\mathtt{FGPIP}(\mathscr{M})$ and note that P is a complex vector space with respect to the action $\lambda \cdot x := (\lambda 1)x$. Since τ (the fixed trace on $\mathscr{M}$) is faithful the sesquilinear form $\tau \circ \langle\cdot|\cdot\rangle$ is a scalar-valued inner product on P and by completing P with respect to the associated norm, we get a Hilbert space. Following Lück (see e.g. [Lüc97]), we denote this Hilbert space by $\nu(P, \langle\cdot|\cdot\rangle)$. When it is clear from the context which inner product P is endowed with, we shall sometimes omit the reference to the inner product and simply denote the completion by $\nu(P)$.
One easily checks that $\nu(\mathscr{M}^n, \langle\cdot|\cdot\rangle_{\mathrm{st}}) = L^2(\mathscr{M})^n$, where the latter is the direct Hilbert-space sum of n copies of $L^2(\mathscr{M})$. We now want to turn $\nu(P, \langle\cdot|\cdot\rangle)$ into an Hilbert $\mathscr{M}$-module and ν into a functor.
Before doing so, we introduce some terminology from category theory which will be needed.

1.3.1 A bit of category theory

In this section we briefly introduce the notion of equivalence of functors and equivalence of categories and we prove the one result needed for our purposes. The reader who is familiar with category theory may therefore skip this part. More details on categories, functors, equivalences of categories, ect. can be found in [MacL].

Definition 1.3.9. *Let $\mathtt{C}$ and $\mathtt{D}$ be categories and assume F and G to be covariant functors from $\mathtt{C}$ to $\mathtt{D}$.*
A natural transformation *from F to G is a family of morphisms $\alpha_c : F(c) \to G(c)$ (one for each object in $\mathtt{C}$), such that for any two objects $c', c'' \in \mathtt{C}$ and any morphism $f \in \mathtt{Mor}^{\mathtt{C}}(c', c'')$ the following diagram commutes*

$$\begin{array}{ccc} F(c') & \xrightarrow{F(f)} & F(c'') \\ {\scriptstyle \alpha_{c'}}\downarrow & & \downarrow{\scriptstyle \alpha_{c''}} \\ G(c') & \xrightarrow[G(f)]{} & G(c'') \end{array}$$

A natural isomorphism *from F to G is a natural transformation $(\alpha_c)_{c \in \mathrm{Obj}(\mathtt{C})}$ from F to G, in which each α_c is an isomorphism. In this case $(\alpha_c^{-1})_{c \in \mathrm{Obj}(\mathtt{C})}$ is a natural isomorphism from G to F and we say that F and G are* naturally isomorphic*. This is denoted by $F \simeq G$.*

Definition 1.3.10. *Let $\mathtt{C}$ and $\mathtt{D}$ be categories. By an* equivalence *from $\mathtt{C}$ to $\mathtt{D}$, we mean a functor $F : \mathtt{C} \to \mathtt{D}$, for which there exists a functor $G : \mathtt{D} \to \mathtt{C}$ such that*

- *The composition $F \circ G$ is naturally isomorphic to the identity functor $\mathrm{id}_{\mathtt{D}}$ on $\mathtt{D}$.*

- *The composition $G \circ F$ is naturally isomorphic to the identity functor $\mathrm{id}_{\mathtt{C}}$ on $\mathtt{C}$.*

In this case, G is an equivalence from $\mathtt{D}$ to $\mathtt{C}$ and we say that $\mathtt{C}$ and $\mathtt{D}$ are equivalent.
If $F \circ G = \mathrm{id}_{\mathtt{D}}$ and $G \circ F = \mathrm{id}_{\mathtt{C}}$, we say that $\mathtt{C}$ and $\mathtt{D}$ are isomorphic *categories and F (respectively G) is said to be an* inverse *of G (respectively F).*

Proposition 1.3.11. [MacL] *Let $\mathtt{C}$ be a category and assume that $\mathtt{C}_0$ is a full subcategory* [4] *of $\mathtt{C}$, with the property that each object $c \in \mathtt{C}$ is isomorphic to exactly one object $c' \in \mathtt{C}_0$.*
Then the inclusion functor $I : \mathtt{C}_0 \longrightarrow \mathtt{C}$ is an equivalence of categories.

Proof. Choose, for each object $c \in \mathtt{C}$, the unique object $c' \in \mathtt{C}_0$ isomorphic to it and an isomorphism $\alpha_c : c \to c'$. If $c \in \mathtt{C}_0$ we choose $\alpha_c = \mathrm{id}_c$. Define $G : \mathtt{C} \to \mathtt{C}_0$ on objects by setting $G(c) := c'$.
If $c_1, c_2 \in \mathtt{C}$ and $f : c_1 \to c_2$ is a morphism, we define $G(f) := \alpha_{c_2} \circ f \circ \alpha_{c_1}^{-1} : c_1' \to c_2'$.
In this way G is turned into a functor (this is easily verified) from $\mathtt{C}$ to $\mathtt{C}_0$ and obviously we have $G \circ I = \mathrm{id}_{\mathtt{C}_0}$. We now need to prove that $I \circ G \simeq \mathrm{id}_{\mathtt{C}}$.
For any $c_1, c_2 \in \mathtt{C}$ and any morphism $f : c_1 \to c_2$ we have the following commutative diagram

$$\begin{array}{ccc} c_1 & \xrightarrow[\sim]{\alpha_{c_1}} & c_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha_{c_2} \circ f \circ \alpha_{c_1}^{-1}} \\ c_2 & \xrightarrow[\alpha_{c_2}]{\sim} & c_2' \end{array}$$

Since $I \circ G(c_1) = c_1'$, $I \circ G(c_2) = c_2'$ and $I \circ G(f) = \alpha_{c_2} \circ f \circ \alpha_{c_1}^{-1}$, this exactly means that $I \circ G \simeq \mathrm{id}_{\mathtt{C}}$. □

We aim to prove that (for any finite von Neumann algebra $\mathscr{M}$) the categories $\mathtt{FGHM}(\mathscr{M})$ and $\mathtt{FGPIP}(\mathscr{M})$ are equivalent. In the light of Proposition 1.3.11, our strategy for this is to find two isomorphic subcategories $\mathtt{C}_0 \subseteq \mathtt{FGPIP}(\mathscr{M})$ and $\mathtt{D}_0 \subseteq \mathtt{FGHM}(\mathscr{M})$, satisfying the requirements in Proposition 1.3.11. This is done in the following section.

1.3.2 The equivalence of $\mathtt{FGHM}(\mathscr{M})$ and $\mathtt{FGPIP}(\mathscr{M})$

Let $\mathscr{M}$ be a finite von Neumann algebra, endowed with a fixed normal, faithful, tracial state τ. For every finitely generated projective $\mathscr{M}$-module with inner product $(P, \langle\cdot|\cdot\rangle)$, we defined (in the last lines of Section 1.3) $\nu(P, \langle\cdot|\cdot\rangle)$ to be the Hilbert space completion of P, with respect to the scalar-valued inner product $\tau \circ \langle\cdot|\cdot\rangle$. This construction has the following properties.

Lemma 1.3.12. [Lüc97] *Consider the sets $\{\mathscr{M}^n | n \in \mathbb{N}\}$ and $\{L^2(\mathscr{M})^n | n \in \mathbb{N}\}$, as full subcategories in $\mathtt{FGPIP}(\mathscr{M})$ and $\mathtt{FGHM}(\mathscr{M})$ respectively.*
For every $n, m \in \mathbb{N}$ and every $\mathscr{M}$-linear map $f : \mathscr{M}^n \to \mathscr{M}^m$, f extends to a bounded $\mathscr{M}$-equivariant operator $\nu(f) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^m$.
In this way ν is turned into a covariant functor from $\{\mathscr{M}^n | n \in \mathbb{N}\}$ to $\{L^2(\mathscr{M})^n | n \in \mathbb{N}\}$ and the following holds.

(i) $\nu : \{\mathscr{M}^n | n \in \mathbb{N}\} \to \{L^2(\mathscr{M})^n | n \in \mathbb{N}\}$ is an isomorphism of categories.

(ii) For any $n, m \in \mathbb{N}$, the map $\nu : \mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGPIP}}(\mathscr{M}^n, \mathscr{M}^m) \to \mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGHM}}(L^2(\mathscr{M})^n, L^2(\mathscr{M}))^m)$ is an isomorphism of vector spaces.

(iii) ν preserves adjoints. I.e. for any morphism $f : \mathscr{M}^n \to \mathscr{M}^m$ we have $\nu(f^*) = \nu(f)^*$.

Almost all of Lemma 1.3.12 is already proven in the preceding part of this section, but since the arguments are spread over four pages we put together the details.

Proof. We already noted that $\nu(\mathscr{M}^n, \langle\cdot|\cdot\rangle_{\mathrm{st}}) = L^2(\mathscr{M})^n$ and hence ν is well-defined on objects.
As shown in Remark 1.1.2, any morphism $f : \mathscr{M}^n \to \mathscr{M}^m$ is given by right multiplication with a unique matrix $A \in M_{n,m}(\mathscr{M})$. In Example 1.3.3 we saw that every such morphism may be

[4] I.e. $\mathtt{Mor}^{\mathtt{C}_0}(c', c'') = \mathtt{Mor}^{\mathtt{C}}(c', c'')$ for all objects $c', c'' \in \mathtt{C}_0$

extended by continuity to an operator $\nu(f) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^m$ and that every $\mathscr{M}$-equivariant operator arises in this way.
Since "extension by continuity" clearly preserves composition of morphisms and extends the identity morphism to the identity morphism, it follows that ν is a covariant functor.
Part (*ii*) of the lemma is proven in Example 1.3.3. (See also Remark 1.3.7)
We now define $\mu(L^2(\mathscr{M})^n) := \mathscr{M}^n$ and for a morphism $F : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^m$ we define $\mu(F) := F|_{\mathscr{M}^n}$. Then, by what is already proven, μ is a functor and by construction an inverse to ν.
We now just need to prove that ν preserves adjoints. Consider any morphism $f : \mathscr{M}^n \to \mathscr{M}^m$. The inner product on $L^2(\mathscr{M})^n$ is given by (extension of) $\tau \circ \langle\cdot|\cdot\rangle_{\text{st}}$ on $\mathscr{M}^n$ and from this it follows, that $\nu(f^*)$ and $\nu(f)^*$ must agree on the dense subspace $\mathscr{M}^n \subseteq L^2(\mathscr{M})^m$ and by continuity therefore everywhere. (A more detailed argument for this fact will be given in the proof of Theorem 1.3.17 part (*iii*))

□

Consider a matrix $A \in M_{n,m}(\mathscr{M})$. To this point we have used the symbol R_A to denote the map $R_A : \mathscr{M}^n \to \mathscr{M}^m$, as well as its extension to $L^2(\mathscr{M})^n$. In the rest of this chapter, we reserve the symbol R_A to denote the map $R_A : \mathscr{M}^n \to \mathscr{M}^m$ and denote by $\nu(R_A)$ its extension to $L^2(\mathscr{M})^n$.

Observation 1.3.13. *Consider a finitely generated projective $\mathscr{M}$-module P. By Lemma 1.1.1, P is isomorphic to $\mathscr{M}^n p$ for some $n \in \mathbb{N}$ and some idempotent matrix $p \in M_n(\mathscr{M})$.*
Since p is idempotent, the extension $\nu(R_p) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$ is idempotent and has therefore closed range and since $\nu(R_p)$ is $\mathscr{M}$-equivariant its range is $\mathscr{M}$-invariant. Thus, $\operatorname{rg}(\nu(R_p))$ is a finitely generated Hilbert $\mathscr{M}$-module. Let π denote the ($\mathscr{M}$-equivariant) projection in $\mathscr{B}(L^2(\mathscr{M})^n)$ onto $\operatorname{rg}(\nu(R_p))$ and let $q \in M_n(\mathscr{M})$ be the unique self-adjoint and idempotent (see e.g. Example 1.3.3) matrix such that $\nu(R_q) = \pi$. We now show that

$$\mathscr{M}^n p = \mathscr{M}^n q.$$

We have

$$\nu((\mathrm{id}_{\mathscr{M}^n} - R_q)R_p) = (\mathrm{id}_{L^2(\mathscr{M})^n} - \pi)\nu(R_p) = 0,$$

and, by Lemma 1.3.12, this implies $(\mathrm{id}_{\mathscr{M}^n} - R_q)R_p = 0$. Thus,

$$\mathscr{M}^n p = \operatorname{rg}(R_p) \subseteq \ker(\mathrm{id}_{\mathscr{M}^n} - R_q) = \operatorname{rg}(R_q) = \mathscr{M}^n q.$$

Similarly we have

$$\nu((\mathrm{id}_{\mathscr{M}^n} - R_p)R_q) = (\mathrm{id}_{L^2(\mathscr{M})^n} - \nu(R_p))\pi = 0,$$

and thus $\mathscr{M}^n q \subseteq \mathscr{M}^n p$.
This proves the following.

Any finitely generated projective $\mathscr{M}$-module is isomorphic to $\mathscr{M}^n q$ for a suitable $n \in \mathbb{N}$ and idempotent **self-adjoint** matrix $q \in M_n(\mathscr{M})$.

Since the trace $\tau_n : M_n(\mathscr{M}) \longrightarrow \mathbb{C}$ is faithful, this implies that the (non-extended)dimension function $\dim_{\mathscr{M}}(\cdot)$ is faithful. That is, if P is a finitely generated projective $\mathscr{M}$-module then $\dim_{\mathscr{M}}(P) = 0$ if, and only if, $P = \{0\}$.

Lemma 1.3.14. *Every finitely generated projective $\mathscr{M}$-module admits an inner product.*

Proof. Since any finitely generated projective $\mathscr{M}$-module, is isomorphic to $\mathscr{M}^n p$ for some $n \in \mathbb{N}$ and some self-adjoint idempotent $p \in M_n(\mathscr{M})$, it suffices to prove the claim for modules of this type. We now claim, that the restriction of $\langle\cdot|\cdot\rangle_{\text{st}}$ to $\mathscr{M}^n p$ is an inner product.
Clearly the restriction of $\langle\cdot|\cdot\rangle_{\text{st}}$ fulfills *1.,2.* and *3.* in Definition 1.3.5, so we only have to check that $\mathscr{M}^n p \ni x \longmapsto \langle\cdot|x\rangle_{\text{st}} \in (\mathscr{M}^n p)^*$ is an isomorphism.

To see that $x \mapsto \langle \cdot | x \rangle_{\mathrm{st}}$ is surjective, we need to prove that every $\mathscr{M}$-linear map $f : \mathscr{M}^n p \to \mathscr{M}$ has the form $f(x) = \langle x | x_0 \rangle_{\mathrm{st}}$ for a suitable $x_0 \in \mathscr{M}^n p$.
The map $f \circ R_p : \mathscr{M}^n \to \mathscr{M}$ is $\mathscr{M}$-linear and hence of the form $f \circ R_p(x) = \langle x | y \rangle_{\mathrm{st}}$ for some $y \in \mathscr{M}^n$. For $x \in \mathscr{M}^n p$ we have

$$f(x) = f \circ R_p(R_p x) = \langle R_p x | y \rangle_{\mathrm{st}} = \langle x | R_p^* y \rangle_{\mathrm{st}} = \langle x | R_{p^*} y \rangle_{\mathrm{st}} = \langle x | R_p y \rangle_{\mathrm{st}},$$

and hence $x_0 := R_p y$ has the desired property. A similar argument shows that $x \mapsto \langle \cdot | x \rangle_{\mathrm{st}}$ is injective.

□

As the proposition below shows, the inner product on a finitely generated projective $\mathscr{M}$-module, constructed in the proof of Lemma 1.3.14, is essentially unique.

Proposition 1.3.15. [Lüc97] *Let P_1 and P_2 be finitely generated projective $\mathscr{M}$-modules, with inner products $\langle \cdot | \cdot \rangle_1$ and $\langle \cdot | \cdot \rangle_2$ respectively. Then $(P_1, \langle \cdot | \cdot \rangle_1)$ and $(P_2, \langle \cdot | \cdot \rangle_2)$ are unitarily isomorphic if, and only if, P_1 and P_2 are isomorphic as $\mathscr{M}$-modules.*

Proof. The "only if" statement is clear.
Assume that P_1 and P_2 are isomorphic as $\mathscr{M}$-modules. We start with some reductions.
By Observation 1.3.13, we may assume, that $P_1 = \mathscr{M}^n p$ for some $n \in \mathbb{N}$ and self-adjoint idempotent $p \in M_n(\mathscr{M})$. Since P_1 and P_2 are isomorphic, we may also assume that $P_1 = P_2$ and that $\langle \cdot | \cdot \rangle_2$ is the standard inner product $\langle \cdot | \cdot \rangle_{\mathrm{st}}$ restricted to $P_2 = \mathscr{M}^n p \subseteq \mathscr{M}^n$. The problem is now reduced to finding a unitary isomorphism from $(P_1, \langle \cdot | \cdot \rangle_1)$ to $(P_1, \langle \cdot | \cdot \rangle_{\mathrm{st}})$.
Define $f : P_1 \to P_1$ be requiring that

$$\langle x | y \rangle_1 = \langle fx | y \rangle_{\mathrm{st}} \text{ for all } x, y \in P_1.$$

(f is the adjoint of the identity $\mathrm{id} : (P_1, \langle \cdot | \cdot \rangle_{\mathrm{st}}) \to (P_1, \langle \cdot | \cdot \rangle_1)$.)
Note, that for any $x \in P_1$ we have $\langle fx | x \rangle_{\mathrm{st}} = \langle x | x \rangle_1 \in \mathscr{M}_+$. Furthermore, f is self-adjoint with respect to $\langle \cdot | \cdot \rangle_{\mathrm{st}}$, since for any $x, y \in P_1$ we have

$$\langle x | fy \rangle_{\mathrm{st}} = \langle fy | x \rangle_{\mathrm{st}}^* = \langle y | x \rangle_1^* = \langle x | y \rangle_1 = \langle fx | y \rangle_{\mathrm{st}}.$$

Let ι denote the inclusion $P_1 \subseteq \mathscr{M}^n$, and consider the morphism $f' := \iota \circ f \circ R_p : \mathscr{M}^n \to \mathscr{M}^n$. Using that R_p is self-adjoint and idempotent, one easily checks that f' is self-adjoint and positive with respect to $\langle \cdot | \cdot \rangle_{\mathrm{st}}$. Hence the operator $\nu(f')$ on $L^2(\mathscr{M})^n$ is positive (in the operator sense) and it therefore has a square root inside $\mathscr{B}(L^2(\mathscr{M})^n)$. Note that $\sqrt{\nu(f')}$ is again $\mathscr{M}$-equivariant by Remark 1.3.2. Define $g := R_p \circ \mu(\sqrt{\nu(f')}) \circ \iota : P_1 \to P_1$, where μ is the inverse functor to ν defined in the proof of Lemma 1.3.12. Using that $\sqrt{\nu(f')}$ is self-adjoint, it is not hard to check that g is self-adjoint with respect to $\langle \cdot | \cdot \rangle_{\mathrm{st}}$ and we now claim that $g^2 = f$.
To see this, we first prove that $\mu(\sqrt{\nu(f')}) \circ R_p = \mu(\sqrt{\nu(f')})$. By construction of f' we have $f' \circ R_p = f'$ and hence $\nu(f') \circ \nu(R_p) = \nu(f')$. Thus, for any $N \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ we have

$$\Big(\sum_{k=1}^{N} \alpha_k (\nu(f'))^k \Big) \circ \nu(R_p) = \sum_{k=1}^{N} \alpha_k (\nu(f'))^k. \qquad (*)$$

Since $\sqrt{0} = 0$, the function $t \mapsto \sqrt{t}$ on $\sigma(\nu(f'))$ can be approximated uniformly by polynomials without constant terms and from the identity $(*)$ we conclude that $\sqrt{\nu(f')} \circ \nu(R_p) = \sqrt{\nu(f')}$.
Applying the functor μ now gives $\mu(\sqrt{\nu(f')}) \circ R_p = \mu(\sqrt{\nu(f')})$.
With this identity established, we are now in position to prove that $g^2 = f$.

Choose any $x \in P_1 = \mathscr{M}^n p$. We then have

$$\begin{aligned} g^2(x) &= R_p \circ \mu(\sqrt{\nu(f')}) \circ \iota \circ R_p \circ \mu(\sqrt{\nu(f')}) \circ \iota(x) \\ &= R_p \circ \mu(\sqrt{\nu(f')}) \circ R_p \circ \mu(\sqrt{\nu(f')})(x) \\ &= R_p \circ (\mu(\sqrt{\nu(f')}\sqrt{\nu(f')}))(x) \\ &= R_p \circ \mu(\nu(f'))(x) \\ &= R_p \circ f'(x) \\ &= R_p(f \circ R_p(x)) \\ &= R_p(fx) && (\text{since } x \in \mathscr{M}^n p) \\ &= f(x), && (\text{since } f(x) \in \mathscr{M}^n p) \end{aligned}$$

and hence $g^2 = f$. Thus, for given $x, y \in P_1$ we have

$$\langle gx \,|\, gy\rangle_{\mathrm{st}} = \langle g^* gx \,|\, y\rangle_{\mathrm{st}} = \langle g^2 x \,|\, y\rangle_{\mathrm{st}} = \langle fx \,|\, y\rangle_{\mathrm{st}} = \langle x \,|\, y\rangle_1.$$

Since f is bijective so is g and hence g is a unitary from $(P_1, \langle\cdot|\cdot\rangle_1)$ to $(P_1, \langle\cdot|\cdot\rangle_{\mathrm{st}})$. This completes the proof. □

Corollary 1.3.16. *Every finitely generated projective $\mathscr{M}$-module possess an inner product, which is unique up to (unitary) isomorphism.*

Proof. The statement is just the combination of Lemma 1.3.14 and Proposition 1.3.15. □

We are now ready to prove the promised equivalence between $\mathtt{FGPIP}(\mathscr{M})$ and $\mathtt{FGHM}(\mathscr{M})$.

Theorem 1.3.17. [Lüc97] *For any $(P_1, \langle\cdot|\cdot\rangle_1)$ and $(P_2, \langle\cdot|\cdot\rangle_2)$ in $\mathtt{FGPIP}(\mathscr{M})$ and any morphism $f : P_1 \to P_2$, the completions $\nu(P_1)$ and $\nu(P_2)$ are finitely generated Hilbert modules over $\mathscr{M}$ and f extends to a bounded $\mathscr{M}$-equivariant operator $\nu(f) : \nu(P_1) \to \nu(P_2)$.*
In this way, ν is turned into a covariant functor from $\mathtt{FGPIP}(\mathscr{M})$ to $\mathtt{FGHM}(\mathscr{M})$.
Furthermore

(i) ν is an equivalence of categories.

(ii) For any $P_1, P_2 \in \mathtt{FGPIP}(\mathscr{M})$, the map $\nu : \mathrm{Hom}^{\mathtt{FGPIP}}_{\mathscr{M}}(P_1, P_2) \to \mathrm{Hom}^{\mathtt{FGHM}}_{\mathscr{M}}(\nu(P_1), \nu(P_2))$ is an isomorphism of vector spaces.

(iii) ν preserves adjoints. I.e. for any morphism f in $\mathtt{FGPIP}(\mathscr{M})$ we have $\nu(f^*) = \nu(f)^*$.

Proof.

Proof of (i): By Proposition 1.3.15 and Observation 1.3.13, it suffices to check that $\nu((\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}})) \in \mathtt{FGHM}(\mathscr{M})$ for any $n \in \mathbb{N}$ and any self-adjoint idempotent $p \in M_n(\mathscr{M})$.
Consider the extension $\nu(R_p) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$ and note that $\nu(R_p)L^2(\mathscr{M})^n \in \mathtt{FGHM}(\mathscr{M})$ since $\nu(R_p)$ is an $\mathscr{M}$-equivariant projection.
When $\mathscr{M}^n p = R_p(\mathscr{M}^n)$ is endowed with the norm induced by the scalar-valued inner product $\tau \circ \langle\cdot|\cdot\rangle_{\mathrm{st}}$, the inclusion $R_p(\mathscr{M}^n) \subseteq \nu(R_p)L^2(\mathscr{M})^n$ is an isometric $\mathscr{M}$-embedding of vector spaces.
Since $R_p(\mathscr{M}^n)$ is dense in $\nu(R_p)L^2(\mathscr{M})^n$, we get $\nu(R_p(\mathscr{M}^n), \langle\cdot|\cdot\rangle_{\mathrm{st}}) = \nu(R_p)L^2(\mathscr{M})^n \in \mathtt{FGHM}(\mathscr{M})$.
We now prove that any $\mathtt{FGPIP}$-morphism $f : (\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}}) \to (\mathscr{M}^m q, \langle\cdot|\cdot\rangle_{\mathrm{st}})$ can be extended to an $\mathtt{FGHM}$-morphism $\nu(f) : \nu(R_p)L^2(\mathscr{M})^n \to \nu(R_q)L^2(\mathscr{M})^m$.
Denote by ι the inclusion $\mathscr{M}^m q \subseteq \mathscr{M}^m$ and consider the $\mathtt{FGPIP}$-morphism

$$g := \iota \circ f \circ R_p : \mathscr{M}^n \to \mathscr{M}^m.$$

By Lemma 1.3.12, g extends to an FGHM-morphism $\nu(g) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^m$.
For $x \in \mathscr{M}^n$ we have

$$\nu(g)(xp) = (\iota \circ f \circ R_p)(xp) = f(xp) \in \mathscr{M}^n q.$$

From this it follows that

- $\nu(g)$ maps $\nu(R_p)L^2(\mathscr{M})^n$ into $\nu(R_q)L^2(\mathscr{M})^m$.
- The restriction $\nu(f) := \nu(g)\Big|_{\nu(R_p)L^2(\mathscr{M})^n} : \nu(R_p)L^2(\mathscr{M})^n \to \nu(R_q)L^2(\mathscr{M})^m$ extends f.

Since "extension by continuity" respects composition of morphisms and the identity morphism extends to the identity morphism, we see that ν defined in this way is a covariant functor.
Before proving that ν is an equivalence of categories, we prove (ii) and (iii).

Proof of (ii): We need to see that

$$\nu : \mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGPIP}}(P_1, P_2) \to \mathrm{Hom}_{\mathscr{M}}^{\mathtt{FGHM}}(\nu(P_1), \nu(P_2)),$$

is an isomorphism of vector spaces, for arbitrary $(P_1, \langle\cdot|\cdot\rangle_1), (P_2, \langle\cdot|\cdot\rangle_2) \in \mathtt{FGPIP}(\mathscr{M})$.
Since ν extends morphisms by continuity, it follows that ν is $\mathbb{C}$-linear and injective and hence we just have to prove that ν is surjective.
As above, it suffices to consider the case where $P_1 = (\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}})$ and $P_2 = (\mathscr{M}^m q, \langle\cdot|\cdot\rangle_{\mathrm{st}})$. So, we need to see that every $\mathscr{M}$-equivariant operator $F : \nu(R_p)L^2(\mathscr{M})^n \to \nu(R_q)L^2(\mathscr{M})^m$ arises as the extension of an $\mathscr{M}$-linear map $f : \mathscr{M}^n p \to \mathscr{M}^m q$. To see this, we let ι denote the inclusion $\nu(R_q)L^2(\mathscr{M})^m \subseteq L^2(\mathscr{M})^m$ and consider the operator

$$G := \iota \circ F \circ \nu(R_p) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^m.$$

By Lemma 1.3.12, there exists a (unique) matrix $A \in M_{n,m}(\mathscr{M})$ such that $\nu(R_A) = G$.
For $x \in \mathscr{M}^n$ we have

$$R_A(xp) = G(xp) = \iota \circ F \circ \nu(R_p)(xp) = F(xp) \in \mathrm{rg}(F) \subseteq \nu(R_q)L^2(\mathscr{M})^m. \tag{$\dagger$}$$

But $R_A(xp) \in \mathscr{M}^m$ and hence R_A maps $\mathscr{M}^n p$ into $\nu(R_q)L^2(\mathscr{M})^m \cap \mathscr{M}^m = \mathscr{M}^m q$. Thus, we can consider the restriction of R_A to $\mathscr{M}^n p$ as a morphism $f : \mathscr{M}^n p \to \mathscr{M}^n q$. The extension $\nu(f) : \nu(R_p)L^2(\mathscr{M})^n \to \nu(R_q)L^2(\mathscr{M})^m$ coincides with F, since the computation ($\dagger$) shows that the two maps agree on the dense subspace $\mathscr{M}^n p$.
This concludes the proof of (ii).

Proof of (iii): By Proposition 1.3.15, it suffices to consider modules of the form $(\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}})$ and $(\mathscr{M}^m q, \langle\cdot|\cdot\rangle_{\mathrm{st}})$ where p, q are idempotent self-adjoint matrices in $M_n(\mathscr{M})$ and $M_m(\mathscr{M})$ respectively. We now consider a morphism $f : \mathscr{M}^n p \to \mathscr{M}^m q$ and wish to prove that $\nu(f^*) = \nu(f)^*$.
Let $x \in \mathscr{M}^n p$ and $y \in \mathscr{M}^m q$ be given and let $\langle\cdot|\cdot\rangle$ denote the inner product on both $L^2(\mathscr{M})^n$ and $L^2(\mathscr{M})^m$. Then

$$\begin{aligned}
\langle x | (\nu(f^*) - \nu(f)^*)y\rangle &= \langle x | \nu(f^*)y\rangle - \langle x | \nu(f)^* y\rangle \\
&= \langle x | f^* y\rangle - \langle \nu(f)x | y\rangle \\
&= \langle x | f^* y\rangle - \langle fx | y\rangle \\
&= \tau(\langle x | f^* y\rangle_{\mathrm{st}} - \langle fx | y\rangle_{\mathrm{st}}) \\
&= 0.
\end{aligned}$$

Hence $(\nu(f^*) - \nu(f)^*)y$ is orthogonal to the dense subspace $\mathscr{M}^n p$ and thus zero. Since this holds for any y in the dense subspace $\mathscr{M}^m q \subseteq \nu(\mathscr{M}^n q, \langle\cdot|\cdot\rangle_{\mathrm{st}})$, we conclude that $\nu(f^*) - \nu(f)^* = 0$. This completes the proof of (iii).
We now prove that ν is an equivalence of categories.

Proof of (i) continued: Consider the full subcategory

$$\mathtt{C} := \{(\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}}) \mid n \in \mathbb{N}, p \text{ a self-adjoint and idempotent matrix in } M_n(\mathscr{M})\},$$

in $\mathtt{FGPIP}(\mathscr{M})$. On $\mathtt{C}$ we consider the equivalence relation "being unitarily isomorphic to" and for each isomorphism class we choose a representative.
Denote by $\mathtt{C}_0$ the full subcategory in $\mathtt{FGPIP}(\mathscr{M})$, consisting of those representatives. By construction $\mathtt{C}_0$ has the following property:

For every $c \in \mathtt{FGPIP}(\mathscr{M})$ there exists a unique $c' \in \mathtt{C}_0$ such that $c \simeq c'$.

Consider

$$\mathtt{D}_0 := \{\nu(R_p)L^2(\mathscr{M})^n \mid (\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}}) \in \mathtt{C}_0\ \}$$

as a full subcategory of $\mathtt{FGHM}(\mathscr{M})$. We now want to prove that

For every $x \in \mathtt{FGHM}(\mathscr{M})$ there exists a unique $x' \in \mathtt{D}_0$ such that $x \simeq x'$.

Any object x in $\mathtt{FGHM}(\mathscr{M})$ is (unitarily) isomorphic to $\pi L^2(\mathscr{M})^n$ for some $n \in \mathbb{N}$ and an $\mathscr{M}$-equivariant projection $\pi : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$.
By Lemma 1.3.12, there exists an idempotent self-adjoint matrix $p \in M_n(\mathscr{M})$ with $\nu(R_p) = \pi$. Then $(\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}}) \in \mathtt{FGPIP}(\mathscr{M})$ and thus there exists a unique $k \in \mathbb{N}$ and self-adjoint idempotent matrix $q \in M_k(\mathscr{M})$ such that $(\mathscr{M}^k q, \langle\cdot|\cdot\rangle_{\mathrm{st}}) \in \mathtt{C}_0$ and $(\mathscr{M}^k q, \langle\cdot|\cdot\rangle_{\mathrm{st}})$ is unitarily isomorphic to $(\mathscr{M}^n p, \langle\cdot|\cdot\rangle_{\mathrm{st}})$. This implies, that also their completions, $\nu(\mathscr{M}^k q)$ and $\nu(\mathscr{M}^n p)$, are unitarily isomorphic and thus

$$x \simeq \nu(R_p)L^2(\mathscr{M})^n \simeq \nu(R_q)L^2(\mathscr{M})^k \in \mathtt{D}_0.$$

This shows, that any object in $\mathtt{FGHM}(\mathscr{M})$ is isomorphic to one in $\mathtt{D}_0$ and we now have to prove that this object is unique. So, consider arbitrary $\mathscr{M}^n p, \mathscr{M}^m q \in \mathtt{C}_0$ and assume that $U : \nu(R_p)L^2(\mathscr{M})^n \to \nu(R_q)L^2(\mathscr{M})^m$ is a unitary $\mathscr{M}$-equivariant operator. By what is already proven, $U|_{\mathscr{M}^n p}$ is an isomorphism from $\mathscr{M}^n p$ to $\mathscr{M}^m q$ and by Proposition 1.3.15 it now follows that $\mathscr{M}^m q$ and $\mathscr{M}^n p$ are unitarily isomorphic. By construction of $\mathtt{C}_0$ we therefore have $\mathscr{M}^n p = \mathscr{M}^n q$.

Applying Proposition 1.3.11, we now have that $\mathtt{FGPIP}(\mathscr{M})$ is equivalent to $\mathtt{C}_0$ and that $\mathtt{FGHM}(\mathscr{M})$ is equivalent to $\mathtt{D}_0$. Hence it suffices to show that $\mathtt{C}_0$ and $\mathtt{D}_0$ are equivalent.
Denote by ν_0 the restriction of ν to $\mathtt{C}_0$. We then define a functor μ_0 from $\mathtt{D}_0$ to $\mathtt{C}_0$ by setting $\mu_0(\nu(R_p)L^2(\mathscr{M})^n) := \mathscr{M}^n p$ and for a morphism $F : \nu(R_p)L^2(\mathscr{M})^n \to \nu(R_q)L^2(\mathscr{M})^m$ we define $\mu_0(F) := F|_{\mathscr{M}^n p}$. (That μ_0 is a functor follows from (ii)) Clearly $\mu_0 \circ \nu_0 = \mathrm{id}_{\mathtt{C}_0}$ and $\nu_0 \circ \mu_0 = \mathrm{id}_{\mathtt{D}_0}$ and hence $\mathtt{C}_0$ and $\mathtt{D}_0$ are isomorphic categories. In particular equivalent.

□

Definition 1.3.18. *By Theorem 1.3.17, $\nu : \mathtt{FGPIP}(\mathscr{M}) \longrightarrow \mathtt{FGHM}(\mathscr{M})$ is an equivalence of categories. We now choose a fixed functor $\nu^{-1} : \mathtt{FGHM}(\mathscr{M}) \to \mathtt{FGPIP}(\mathscr{M})$ such that $\nu \circ \nu^{-1} \simeq \mathrm{id}_{\mathtt{FGHM}}$ and $\nu^{-1} \circ \nu \simeq \mathrm{id}_{\mathtt{FGPIP}}$.*
If $\mathscr{H} \in \mathtt{FGHM}(\mathscr{M})$, we define its Murray-von Neumann dimension over $\mathscr{M}$ as $\dim_{\mathscr{M}}(\nu^{-1}(\mathscr{H}))$ and we denote it $\dim_{\mathscr{M}}(\mathscr{H})$. (Compare e.g. to [Lüc02] Section 1.1)
This is well-defined by Theorem 1.3.17.

Remark 1.3.19. *As our notation is set up, a minor ambiquity arises: Consider $(P_1, \langle\cdot|\cdot\rangle_1)$ and $(P_2, \langle\cdot|\cdot\rangle_2)$ in $\mathtt{FGPIP}(\mathscr{M})$ and an $\mathscr{M}$-equivariant operator $F : \nu(P_1) \to \nu(P_2)$.*
Then the symbol $\nu^{-1}(F)$ carries the following two meanings

(1) The unique map $f : P_1 \to P_2$ such that $\nu(f) = F$, provided by the isomorphism of vectorspaces from Theorem 1.3.17 part (ii).

(2) The morphism $\nu^{-1}(F) : \nu^{-1}(\nu(P_1)) \to \nu^{-1}(\nu(P_2))$.

However, since $\nu^{-1} \circ \nu \simeq \mathrm{id}_{\mathtt{FGPIP}}$ we have isomorphisms $\alpha_{P_1} : \nu^{-1}(\nu(P_1)) \to P_1$ and $\alpha_{P_2} : \nu^{-1}(\nu(P_2)) \to P_2$ making the following diagram commutative

$$\begin{array}{ccc} \nu^{-1}(\nu(P_1)) & \xrightarrow{\nu^{-1}(F)} & \nu^{-1}(\nu(P_2)) \\ {\scriptstyle \alpha_{P_1}} \downarrow \wr & & \wr \downarrow {\scriptstyle \alpha_{P_2}} \\ P_1 & \xrightarrow[f]{} & P_2 \end{array}$$

We will therefore not develop notational distinction between the two maps, but distinguish between them by specifying there domain- and range-space, if necessary.

Lemma 1.3.20. *Let $n, m, l \in \mathbb{N}$ and consider a sequence of $\mathscr{M}$-homomorphisms*

$$\mathscr{M}^n \xrightarrow{f} \mathscr{M}^m \xrightarrow{g} \mathscr{M}^l, \qquad (*)$$

and the induced sequence

$$L^2(\mathscr{M})^n \xrightarrow{\nu(f)} L^2(\mathscr{M})^m \xrightarrow{\nu(g)} L^2(\mathscr{M})^l. \qquad (**)$$

If $(**)$ is exact, then so is $(*)$.

Proof. Since $f = \nu(f)|_{\mathscr{M}^n}$ and $g = \nu(g)|_{\mathscr{M}^m}$, it is clear that $g \circ f = 0$.
Let π denote the projection onto $\ker(\nu(g))$ and let ρ denote the projection onto $\ker(\nu(f))^{\perp}$. Then

$$T := \nu(f)|_{\rho L^2(\mathscr{M})^n} : \rho L^2(\mathscr{M})^n \longrightarrow \pi L^2(\mathscr{M})^m,$$

is a bounded, invertible, $\mathscr{M}$-equivariant operator. Thus the same is true for T^{-1} and by Theorem 1.3.17 we know that T^{-1} maps $\pi L^2(\mathscr{M})^m \cap \mathscr{M}^m$ into $\rho L^2(\mathscr{M})^n \cap \mathscr{M}^n$.
Consider any $x \in \mathscr{M}^m$ and assume that $gx = 0$. Then

$$x \in \ker(\nu(g)) \cap \mathscr{M}^m = \pi L^2(\mathscr{M})^m \cap \mathscr{M}^m,$$

and hence $y := T^{-1}x \in \rho L^2(\mathscr{M})^n \cap \mathscr{M}^n$. We now get

$$fy = \nu(f)y = \nu(f)T^{-1}x = x,$$

and the proof is complete.

□

We are now able to prove the promised semi-heriditaryness of finite von Neumann algebras.

Corollary 1.3.21. [Lüc97] *Any finite von Neumann algebra $\mathscr{M}$ is semi-hereditary. That is, every finitely generated left ideal in $\mathscr{M}$ is projective, when considered as an $\mathscr{M}$-module.*
Hence every projective $\mathscr{M}$-module is semi-hereditary.

Proof. Let $\mathscr{J}$ be any finitely generated left ideal in $\mathscr{M}$ and choose a suitable $n \in \mathbb{N}$ and a homomorphism $f : \mathscr{M}^n \to \mathscr{M}$ with $\mathscr{J}$ as its range. If $\ker(f)$ is a direct summand in $\mathscr{M}^n$, then $\mathscr{J} \simeq \mathscr{M}^n / \ker(f)$ is isomorphic to a direct summand in $\mathscr{M}^n$ and therefore projective. Hence we aim to show that $\ker(f)$ is a direct summand in $\mathscr{M}^n$.
Applying the functor ν, we get $\nu(f) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})$ and we denote by p, the orthogonal projection in $\mathscr{B}(L^2(\mathscr{M})^n)$ onto $\ker(\nu(f))$. Since $\nu(f)$ is $\mathscr{M}$-equivariant, the kernel $\ker(\nu(f))$ is invariant under the action of $\mathscr{M}$ and hence p is $\mathscr{M}$-equivariant. By Lemma 1.3.12 there exists

a self-adjoint idempotent matrix $q \in M_n(\mathscr{M})$, such that $\nu(R_q) = p$. Since R_q is idempotent, its range is a direct summand in $\mathscr{M}^n$ and thus it suffices to show that $\mathrm{rg}\,(R_q) = \ker(f)$.
To see this, we consider the (by construction) exact sequence

$$L^2(\mathscr{M})^n \overset{\nu(R_q)}{\longrightarrow} L^2(\mathscr{M})^n \overset{\nu(f)}{\longrightarrow} L^2(\mathscr{M}),$$

and by applying Lemma 1.3.20, we get exactness of the sequence

$$\mathscr{M}^n \overset{R_q}{\longrightarrow} \mathscr{M}^n \overset{f}{\longrightarrow} \mathscr{M}.$$

This means precisely that $\ker(f) = \mathrm{rg}\,(q)$ and the proof is complete.
The last claim in Corollary 1.3.21 follows directly from Proposition 1.1.6. □

1.4 Properties of the generalized dimension function

Having constructed the functor ν and established some of its properties, we are now able to prove some practical results about the extended dimension function defined in section 1.1.1.
The advantage of the work in the previous section is, that we have the opportunity to transport problems about finitely generated projective modules, into the Hilbert-module-framework — this, for instance, allows us to make use of topological arguments.
As usual, $\mathscr{M}$ denotes a finite von Neumann algebra, endowed with a fixed normal, faithful, tracial state τ.
As we saw in Section 1.3, we get a $*$-algebra-isomorphism

$$\varphi : M_n(\mathscr{M})^{\mathrm{op}} \longrightarrow \mathrm{Hom}^{\mathtt{FGHM}}_{\mathscr{M}}(L^2(\mathscr{M})^n, L^2(\mathscr{M})^n) =: \mathscr{B}(L^2(\mathscr{M})^n)^{\mathscr{M}},$$

which, on a matrix A^{op}, simply extends $R_A : \mathscr{M}^n \to \mathscr{M}^n$ by continuity.
The trace τ on $\mathscr{M}$ gives rise to a positive, faithful and tracial functional τ_n on $M_n(\mathscr{M})^{\mathrm{op}}$ given by

$$\tau_n(\{a_{ij}\}^{\mathrm{op}}_{i,j}) = \sum_{i=1}^{n} \tau(a_{ii}).$$

Since φ is a $*$-algebra-isomorphism, the functional $\sigma_n := \tau_n \circ \varphi^{-1}$ on $\mathscr{B}(L^2(\mathscr{M})^n)^{\mathscr{M}}$ is faithful, positive and tracial.

Lemma 1.4.1. *The functional $\sigma_n := \tau_n \circ \varphi^{-1} : \mathscr{B}(L^2(\mathscr{M})^n)^{\mathscr{M}} \to \mathbb{C}$ is normal.*

Proof. Let η denote the inclusion $\mathscr{M} \subseteq L^2(\mathscr{M})$ and denote by ξ_i the vector $(0, 0, \dots, 0, \eta(1), 0, \dots, 0)$ in $L^2(\mathscr{M})^n$, where $\eta(1)$ is in the i'th position.
We now claim that

$$\sigma_n(T) = \sum_{i=1}^{n} \langle T\xi_i \,|\, \xi_i \rangle.$$

To see this, we consider $T = \varphi(\{a_{ij}\}^{\mathrm{op}}) \in \mathscr{B}(L^2(\mathscr{M})^n)^{\mathscr{M}}$ and calculate:

$$\begin{aligned} \sum_{i=1}^{n} \langle T\xi_i \,|\, \xi_i \rangle &= \sum_{i=1}^{n} \langle \eta(a_{i1}), \cdots, \eta(a_{in}) \,|\, \xi_i \rangle \\ &= \sum_{i=1}^{n} \tau(a_{ii}) \\ &= \tau_n(\varphi^{-1}(T)) \\ &= \sigma_n(T). \end{aligned}$$

So, σ_n is a finite sum of vector-states and hence normal. □

Remark 1.4.2. *If P is a finitely generated projective $\mathscr{M}$-module and $p \in M_n(\mathscr{M})$ is an idempotent matrix such that $P \simeq \mathscr{M}^n p$, then $\dim_{\mathscr{M}}(P) = \sigma_n(\nu(R_p))$, where σ_n is the trace from Lemma 1.4.1. Conversely, if $\pi : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$ is an $\mathscr{M}$-equivariant projection, such that $\mathrm{rg}\,(\nu^{-1}(\pi)) \simeq P$, then $\dim_{\mathscr{M}}(P) = \sigma_n(\pi)$.*

Proposition 1.4.3. [Lüc98] *Let P be a finitely generated projective $\mathscr{M}$-module and let K be any submodule of P. Then the following holds.*

1. *The algebraic closure $\overline{K}^{\mathrm{alg}}$ is a direct summand in P. Hence both $\overline{K}^{\mathrm{alg}}$ and $P/\overline{K}^{\mathrm{alg}}$ are finitely generated and projective.*
2. *We have $\dim_{\mathscr{M}}(\overline{K}^{\mathrm{alg}}) = \dim'_{\mathscr{M}}(K)$.*

Proof. Let $\Omega := \{P_i | i \in I\}$ be the family of all finitely generated projective submodules of K. Then Ω is directed by inclusion, since for any $P_1, P_2 \in \Omega$ the submodule in K, generated by P_1 and P_2, is obviously finitely generated and hence projective since P is semi-hereditary.
Choose an inner product $\langle\cdot|\cdot\rangle_i$ on P_i for each $i \in I$ and let ι_i be the inclusion $P_i \subseteq P$. Choose also an inner product $\langle\cdot|\cdot\rangle$ on P. Applying the functor ν, we get (for each $i \in I$) a subspace $\mathrm{rg}\,\nu(\iota_i)$ in the Hilbert space $\nu(P)$. Let p_i denote the orthogonal projection onto the norm-closure $\overline{\mathrm{rg}\,\nu(\iota_i)}$ and put $p := \bigvee_{i\in I} p_i$. Note that all the p_i's, and hence p, are $\mathscr{M}$-equivariant operators.
Consider the homomorphism $\nu^{-1}(p) : P \to P$. We now prove that

$$\mathrm{rg}\,\nu^{-1}(p) = \overline{K}^{\mathrm{alg}}.$$

This is sufficient since $\nu^{-1}(p) : P \to P$ is an idempotent morphism, so its range is a direct summand in P.

"$\subseteq$" Let $f : P \to \mathscr{M}$ be a homomorphism vanishing on K.
Since $\overline{K}^{\mathrm{alg}}$ (by definition) is the intersection of the kernels of all such homomorphisms, we need to see that f vanishes on $\mathrm{rg}\,\nu^{-1}(p)$.
Choose any $i \in I$. Since $\mathrm{rg}\,(\iota_i) \subseteq K$ we have $f \circ \iota_i = 0$ and by functoriality $\nu(f) \circ \nu(\iota_i) = 0$. Therefore $\mathrm{rg}\,\nu(\iota_i) \subseteq \ker \nu(f)$ and hence also

$$\mathrm{rg}\,p = \overline{\mathrm{span}_{\mathbb{C}}\Big(\bigcup_{i\in I} \mathrm{rg}\,(p_i)\Big)} = \overline{\mathrm{span}_{\mathbb{C}}\Big(\bigcup_{i\in I} \overline{\mathrm{rg}\,\nu(\iota_i)}\Big)} \subseteq \ker \nu(f),$$

Thus we have $\nu(f) \circ p = 0$ and applying ν^{-1} yields $f \circ \nu^{-1}(p) = 0$. Thus

$$\mathrm{rg}\,\nu^{-1}(p) \subseteq \ker f.$$

"$\supseteq$" For any $x \in K$, the module $\mathscr{M}x \subseteq K$ is a finitely generated submodule of P and therefore an element in Ω by Corollary 1.3.21. Hence $px = x$, which shows that $1_{\nu(P)} - p = 0$ on K. Hence $1_P - \nu^{-1}(p) = 0$ on K and therefore $K \subseteq \ker(1_P - \nu^{-1}(p))$. Since P is assumed projective, Proposition 1.1.9 implies that $\ker(1_P - \nu^{-1}(p))$ is closed in P and hence

$$\overline{K}^{\mathrm{alg}} \subseteq \ker(1_P - \nu^{-1}(p)) = \mathrm{rg}\,(\nu^{-1}(p)).$$

Since $P = \mathrm{rg}\,(\nu^{-1}(p)) \oplus \mathrm{rg}\,(1 - \nu^{-1}(p))$ is finitely generated and projective, it is clear that $\overline{K}^{\mathrm{alg}} = \mathrm{rg}\,(\nu^{-1}(p))$ is finitely generated and projective.

For the same reason, $P/\mathrm{rg}\,(\nu^{-1}(p)) = P/\overline{K}^{\mathrm{alg}}$ is finitely generated and projective.
We now show the equality

$$\dim_{\mathscr{M}}(\overline{K}^{\mathrm{alg}}) = \dim'_{\mathscr{M}}(K),$$

which is well-defined, since we now know that $\overline{K}^{\mathrm{alg}}$ is finitely generated and projective.
Because $\nu(P) \in \mathtt{FGHM}(\mathscr{M})$ it has the form $\pi L^2(\mathscr{M})^n$ for an $\mathscr{M}$-equivariant projection $\pi : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$. Then the $\mathscr{M}$-equivariant operator $\tilde{p} : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$ given by

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} : \pi L^2(\mathscr{M})^n \oplus (1-\pi)L^2(\mathscr{M})^n \longrightarrow \pi L^2(\mathscr{M})^n \oplus (1-\pi)L^2(\mathscr{M})^n,$$

is a projection whose range is isomorphic to $\mathrm{rg}\,(p)$.
If we denote by σ_n the trace induced on $\mathscr{B}(L^2(\mathscr{M})^n)^{\mathscr{M}}$ by the standard trace on $M_n(\mathscr{M})$ (see e.g. Lemma 1.4.1) we get

$$\dim_{\mathscr{M}}(\overline{K}^{\mathrm{alg}}) = \dim_{\mathscr{M}}(\mathrm{rg}\,(\nu^{-1}(\tilde{p}))) = \sigma_n(\tilde{p}).$$

Fix an arbitrary $i \in I$ and extend $p_i : \nu(P) \to \nu(P)$ to an $\mathscr{M}$-equivariant projection $\tilde{p}_i : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$, in the same way as with p. We now want to prove that

$$\dim_{\mathscr{M}}(P_i) = \sigma_n(\tilde{p}_i).$$

Consider $\nu(\iota_i) : \nu(P_i) \to \nu(P)$ (which is one-to-one and maps $\nu(P_i)$ onto a dense subspace in $\mathrm{rg}\,(p_i)$) and its polar decomposition $V|\nu(\iota_i)|$. (see e.g. [MV] Prop. 18.18) Then V is an isometry from $\overline{\mathrm{rg}\,(\nu(\iota_i)^*)} = \overline{\mathrm{rg}\,|\nu(\iota_i)|}$ to $\overline{\mathrm{rg}\,(\nu(\iota_i))} = \mathrm{rg}\,(p_i)$.
Since $\nu(\iota_i)$ is injective, $\nu(\iota_i)^* : \nu(P) \to \nu(P_i)$ has dense image and hence V is an isometry from $\nu(P_i)$ onto $\mathrm{rg}\,(p_i)$. To see that $\nu(P_i)$ and $\mathrm{rg}\,(p_i)$ are isomorphic as Hilbert $\mathscr{M}$-modules, it is therefore sufficient to prove that V is $\mathscr{M}$-equivariant.
Since $\nu(\iota_i)$ and $\nu(\iota_i)^*$ are $\mathscr{M}$-equivariant, every polynomial in $\nu(\iota_i)^*\nu(\iota_i)$ is $\mathscr{M}$-equivariant and hence the same is true for $\sqrt{\nu(\iota_i)^*\nu(\iota_i)} = |\nu(\iota_i)|$. Given $m \in \mathscr{M}$ and $x \in \nu(P_i)$, we therefore have

$$(mV)(|\nu(\iota_i)|x) = m\nu(\iota_i)(x) = \nu(\iota_i)mx = V|\nu(\iota_i)|mx = (Vm)(|\nu(\iota_i)|x),$$

so that mV and Vm agrees on $\mathrm{rg}\,(|\nu(\iota_i)|)$, which is dense in $\nu(P_i)$. This shows that V is $\mathscr{M}$-equivariant and we conclude that $\nu(P_i)$ and $\mathrm{rg}\,(p_i)$ are isomorphic as Hilbert $\mathscr{M}$-modules.
Thus

$$\dim_{\mathscr{M}}(P_i) = \sigma_n(\tilde{p}_i).$$

To finish the proof, we have to show that

$$\sigma_n(\tilde{p}) = \sup_{i \in I} \sigma_n(\tilde{p}_i).$$

Since $\tilde{p}_i \leq \tilde{p}$ for every $i \in I$, the inequality "$\geq$" is evident.
Define, for each finite subset J in I, $q_J := \vee_{j \in J} p_i$ and extend q_J to a projection $\tilde{q}_J : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$ in the same way as above. Then $(\tilde{q}_J)_{J \in \mathcal{P}_e(I)}$ is a monotone increasing net of projections and since $p := \vee_{i \in I} p_i$ we get that $(\tilde{q_J})_{J \in \mathcal{P}_e(I)}$ converges strongly to $\tilde{p}$.
Since σ_n is normal, this implies that $\sigma_n(\tilde{p}) = \sup_{J \in \mathcal{P}_e(I)} \sigma_n(\tilde{q}_J)$. Because $(P_i)_{i \in I}$ is directed by inclusion, every $\tilde{q}_J$ is dominated by an element in $(\tilde{p}_i)_{i \in I}$ and the opposite inequality follows.

□

Corollary 1.4.4. [Lüc98] *Let P be a finitely generated projective $\mathscr{M}$-module and let Q be a finitely generated projective $\mathscr{M}$-submodule of P. Then $\dim_{\mathscr{M}}(Q) \leq \dim_{\mathscr{M}}(P)$.*
This implies that $\dim'_{\mathscr{M}}(\cdot)$ actually extends $\dim_{\mathscr{M}}(\cdot)$.

Proof. From Proposition 1.4.3, we know that $\overline{Q}$ is a direct summand in P and by Proposition 1.4.3 combined with additivity of $\dim_{\mathscr{M}}(\cdot)$ we have

$$\dim_{\mathscr{M}}(P) \geq \dim_{\mathscr{M}}(\overline{Q}) = \dim'_{\mathscr{M}}(\overline{Q}) \geq \dim_{\mathscr{M}}(Q).$$

Hence

$$\dim'_{\mathscr{M}}(P) = \sup\{\dim_{\mathscr{M}}(P_i) \mid P_i \subseteq P;\ P_i \text{ finitely generated and projective }\} = \dim_{\mathscr{M}}(P).$$

□

Before stating and proving the main theorem about the extended dimension function, we make a small detour to prove some exactness-results for the functors ν and ν^{-1}, which will become very useful in the sequel. To this end, first some notation.

Definition 1.4.5. *A sequence $\mathscr{H}_1 \xrightarrow{F} \mathscr{H}_2 \xrightarrow{G} \mathscr{H}_3$ of finitely generated Hilbert $\mathscr{M}$-modules is said to be* weakly exact *if* $\operatorname{rg}(F)$ *is a dense subspace in* $\ker(G)$*.*
Similarly, a sequence $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3$ of finitely generated projective $\mathscr{M}$-modules is said to be weakly exact *if* $\overline{\operatorname{rg}(f)}^{\text{alg}} = \ker(g)$*.*
A morphism between two finitely generated Hilbert $\mathscr{M}$-modules is called a weak isomorphism*, if it is injective and has dense image.*
Similarly, a morphism $f : P \longrightarrow Q$ of finitely generated projective $\mathscr{M}$-modules, is called a weak isomorphism *if f is injective and* $\overline{\operatorname{rg}(f)}^{\text{alg}} = Q$*.*

Lemma 1.4.6. [Lüc97] *Both ν and ν^{-1} preserves exactness and weak exactness.*

Proof. Consider sequences

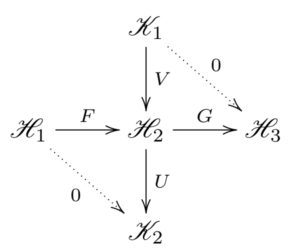
$$P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3 \quad \text{in } \texttt{FGPIP}(\mathscr{M}). \tag{†}$$

$$\mathscr{H}_1 \xrightarrow{F} \mathscr{H}_2 \xrightarrow{G} \mathscr{H}_3 \quad \text{in } \texttt{FGHM}(\mathscr{M}). \tag{‡}$$

Claim 1: *The sequence (‡) is weakly exact if, and only if, $GF = 0$ and for every $\mathscr{K}_1, \mathscr{K}_2 \in$ $\texttt{FGHM}(\mathscr{M})$ and morphisms $U : \mathscr{H}_2 \to \mathscr{K}_2$ and $V : \mathscr{K}_1 \to \mathscr{H}_2$ we have*

$$(UF = 0 \text{ and } GV = 0) \Rightarrow UV = 0.$$

In the shape of a diagram, this looks like



Proof: Assume first that (‡) is weakly exact and let $\mathscr{K}_1, \mathscr{K}_2, U$ and V be as in Claim 1, with $UF = 0$ and $GV = 0$.
Then $\operatorname{rg}(V) \subseteq \ker(G) = \overline{\operatorname{rg}(F)}$. From $UF = 0$ we get $\operatorname{rg}(F) \subseteq \ker(U) = \overline{\ker(U)}$ and thus

$$\operatorname{rg}(V) \subseteq \overline{\operatorname{rg}(F)} \subseteq \ker(U).$$

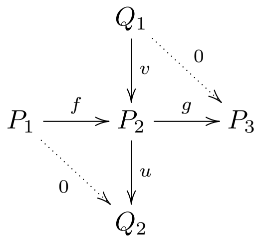
Conversely, assume that $\operatorname{rg}(F) \subseteq \ker(G)$ and $\overline{\operatorname{rg}(F)} \subsetneq \ker(G)$ and fix a non-zero vector x in $\ker(G) \setminus \overline{\operatorname{rg}(F)}$. Put $\mathscr{K}_1 := \overline{\mathscr{M}x}$ and let V be the inclusion into $\mathscr{H}_2$. Note that $\mathscr{K}_1$ is a closed

$\mathscr{M}$-invariant subspace of the finitely generated Hilbert module $\mathscr{H}_2$ and hence a finitely generated Hilbert $\mathscr{M}$-module it self. Define $\mathscr{K}_2 := \mathscr{H}_2/\overline{\mathrm{rg}\,(F)}$ (which is again a finitely generated Hilbert $\mathscr{M}$-module) and let $U : \mathscr{H}_2 \to \mathscr{K}_2$ be the quotient mapping.
Then $UF = 0$ and $GV = 0$ but $UV(x) = U(x) \neq 0$, since $x \notin \overline{\mathrm{rg}\,(F)}$.
This proves Claim 1.

Claim 2: *The sequence (†) is weakly exact if, and only if, $gf = 0$ and for every $Q_1, Q_2 \in$ FGPIP($\mathscr{M}$) and morphisms $u : P_2 \to Q_2$ and $v : Q_1 \to P_2$ we have*

$$(uf = 0 \text{ and } gv = 0) \Rightarrow uv = 0.$$

In the shape of a diagram, this looks like



Proof: The proof of Claim 2 is basically a copy of the proof of Claim 1. The details are as follows:
Assume first that (†) is weakly exact and let Q_1, Q_2, u, v be as in Claim 2. Since $uf = 0$, we have $\mathrm{rg}\,(f) \subseteq \ker(u)$ and since Q_2 is projective, $\ker(u)$ is closed by Proposition 1.1.9. Thus $\overline{\mathrm{rg}\,(f)} \subseteq \ker(u)$. Since $gv = 0$ by assumption, we get

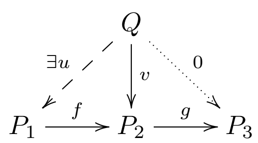
$$\mathrm{rg}\,(v) \subseteq \ker(g) = \overline{\mathrm{rg}\,(f)} \subseteq \ker(u).$$

Conversely, assume that $g \circ f = 0$ and that $\overline{\mathrm{rg}\,(f)} \subsetneqq \ker(g)$ and fix a non-zero element $x \in \ker(g) \setminus \overline{\mathrm{rg}\,(f)}$. Define $Q_1 := \mathscr{M}x$ and v to be the inclusion $\mathscr{M}x \subseteq P_2$. Note, that Q_1 is finitely generated and hence projective by Corollary 1.3.21.
By Proposition 1.4.3, the module $Q_2 := P_2/\overline{\mathrm{rg}\,(f)}$ is finitely generated and projective and we define u to be the quotient mapping. Then $uf = 0$ and $g \circ v = 0$, but $uv(x) = u(x) \neq 0$, since $x \notin \overline{\mathrm{rg}\,(f)}$.

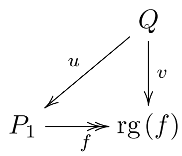
Since $\nu \circ \nu^{-1} \simeq \mathrm{id}_{\mathtt{FGHM}}$ and $\nu^{-1} \circ \nu \simeq \mathrm{id}_{\mathtt{FGPIP}}$, it is not hard to check that both $\nu \circ \nu^{-1}$ and $\nu^{-1} \circ \nu$ preserves weak exactness. Using this, and Claim 1 and Claim 2, it follows that both ν and ν^{-1} preserves weak exactness.

Claim 3: *The sequence (†) is exact if, and only if, $gf = 0$ and for every $Q \in$ FGPIP($\mathscr{M}$) and morphism $v : Q \to P_2$ with $gv = 0$, there exists a morphism $u : Q \to P_1$ with $fu = v$*

In the shape of a diagram, this looks like



Proof: Assume first that (†) is exact and let Q, v be as in Claim 3. Then $\mathrm{rg}\,(v) \subseteq \ker(g) = \mathrm{rg}\,(f)$ and by projectivity of Q, there exists $u : Q \to P_1$ making the following diagram commutative

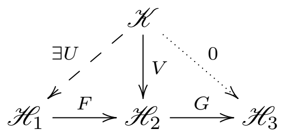


Thus u fulfills the requirements.
Conversely, assume that $gf = 0$ and that the lifting-property from Claim 3 is fulfilled for every pair (Q, v). Since g is map between finitely generated projective modules, Proposition 1.1.9 together with Proposition 1.4.3, implies that $\ker(g)$ is finitely generated an projective. We may therefore choose $Q := \ker(g)$ and v to be the inclusion. Then, by assumption, there exists $u : \ker(g) \to P_1$ such that $fu = v$ and therefore

$$\ker(g) = \mathrm{rg}\,(v) = \mathrm{rg}\,(fu) \subseteq \mathrm{rg}\,(f).$$

Since $gf = 0$ by assumption, this shows that (†) is exact.

Claim 4: *The sequence (‡) is exact if, and only if, $GF = 0$ and for every $\mathscr{K} \in \mathtt{FGHM}(\mathscr{M})$ and morphism $V : \mathscr{K} \to \mathscr{H}_2$ with $GV = 0$, there exists a morphism $U : \mathscr{K} \to \mathscr{H}_2$ with $FU = V$*
In the shape of a diagram, this looks like:



Proof: Assume first that the $GF = 0$ and that the lifting-property from Claim 4 is fulfilled for every pair $(\mathscr{K}, V)$.
The kernel of G is a closed $\mathscr{M}$-invariant subspace of $\mathscr{H}_2$ and therefore in $\mathtt{FGHM}(\mathscr{M})$. We now put $\mathscr{K} := \ker(G)$ and let V denote the inclusion $\mathscr{K} \subseteq \mathscr{H}_2$. Then, by assumption, there exists $U : \mathscr{K} \to \mathscr{H}_1$ such that $V = FU$ and hence

$$\ker(G) = \mathrm{rg}\,(V) = \mathrm{rg}\,(FU) \subseteq \mathrm{rg}\,(F).$$

Assume conversely, that the sequence (†) is exact. Then $\mathrm{rg}\,(F) = \ker(G) \in \mathtt{FGHM}(\mathscr{M})$.
The Hilbert $\mathscr{M}$-module $\mathscr{H}_1$ splits as

$$\mathscr{H}_1 = \ker(F) \oplus \ker(F)^{\perp},$$

an the $\mathscr{M}$-equivariant operator

$$F_0 := F|_{\ker(F)^{\perp}} : \ker(F)^{\perp} \longrightarrow \mathrm{rg}\,(F)$$

is invertible. (with bounded $\mathscr{M}$-equivariant inverse) By assumption, we have $\mathrm{rg}\,(V) \subseteq \mathrm{rg}\,(F)$ and we can therefore define $U := F_0^{-1}V$. Then U is bounded and $\mathscr{M}$-equivariant and for every $x \in \mathscr{K}$ we have

$$FU(x) = FF_0^{-1}Vx = Vx.$$

Since $\nu \circ \nu^{-1} \simeq \mathrm{id}_{\mathtt{FGHM}}$ and $\nu^{-1} \circ \nu \simeq \mathrm{id}_{\mathtt{FGPIP}}$, it is not hard to check that both $\nu \circ \nu^{-1}$ and $\nu^{-1} \circ \nu$ preserves exactness. Using this, and Claim 3 and Claim 4, it follows that both ν and ν^{-1} preserves exactness.

□

We can now prove the main theorem of this section, which consist of a list of properties of the extended dimension function.

Theorem 1.4.7 (Properties of the dimension function). [Lüc98] *The extended dimension function* $\dim'_{\mathscr{M}}(\cdot)$ *has the following properties.*

1. *Let M be an $\mathscr{M}$-module and let $(M_i)_{i\in I}$ be a system of $\mathscr{M}$-submodules of M. Assume moreover, that $(M_i)_{i\in I}$ is directed by inclusion*[5] *and that $M = \cup_{i\in I} M_i$. Then*

$$\dim'_{\mathscr{M}}(M) = \sup_{i\in I} \dim'_{\mathscr{M}}(M_i). \qquad (*)$$

Such a system $(M_i)_{i\in I}$ is called a cofinal *system of submodules and the formula $(*)$ will therefore be referred to as* cofinality *of the (extended) dimension function.*

2. *Let $0 \to M' \to M \to M'' \to 0$ be a short-exact sequence of $\mathscr{M}$-modules. Then*

$$\dim'_{\mathscr{M}}(M) = \dim'_{\mathscr{M}}(M') + \dim'_{\mathscr{M}}(M'').$$

(With addition in $[0,\infty]$ being the obvious one)
We will be refer to this property as additivity *of the (extended) dimension function.*

3. *Let M be a finitely generated $\mathscr{M}$-module and let K be a submodule of M. Then $\overline{K}$ is a direct summand in M, and the quotient $M/\overline{K}$ is finitely generated and projective.*
Moreover

$$\dim'_{\mathscr{M}}(\overline{K}) = \dim'_{\mathscr{M}}(K).$$

This formula will be referred to as continuity *of the (extended) dimension function.*

4. *If M is a finitely generated $\mathscr{M}$-module, then PM is finitely generated and projective, $M \simeq TM \oplus PM$ and*

$$\dim'_{\mathscr{M}}(M) = \dim'_{\mathscr{M}}(PM) = \dim_{\mathscr{M}}(PM).$$

In particular $\dim'_{\mathscr{M}}(M)$ is finite.

Proof.

Proof of *1.* Consider any $i \in I$. Since every finitely generated projective submodule of M_i is also a submodule of M, the inequality "$\geq$" is evident.
To prove the other inequality, it suffices to prove, that for any finitely generated projective submodule P of M, we can find a $k \in I$ such that $P \subseteq M_k$. To see this, we fix a finite set of generators $\{\xi_1, \dots, \xi_n\}$ of P. Since $M = \cup_{i\in I} M_i$ we can find $i_1, \dots, i_n \in I$ with $\xi_{i_j} \in M_{i_j}$ and since $(M_i)_{i\in I}$ is directed by inclusion, there exists a $k \in I$ with $\cup_{j=1}^n M_{i_j} \subseteq M_k$. Hence $P \subseteq M_k$.

Proof of *2.* Consider a short-exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{p} M'' \to 0$ and let $P \subseteq M''$ be any finitely generated projective $\mathscr{M}$-module.
Since $0 \in P$ we have $\mathrm{rg}\,(\iota) \subseteq p^{-1}(P)$ and by restriction we obtain a short-exact sequence

$$0 \longrightarrow M' \xrightarrow{\iota} p^{-1}(P) \xrightarrow{p} P \longrightarrow 0.$$

[5] I.e. $\forall i, j \in I\ \exists k \in I$ s.t. $M_i, M_j \subseteq M_k$.

Because P is projective this sequence splits, such that $p^{-1}(P) \simeq P \oplus M'$.
We therefore have

$$\dim'_{\mathscr{M}}(M') + \dim_{\mathscr{M}}(P) = \dim'_{\mathscr{M}}(M') + \dim'_{\mathscr{M}}(P) \leq \dim'_{\mathscr{M}}(p^{-1}(P)) \leq \dim'_{\mathscr{M}}(M).$$

Since P was arbitrary, we conclude that

$$\dim'_{\mathscr{M}}(M') + \dim'_{\mathscr{M}}(M'') \leq \dim'_{\mathscr{M}}(M).$$

To prove the opposite inequality, we consider an arbitrary finitely generated projective submodule Q in M. We then get the following two short-exact sequences:

$$0 \longrightarrow \iota(M') \cap Q \longrightarrow Q \longrightarrow p(Q) \longrightarrow 0 \tag{$*$}$$

$$0 \longrightarrow \overline{\iota(M') \cap Q} \longrightarrow Q \longrightarrow Q/\overline{\iota(M') \cap Q} \longrightarrow 0 \tag{$**$}$$

(All morphisms are the canonical ones and the closure is performed relative to Q)
Since Q is assumed to be finitely generated and projective, Proposition 1.4.3 implies that

$$Q = \overline{\iota(M') \cap Q} \bigoplus Q/\overline{\iota(M') \cap Q},$$

and that both summands are finitely generated and projective.
By additivity of $\dim_{\mathscr{M}}(\cdot)$, we have

$$\dim_{\mathscr{M}}(Q) = \dim_{\mathscr{M}}(\overline{\iota(M') \cap Q}) + \dim_{\mathscr{M}}(Q/\overline{\iota(M') \cap Q}).$$

Moreover, Proposition 1.4.3 implies that $\dim_{\mathscr{M}}(\overline{\iota(M') \cap Q}) = \dim'_{\mathscr{M}}(\iota(M') \cap Q)$.
Because of the two short-exact sequences ($*$) and ($**$), we get an epimorphism
$\pi : p(Q) \to Q/\overline{\iota(M') \cap Q}$ by composing through the diagram

$$p(Q) \simeq Q/\iota(M') \cap Q \twoheadrightarrow Q/\overline{\iota(M') \cap Q},$$

and hence another short-exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow p(Q) \longrightarrow Q/\overline{\iota(M') \cap Q} \longrightarrow 0.$$

Because the right-most module in this sequence is projective, the sequence is split-exact and hence $Q/\overline{\iota(M') \cap Q}$ is isomorphic to a submodule in $p(Q)$.
Since $Q/\overline{\iota(M') \cap Q}$ is finitely generated and projective, we now get

$$\dim_{\mathscr{M}}(Q/\overline{\iota(M') \cap Q}) \leq \dim'_{\mathscr{M}}(p(Q)). \tag{$***$}$$

By combining these facts, we see that

$$\begin{aligned}\dim_{\mathscr{M}}(Q) &= \dim_{\mathscr{M}}(\overline{\iota(M') \cap Q}) + \dim_{\mathscr{M}}(Q/\overline{\iota(M') \cap Q}) && \text{(by ($**$))}\\ &\leq \dim'_{\mathscr{M}}(\iota(M') \cap Q) + \dim'_{\mathscr{M}}(p(Q)) && \text{(by ($***$))}\\ &\leq \dim'_{\mathscr{M}}(M') + \dim'_{\mathscr{M}}(M''),\end{aligned}$$

where the last inequality follows from the fact that $p(Q) \subseteq M''$ and that $\iota(M') \cap Q$ is isomorphic to a submodule in M'.
Since this holds for any finitely generated projective $Q \subseteq M$, we have

$$\dim'_{\mathscr{M}}(M) \leq \dim'_{\mathscr{M}}(M') + \dim'_{\mathscr{M}}(M''),$$

and the claim follows.

Proof of *3.* We first notice, that in the case where M is also projective, the claim was already proved in Proposition 1.4.3. Since M is finitely generated, we can choose an $n \in \mathbb{N}$ and a surjective homomorphism $p : \mathscr{M}^n \to M$. We now claim that

$$p^{-1}(\overline{K}) = \overline{p^{-1}(K)}^{\mathscr{M}^n} \quad \text{and} \quad \mathscr{M}^n/p^{-1}(\overline{K}) \simeq M/\overline{K}.$$

Let $x \in p^{-1}(\overline{K})$ be given and consider an arbitrary homomorphism $f : \mathscr{M}^n \to \mathscr{M}$ vanishing on $p^{-1}(K)$. We need to show that $f(x) = 0$.
Since f vanishes on $p^{-1}(K) \supseteq \ker p$, we get an induced homomorphism $\tilde{f} : \mathscr{M}^n/\ker p \to \mathscr{M}$.
Letting φ denote the isomorphism $\mathscr{M}^n/\ker p \to M$ induced by p, we get a homomorphism $g := \tilde{f} \circ \varphi^{-1} : M \to \mathscr{M}$. One easily checks that g vanishes on K and, by construction of x, we get

$$0 = g(p(x)) = \tilde{f}(\varphi^{-1}(px)) = \tilde{f}(x + \ker p) = f(x).$$

This shows that $p^{-1}(\overline{K}) \subseteq \overline{p^{-1}(K)}^{\mathscr{M}^n}$.
For the opposite inclusion it suffices to prove that $p^{-1}(\overline{K})$ is closed in $\mathscr{M}^n$.
So, choose any $x \in \overline{p^{-1}(\overline{K})}^{\mathscr{M}^n}$ and any homomorphism $f : M \to \mathscr{M}$ vanishing on K. We need to prove that $f(p(x)) = 0$, and by definition of x, it suffices to show that $f \circ p : \mathscr{M}^n \to \mathscr{M}$ vanishes on $p^{-1}(\overline{K})$. But this is true by construction of f.
To prove the isomorphism-statement in the claim, one easily checks that $\psi : \mathscr{M}^n/p^{-1}(\overline{K}) \to M/\overline{K}$ defined by

$$x + p^{-1}(\overline{K}) \longmapsto p(x) + \overline{K}$$

is well-defined and indeed an isomorphism.
We now proceed with the proof of *3.* By what was just proven, we get

$$\mathscr{M}^n/\overline{p^{-1}(K)}^{\mathscr{M}^n} = \mathscr{M}^n/p^{-1}(\overline{K}) \simeq M/\overline{K},$$

and by Proposition 1.4.3 the left-most module is finitely generated and projective. Hence the same holds for $M/\overline{K}$. Proposition 1.4.3 also gives

$$\dim'_{\mathscr{M}}(p^{-1}(K)) = \dim_{\mathscr{M}}(\overline{p^{-1}(K)}^{\mathscr{M}^n}) = \dim_{\mathscr{M}}(p^{-1}(\overline{K})) = \dim'_{\mathscr{M}}(p^{-1}(\overline{K})).$$

Using the additivity of $\dim'_{\mathscr{M}}(\cdot)$ on the two short-exact sequences

$$0 \longrightarrow \ker p \xrightarrow{\iota} p^{-1}(K) \xrightarrow{p} K \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \ker p \xrightarrow{\iota} p^{-1}(\overline{K}) \xrightarrow{p} \overline{K} \longrightarrow 0,$$

it follows that $\dim'_{\mathscr{M}}(K) = \dim'_{\mathscr{M}}(\overline{K})$.
To see that $\overline{K}$ is a direct summand in M, we just note that the short-exact sequence

$$0 \longrightarrow \overline{K} \longrightarrow M \longrightarrow M/\overline{K} \longrightarrow 0,$$

is split-exact since $M/\overline{K}$ is projective.

Proof of *4*: This follows directly from *3.* applied to $K = \{0\}$. □

Corollary 1.4.8. *If M is a finitely generated $\mathscr{M}$-module, then*

$$\dim'_{\mathscr{M}}(M) = \dim'_{\mathscr{M}}(M^*),$$

where M^* is the dual module defined in Definition 1.1.7.

Proof. By definition, $M^* = \mathrm{Hom}_{\mathscr{M}}(M, \mathscr{M})$ with $\mathscr{M}$-module structure given by

$$(mf)(x) := f(x)m^*,$$

for $m \in \mathscr{M}$, $f \in \mathrm{Hom}_M(M, \mathscr{M})$ and $x \in M$.
Since M is finitely generated, PM is projective by Theorem 1.4.7 and by Corollary 1.3.16 the module PM therefore possess an inner product $\langle \cdot | \cdot \rangle$, which is unique up to unitary isomorphism. Note, that this implies that $PM \ni x \mapsto \langle \cdot | x \rangle \in (PM)^*$ is an isomorphism.
By Theorem 1.4.7, we have $\dim'_{\mathscr{M}}(M) = \dim'_{\mathscr{M}}(PM)$ and it is therefore sufficient to prove that M^* and $(PM)^*$ are isomorphic. Let κ denote the qoutient-map $M \to M/\overline{\{0\}} =: PM$. Then

$$\varphi : (PM)^* \longrightarrow M^*,$$

given by $f \mapsto f \circ \kappa$ is $\mathscr{M}$-linear and since κ is surjective φ is injective.
To see that φ is also surjective, we need to see that every morphism $f : M \to \mathscr{M}$ factorizes (via κ) through a morphism $\tilde{f} : PM \to \mathscr{M}$. For this it is sufficient to see that f vanishes on $\overline{\{0\}}$, which follows directly from the definition of closure.

□

Corollary 1.4.9. [Lüc98] *The extended dimension function* $\dim'_{\mathscr{M}}(\cdot)$ *is the only extension of* $\dim_{\mathscr{M}}(\cdot)$ *to arbitrary* $\mathscr{M}$*-modules, satisfying continuity, additivity and cofinality.*

Proof. Since the system of finitely generated submodules of any module M is cofinal, it suffices to show that $\dim_{\mathscr{M}}(\cdot)$ is the only extension to finitely generated modules, satisfying additivity and continuity. By Theorem 1.4.7 part *4.*, we have $\dim'_M(M) = \dim'_{\mathscr{M}}(PM) = \dim_{\mathscr{M}}(PM)$ for any finitely generated module M and since part *4* (via part *3*) is a consequence of only additivity and continuity, the uniqueness follows. □

Remark 1.4.10. *In the light of Corollary 1.4.4 we will often omit the distinction between* $\dim_{\mathscr{M}}(\cdot)$ *and* $\dim'_{\mathscr{M}}(\cdot)$ *and just use the symbol* $\dim_{\mathscr{M}}(\cdot)$ *to denote them both.*

Corollary 1.4.11. [Lüc02] *Let P and Q be finitely generated projective $\mathscr{M}$-modules and let $f : P \longrightarrow Q$ be a homomorphism. Assume moreover, that* $\dim_{\mathscr{M}}(P) = \dim_{\mathscr{M}}(Q)$*. Then the following statements are equivalent*

(i) f *is injective.*

(ii) $\overline{\mathrm{rg}\,(f)} = Q$.

(iii) f *is a weak isomorphism.*

Proof.

"(i) ⇒ (ii)" Assume f is injective. Since Q is projective, $\overline{\mathrm{rg}\,(f)}$ is a direct summand in Q by Proposition 1.4.3 and hence $Q/\overline{\mathrm{rg}\,(f)}$ is also finitely generated and projective.
Using Theorem 1.4.7 we get

$$\begin{aligned}\dim_{\mathscr{M}}(Q/\overline{\mathrm{rg}\,(f)}) &= \dim_{\mathscr{M}}(Q) - \dim_{\mathscr{M}}(\overline{\mathrm{rg}\,(f)})\\ &= \dim_{\mathscr{M}}(Q) - \dim_{\mathscr{M}}(\mathrm{rg}\,(f))\\ &= \dim_{\mathscr{M}}(Q) - \dim_{\mathscr{M}}(P)\\ &= 0.\end{aligned}$$

Since $\dim_{\mathscr{M}}(\cdot)$ is faithful on the class of finitely generated projective modules, we conclude that $Q = \overline{\mathrm{rg}\,(f)}$.

"(ii)⇒ (iii)" Assume $\overline{\mathrm{rg}\,(f)} = Q$ and note that

$$\dim_{\mathscr{M}}(P) = \dim_{\mathscr{M}}(Q) = \dim_{\mathscr{M}}(\overline{\mathrm{rg}\,(f)}) = \dim_{\mathscr{M}}(\mathrm{rg}\,(f)).$$

Using additivity of the dimension function on the short-exact sequence

$$0 \longrightarrow \ker(f) \xrightarrow{\iota} P \xrightarrow{f} \mathrm{rg}\,(f) \longrightarrow 0,$$

we conclude that $\dim_{\mathscr{M}}(\ker(f)) = 0$. Since f is a map between finitely generated projective modules, its kernel is closed and hence by Proposition 1.4.3 finitely generated and projective. Hence $\ker(f) = \{0\}$ and f is a weak isomorphism.
Since $(iii) \Rightarrow (i)$ is trivial, the proof is complete.

□

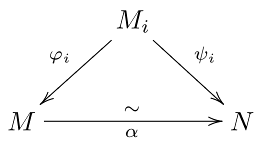
We now proceed with an investigation of the behavior of the dimension function with respect inductive limits. To this end, note that Theorem 1.4.7 part *1.* is a special case of an inductive limit process. We briefly recall the general notion of inductive limits. For this purpose, we consider a directed set $(I, \leq)$ and a family of modules $(M_i)_{i\in I}$ over some unital ring R. Assume moreover the following:
For each $i, j \in I$ with $i \leq j$ there exists a homomorphism $\varphi_{ji} : M_i \to M_j$ and if $i, j, k \in I$ with $i \leq j \leq k$ then $\varphi_{kj}\varphi_{ji} = \varphi_{ki}$. Then the following holds:

Proposition 1.4.12. *There exists an R-module M, called the* inductive limit of the system $(M_i)_{i\in I}$ *and a family of homomorphisms $\varphi_i : M_i \to M$, such that*

(i) $M = \cup_{i\in I}\varphi_i(M_i)$.

(ii) For each pair $i, j \in I$ with $i \leq j$ we have $\varphi_i = \varphi_j\varphi_{ji}$.

(iii) The inductive limit is unique in the following sense: If $(N, (\psi_i))_{i\in I}$ fulfills (the analogue of) (i) and (ii), then there exists a (unique) isomorphism $\alpha : M \to N$ making the following family of diagrams commutative:



(iv) For each $i_0 \in I$ we have $\ker\varphi_{i_0} = \cup_{i\geq i_0}\ker(\varphi_{ii_0})$.

The inductive limit M is often denoted $\varinjlim M_i$.

This is a standard algebraic construction and we omit the proof here. More details on inductive limits can be found in [WO] and [RLL].

Assume, as before, that $\mathscr{M}$ is a finite von Neumann algebra endowed with a fixed normal, faithful, tracial state τ. Then the following holds.

Theorem 1.4.13. [Lüc98] *Consider a family $(M_i)_{i\in I}$ of $\mathscr{M}$-modules with homomorphisms $\varphi_{ji} : M_i \to M_j$ whenever $j \geq i$, satisfying the compatibility conditions $\varphi_{kj}\varphi_{ji} = \varphi_{ki}$, when $k \geq j \geq i$. Let $\varphi_i : M_i \to \varinjlim M_i =: M$ be the morphisms from Proposition 1.4.12. Then the following holds.*

(i) $\dim_{\mathscr{M}}(M) = \sup_{i\in I}\dim_{\mathscr{M}}(\varphi_i(M_i))$.

(ii) If moreover each $i \in I$ is dominated by an $i' \in I$ (i.e. $i \le i'$) such that $\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{i'i}))$ is finite, then

$$\dim_{\mathscr{M}}(M) = \sup_{i\in I} \inf_{j\ge i}(\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji}))).$$

During the reading of the proof, it might be helpful to take a look at the diagram placed at the end of the proof, which encodes the relative position (with respect to the directed set I), of some of the modules appearing in the proof.

Proof. By Proposition 1.4.12, we have $M = \cup_{i\in I}\mathrm{rg}\,(\varphi_i)$ and since $\varphi_i = \varphi_j \circ \varphi_{ji}$ whenever $j \ge i$, the system $(\mathrm{rg}\,(\varphi_i))_{i\in I}$ is directed by inclusion. Applying cofinality of the dimension function (Theorem 1.4.7) to the system $(\mathrm{rg}\,(\varphi_i))_{i\in I}$ yields the equality in (i).
To prove (ii) it suffices to prove the following identity

$$\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_i)) = \inf_{j\ge i} \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji})),$$

for each $i \in I$. Fix an $i \in I$ and choose (according to assumptions) an $i' \in I$ such that $i' \ge i$ and $\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{i'i})) < \infty$. By Proposition 1.4.12 we have $\ker(\varphi_{i'}) = \cup_{j\ge i'} \ker(\varphi_{ji'})$ and hence

$$\ker(\varphi_{i'}) \cap \mathrm{rg}\,(\varphi_{i'i}) = \cup_{j\ge i'} \ker(\varphi_{ji'}) \cap \mathrm{rg}\,(\varphi_{i'i}).$$

Applying cofinality to the system $(\ker(\varphi_{ji'}) \cap \mathrm{rg}\,(\varphi_{i'i}))_{j\ge i'}$ we get

$$\dim_{\mathscr{M}}(\ker(\varphi_{i'}) \cap \mathrm{rg}\,(\varphi_{i'i})) = \sup_{j\ge i'} \dim_{\mathscr{M}}(\ker(\varphi_{ji'}) \cap \mathrm{rg}\,(\varphi_{i'i})). \qquad (\dagger)$$

Since $\varphi_i = \varphi_{i'}\varphi_{i'i}$ we get

$$\begin{aligned}
\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_i)) &= \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{i'}|_{\mathrm{rg}\,(\varphi_{i'i})})) \\
&= \dim_{\mathscr{M}} \mathrm{rg}\,(\varphi_{i'i}) - \dim_{\mathscr{M}}(\ker \varphi_{i'} \cap \mathrm{rg}\,(\varphi_{i'i})) && \text{(Additivity)} \\
&= \dim_{\mathscr{M}} \mathrm{rg}\,(\varphi_{i'i}) - \sup_{j\ge i'} \dim_{\mathscr{M}}(\ker(\varphi_{ji'}) \cap \mathrm{rg}\,(\varphi_{i'i})) && \text{(by } (\dagger)) \\
&= \inf_{j\ge i'} (\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{i'i})) - \dim_{\mathscr{M}}(\ker(\varphi_{ji'}) \cap \mathrm{rg}\,(\varphi_{i'i}))) \\
&= \inf_{j\ge i'} \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji'}|_{\mathrm{rg}\,(\varphi_{i'i})})) && \text{(Additivity)} \\
&= \inf_{j\ge i'} \dim_{\mathscr{M}} \mathrm{rg}\,(\varphi_{ji}) && (\text{since } \varphi_{ji'}\varphi_{i'i} = \varphi_{ji}) \\
&\ge \inf_{j\ge i} \dim_{\mathscr{M}} \mathrm{rg}\,(\varphi_{ji}). && (\text{since } i' \ge i)
\end{aligned}$$

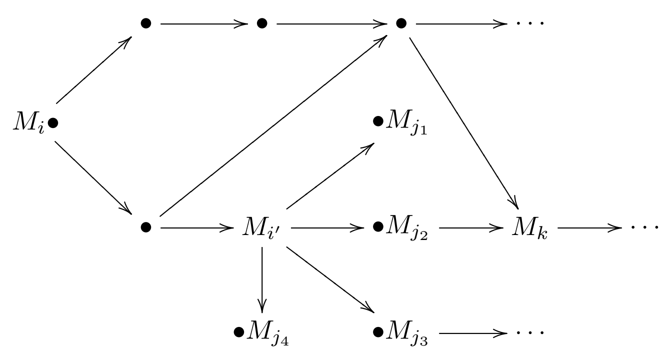
To see the opposite inequality, we note that for any $j \ge i$ there exists $k \ge j, i'$ and we therefore get

$$\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ki})) = \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{kj}\varphi_{ji})) = \dim_{\mathscr{M}}(\varphi_{kj}(\mathrm{rg}\,(\varphi_{ji}))) \le \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji})).$$

This, combined with the two last lines in the above computation, gives

$$\inf_{j\ge i'} \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji})) = \inf_{j\ge i} \dim_{\mathscr{M}} \mathrm{rg}\,(\varphi_{ji}).$$

We now have $\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_i)) = \inf_{j\ge i} \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji}))$ and (ii) follows.



□

Remark 1.4.14. *The formula in Theorem 1.4.13 (ii) does not hold (in general) without the assumption that each $i \in I$ is dominated by an i' with $\dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{i'i})) < \infty$.*
To see this, we consider the case $I = \mathbb{N}$ and define

$$M_i := \mathscr{M}^{(\{i,i+1,\dots\})} = \{x : \{i, i+1, \dots\} \to \mathscr{M} \mid \mathrm{supp}(x) \text{ is finite }\},$$

and for $j \geq i$ we define

$$\varphi_{ji}((x_k)_{k=i}^{\infty}) = (x_k)_{k=j}^{\infty}.$$

This turns $(M_i)_{i\in\mathbb{N}}$ into an inductive system and we denote by M its inductive limit and by $(\varphi_i)_{i\in\mathbb{N}}$ the associated maps $\varphi_i : M_i \to M$. We now claim that $M = \{0\}$.
To see this, consider any $i \in \mathbb{N}$ and an $x = (x_k)_{k=i}^{\infty} \in M_i$. Then, since x has finite support, we can choose $N \in \mathbb{N}$ such that $N > i$ and $\mathrm{supp}(x) \subseteq \{i, \dots, N-1\}$. We then get

$$\varphi_i(x) = \varphi_N \varphi_{Ni}(x) = \varphi_N(0) = 0,$$

and since $M = \cup_{i\in\mathbb{N}} \mathrm{rg}\,(\varphi_i)$ we conclude that $M = \{0\}$.
On the other hand, for each $i \in I$ we have $\dim_{\mathscr{M}}(M_i) = \infty$, since for any given $K \in \mathbb{N}$ the submodule

$$\{x \in M_i | \mathrm{supp}(x) \subseteq \{i, i+1, \dots, i+K-1\}\} \subseteq M_i,$$

is free on K generators and hence of dimension K.
Because each φ_{ji} is surjective, we see from this that

$$\sup_{i\in\mathbb{N}} \inf_{j>i} \dim_{\mathscr{M}}(\mathrm{rg}\,(\varphi_{ji})) = \infty.$$

We end this section, by introducing a special class of modules, which will be needed in the investigation of the so-called *induction-functor* which we introduce in the following section.
As before, $\mathscr{M}$ denotes a finite von Neumann algebra, endowed with a fixed, faithful, tracial state τ.

Definition 1.4.15. *An $\mathscr{M}$-module M is called* finitely presented *if there exists an exact sequence of morphisms*

$$\mathscr{M}^n \xrightarrow{f} \mathscr{M}^m \xrightarrow{g} M \longrightarrow 0, \tag{$*$}$$

and, in this case, the sequence $(*)$ is called a finite presentation *of M.*

The properties of finitely presented modules, which is of relevance to us, is collected in the following lemma. For more results on finitely presented modules, we refer to [Lüc97].

Lemma 1.4.16. [Lüc02] *Let M be an $\mathscr{M}$-module. Then the following holds.*

(i) M is finitely presented, if, and only if, there exists an exact sequence of the form

$$0 \longrightarrow P \longrightarrow \mathscr{M}^m \longrightarrow M \longrightarrow 0,$$

where P is finitely generated and projective.

(ii) If M and N are finitely presented $\mathscr{M}$-modules and $f : M \to N$ is a homomorphism, then $\mathrm{rg}\,(f)$ is finitely presented.

(iii) If M is finitely presented and $\dim_{\mathscr{M}}(M) = 0$, then there exists an exact sequence of the form

$$0 \longrightarrow \mathscr{M}^n \xrightarrow{f} \mathscr{M}^n \longrightarrow M \longrightarrow 0,$$

where f is self-adjoint, with respect to the standard inner product $\langle\cdot|\cdot\rangle_{\mathrm{st}}$ on $\mathscr{M}^n$.

Proof.

Proof of (i): Assume that $0 \longrightarrow P \xrightarrow{f} \mathscr{M}^m \xrightarrow{q} M \longrightarrow 0$ is exact and that P is finitely generated and projective. Then P is (isomorphic to) a direct summand in $\mathscr{M}^n$ for suitable $n \in \mathbb{N}$. That is, $\mathscr{M}^n = P \oplus Q$ for some module Q. Define $F : \mathscr{M}^n \to \mathscr{M}^m$ by $F(x,y) = f(x)$. Then

$$\mathscr{M}^n \xrightarrow{F} \mathscr{M}^m \xrightarrow{q} M \longrightarrow 0,$$

is a finite presentation of M.
Conversely, if $\mathscr{M}^n \xrightarrow{f} \mathscr{M}^m \xrightarrow{g} M \longrightarrow 0$ is a finite presentation of M, then $\ker(g)$ is finitely generated and hence projective by Corollary 1.3.21. Then

$$0 \longrightarrow \ker(g) \xrightarrow{\iota} \mathscr{M}^m \xrightarrow{q} M \longrightarrow 0,$$

has the required properties.

Proof of (ii): Choose a finite presentation $\mathscr{M}^n \xrightarrow{h} \mathscr{M}^m \xrightarrow{g} N \longrightarrow 0$ of N. We then have an exact sequence

$$\mathscr{M}^n \xrightarrow{h} g^{-1}(\mathrm{rg}\,(f)) \xrightarrow{g} \mathrm{rg}\,(f) \longrightarrow 0.$$

Since both $\mathrm{rg}\,(f)$ and $\ker(g)$ are finitely generated, one easily checks that $P := g^{-1}(\mathrm{rg}\,(f))$ is finitely generated and, by Corollary 1.3.21, therefore projective. Thus, up to isomorphism, we have $\mathscr{M}^l = P \oplus Q$ for some module Q and some $l \in \mathbb{N}$.
Define $\alpha : \mathscr{M}^n \oplus \mathscr{M}^l \longrightarrow \mathscr{M}^l$ by

$$\mathscr{M}^n \oplus P \oplus Q \ni (x,y,z) \longmapsto (h(x), z) \in P \oplus Q,$$

and $\beta : \mathscr{M}^l \to \mathrm{rg}\,(f)$ by

$$P \oplus Q \ni (x,y) \longmapsto gx \in \mathrm{rg}\,(f).$$

Then $\mathscr{M}^n \oplus \mathscr{M}^l \xrightarrow{\alpha} \mathscr{M}^l \xrightarrow{\beta} \mathrm{rg}\,(f) \longrightarrow 0$ is a finite presentation of $\mathrm{rg}\,(f)$.

Proof of (iii:) Assume that M is finitely presented with $\dim_{\mathscr{M}}(M) = 0$ and choose (according to (i)) an exact sequence

$$0 \longrightarrow P \xrightarrow{g} \mathscr{M}^n \xrightarrow{p} M \longrightarrow 0,$$

with P finitely generated and projective. Since $\dim_{\mathscr{M}}(M) = 0$ and g is injective, additivity of the dimension function and Corollary 1.4.11 implies that g is a weak isomorphism.
Choose an inner product on P and consider the induced map

$$\nu(g) : \nu(P) \longrightarrow L^2(\mathscr{M})^n.$$

Then, since ν preserves weak exactness, $\nu(g)$ is also a weak isomorphism; i.e. injective with dense range. If we do the polar decomposition $\nu(g) = |\nu(g)^*|v$, then v is an isomorphism from $\nu(P)$ to $L^2(\mathscr{M})^n$ and $|\nu(g)^*| : L^2(\mathscr{M})^n \longrightarrow L^2(\mathscr{M})^n$ is an injection. (see e.g. the proof of Proposition 1.4.3) Therefore $u := \nu^{-1}(v)$ is an isomorphism from P to $\mathscr{M}^n$ and $f := \nu^{-1}(|\nu(g)^*|) : \mathscr{M}^n \to \mathscr{M}^n$ is a self-adjoint injection. The sequence

$$0 \longrightarrow \mathscr{M}^n \xrightarrow{f} \mathscr{M}^n \xrightarrow{p} M \longrightarrow 0,$$

is exact, since $\mathrm{rg}\,(f) = \mathrm{rg}\,(gu^{-1}) = \mathrm{rg}\,(g) = \ker(p)$.
This completes the proof.

□

1.5 The induction functor

Let $\mathscr{M}$ and $\mathscr{N}$ be finite von Neumann algebras with faithful, normal, tracial states τ and σ respectively. Let $\varphi : \mathscr{N} \to \mathscr{M}$ be a unital, trace-preserving, $*$-algebra-homomorphism, where trace-preserving means that $\sigma = \tau \circ \varphi$. Via the homomorphism φ, $\mathscr{M}$ can be considered a right $\mathscr{N}$-module[6] and the functor $\mathscr{M} \otimes_{\mathscr{N}} -$ from $\mathrm{Mod}(\mathscr{N})$ to $\mathrm{Mod}(\mathscr{M})$ is called *induction with* φ and we denote it by φ_*.
Note, that since φ is assumed to be trace-preserving, we have

$$\|\eta_\tau(\varphi(x))\|_2 = \tau(\varphi(x)^*\varphi(x)) = \tau(\varphi(x^*x)) = \sigma(x^*x) = \|\eta_\sigma(x)\|_2,$$

for any $x \in \mathscr{N}$.
In the following, φ_n will denote $(\varphi, \varphi, \dots, \varphi) : \mathscr{N}^n \to \mathscr{M}^n$ and η_τ and η_σ will (also) denote the inclusions of $\mathscr{M}^n$ and $\mathscr{N}^n$ respectively into $L^2(\mathscr{M}, \tau)^n$ and $L^2(\mathscr{N}, \sigma)^n$.
Moreover, $e_1, \dots, e_n$ denotes the canonical basis of $\mathscr{N}^n$, where

$$e_i = (0, \dots, 0, 1, 0 \dots, 0),$$

with the 1 in the i'th position. To avoid confusingly many appearances of the letter "M"; throughout this section, we choose the last letters in the alphabet $(X, Y, Z, \cdots)$ to denote modules. The following theorem is a slight generalization of [Lüc98] Theorem 3.3.

Theorem 1.5.1. *Induction with φ is a has the following properties.*

(i) It preserves direct sums and projectivity and if X is a finitely generated $\mathscr{N}$-module, then $\varphi_*(X)$ is a finitely generated $\mathscr{M}$-module.

(ii) It preserves exactness.

(iii) For any $X \in \mathrm{Mod}(\mathscr{N})$ we have

$$\dim_{\mathscr{N}}(X) = \dim_{\mathscr{M}} \varphi_*(X)$$

Proof.

Proof of (i): Since the tensor product distributes over direct sums (up to canonical isomorphism), we get that φ_* preserves direct sums.
If $X \in \mathrm{Mod}(\mathscr{N})$ is finitely generated, say by $\xi_1, \dots, \xi_n$, then $\varphi_*(X) := \mathscr{M} \otimes_{\mathscr{N}} X$ is generated as an $\mathscr{M}$-module by $1 \otimes \xi_1, \dots, 1 \otimes \xi_n$ and is therefore especially finitely generated.
An $\mathscr{N}$-module X is projective if, and only if, it is isomorphic to a direct summand in a free $\mathscr{N}$-module. One easily checks that φ_* preserves free-ness and since it also preserves direct sums, it preserves projectivity.

Proof of (ii): This is the most technical part of the proof. It is divided into four parts, in which we gradually obtain the desired result for an increasing class of modules. The details are as follows.

Part 1: *If X is a finitely generated and projective $\mathscr{N}$-module, then* $\dim_{\mathscr{N}}(X) = \dim_{\mathscr{M}}(\varphi_*(X))$

Since X is finitely generated and projective, we may assume that $X = \mathscr{N}^n p$ for some $n \in \mathbb{N}$ and some self-adjoint idempotent $p \in M_n(\mathscr{N})$. Let ι denote the inclusion $\mathscr{N}^n p \subseteq \mathscr{N}^n$ and let α denote the canonical isomorphism $\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n \to \mathscr{M}^n$ given by

$$T \otimes (x_1, \dots, x_n) \longmapsto (T\varphi(x_1), \dots, T\varphi(x_n)).$$

[6] The action being $x \cdot n := x\varphi(n)$ for $x \in \mathscr{M}$ and $n \in \mathscr{N}$

Since p is idempotent, $\alpha \circ (1 \otimes \iota)$ is an injective $\mathscr{M}$-linear map from $\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n p$ into $\mathscr{N}^n$.
Let p_{ij} denote the (i,j)'th entrance in p and denote by $\varphi(p)$ the matrix $\{\varphi(p_{ij})\}_{i,j=1}^n \in M_n(\mathscr{M})$.
Then, for $(x_1, \ldots, x_n) \in \mathscr{N}^n$ and $T \in \mathscr{M}$, we get

$$\begin{aligned}
\alpha \circ (1 \otimes \iota)(T \otimes (x_1, \ldots, x_n)p) &= \alpha(T \otimes (\sum_{i=1}^n x_i p_{i1}, \ldots, \sum_{i=1}^n x_i p_{in})) \\
&= (T \sum_{i=1}^n \varphi(x_i)\varphi(p_{i1}), \ldots, T \sum_{i=1}^n \varphi(x_i)\varphi(p_{in})) \\
&= T(\varphi(x_1), \ldots, \varphi(x_n))\varphi(p).
\end{aligned}$$

This shows, that $\alpha \circ (1 \otimes \iota)$ maps $\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n p$ injectively into $\mathscr{M}^n \varphi(p)$ and since

$$\alpha \circ (1 \otimes \iota)(T \otimes e_i) = (T\varphi_n(e_i))\varphi(p),$$

it follows that $\alpha \circ (1 \otimes \iota)$ is surjective. Note, that since φ is a $*$-algebra-homomorphism, $\varphi(p)$ is a self-adjoint idempotent in $M_n(\mathscr{M})$ and since φ is assumed trace-preserving we get

$$\dim_{\mathscr{M}}(\varphi_*(X)) = \dim_{\mathscr{M}}(\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n p) = \dim_{\mathscr{M}}(\mathscr{M}^n \varphi(p)) = \sum_{i=1}^n \tau(\varphi(p_{ii})) = \sum_{i=1}^n \sigma(p_{ii}) = \dim_{\mathscr{N}}(X).$$

Part 2: *If X is a finitely presented $\mathscr{N}$-module, then* $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X) = 0$ *and*

$$\dim_{\mathscr{N}}(X) = \dim_{\mathscr{M}}(\varphi_*(X)).$$

Since X is finitely presented, it is in particular finitely generated and hence it splits as $X \simeq TX \oplus PX$ by Theorem 1.4.7.
By Lemma 1.4.16 (ii), also TX is finitely presented and since $\dim_{\mathscr{N}}(TX) = 0$, there exists (Lemma 1.4.16 (iii)) an exact sequence

$$0 \longrightarrow \mathscr{N}^n \xrightarrow{f} \mathscr{N}^n \xrightarrow{q} TX \longrightarrow 0, \tag{†}$$

where f is represented by a self-adjoint matrix $A \in M_n(\mathscr{N})$. (I.e. $f = R_A$.)
Since f is injective and $\dim_{\mathscr{N}}(TX) = 0$, f is a weak isomorphism (Corollary 1.4.11) and since ν preserves weak exactness, we see that $\mathrm{rg}\,(\nu(f))$ is dense in $L^2(\mathscr{N})^n$.

By general homological algebra, the tensor-product is right-exact (see e.g. [Fox] 19.19), and by applying the induction-functor φ_* to (†), we therefore get the following exact sequence:

$$\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n \xrightarrow{1 \otimes R_A} \mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n \xrightarrow{1 \otimes q} \mathscr{M} \otimes_{\mathscr{N}} TM \longrightarrow 0.$$

We now aim to show that $1 \otimes R_A$ is injective.
A direct computation reveals that $\alpha \circ (1 \otimes R_A) = R_{\varphi(A)} \circ \alpha$ and hence we get the following commutative diagram

$$\begin{array}{ccccc}
\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n & \xrightarrow[\sim]{\alpha} & \mathscr{M}^n & \xrightarrow{\eta_\tau} & L^2(\mathscr{M})^n \\
\downarrow{\scriptstyle 1 \otimes R_A} & & \downarrow{\scriptstyle R_{\varphi(A)}} & & \downarrow{\scriptstyle \nu(R_{\varphi(A)})} \\
\mathscr{M} \otimes_{\mathscr{N}} \mathscr{N}^n & \xrightarrow[\alpha]{\sim} & \mathscr{M}^n & \xrightarrow[\eta_\tau]{} & L^2(\mathscr{M})^n
\end{array} \tag{◇}$$

We now claim that $\nu(R_{\varphi(A)})$ has dense range.
To see this, let $\xi \in L^2(\mathscr{M})^n$ and $\varepsilon > 0$ be given and choose $x \in \mathscr{M}^n$ such that $\|\eta_\tau(x) - \xi\|_2 < \frac{\varepsilon}{2}$. Then x has the form $\alpha(\sum_{i=1}^l s_i \otimes z_i)$, for $s_i \in \mathscr{M}$ and $z_i \in \mathscr{N}^n$. Assume, without loss of generality, that $x \neq 0$. By what was proven above, $\nu(R_A)$ has dense range and since $\nu(R_A)$ is the extension of R_A by continuity, we have that $\mathrm{rg}\,(\nu(R_A)) \subseteq \overline{\eta_\sigma(\mathrm{rg}\,(R_A))}^{L^2}$ and hence that $\eta_\sigma(\mathrm{rg}\,(R_A)$ is dense in $L^2(\mathscr{N})^n$. So we may choose $y_1, \dots, y_l \in \mathscr{N}^n$ such that

$$\|\eta_\sigma(R_A(y_i)) - \eta_\sigma(z_i)\|_2 < \frac{\varepsilon}{2l \sup_{i=1}^l(\|s_i\|_\infty)},$$

where $\|s_i\|_\infty$ denotes the operator-norm of s_i.
We now have

$$\begin{aligned}
\|\eta_\tau \circ \alpha \circ (1 \otimes R_A)(\sum_{i=1}^l s_i \otimes y_i) - \xi\|_2 &= \|\eta_\tau \circ \alpha \circ (1 \otimes R_A)(\sum_{i=1}^l s_i \otimes y_i) - \eta_\tau(x) + \eta_\tau(x) - \xi\|_2 \\
&\leq \|\eta_\tau \circ \alpha(\sum_{i=1}^l s_i \otimes (R_A(y_i) - z_i))\|_2 + \frac{\varepsilon}{2} \\
&= \|\eta_\tau(\sum_{i=1}^l s_i \varphi_n(R_A(y_i) - z_i))\|_2 + \frac{\varepsilon}{2} \\
&\leq \sum_{i=1}^l \|\eta_\tau(s_i \varphi_n(R_A(y_i) - z_i))\|_2 + \frac{\varepsilon}{2} \\
&\leq \sum_{i=1}^l \|s_i\|_\infty \|\eta_\tau(\varphi_n(R_A(y_i) - z_i))\|_2 + \frac{\varepsilon}{2} \\
&\leq \sup_i(\|s_i\|_\infty) \sum_{i=1}^l \|\eta_\sigma((R_A(y_i) - z_i))\|_2 + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

By commutativity of the diagram ($\diamondsuit$), this implies that

$$\|\nu(R_A) \circ \eta_\tau \circ \alpha(\sum_{i=1}^l s_i \otimes y_i) - \xi\|_2 < \varepsilon,$$

and hence $\mathrm{rg}\,(\nu(R_{\varphi(A)}))$ is dense in $L^2(\mathscr{M})^n$.
Since $f = R_A$ is self-adjoint, $A \in M_n(\mathscr{N})$ is self-adjoint and since φ is a $*$-algebra-homomorphism, $\varphi(A) \in M_n(\mathscr{M})$ is also self-adjoint. Therefore $R_{\varphi(A)} : \mathscr{M}^n \to \mathscr{M}^n$ is self-adjoint and hence $\nu(R_{\varphi(A)}) : L^2(\mathscr{M})^n \to L^2(\mathscr{M})^n$ is self-adjoint. From this it follows that

$$\ker(\nu(R_{\varphi(A)})) = \mathrm{rg}\,(\nu(R_{\varphi(A)})^*)^\perp = \mathrm{rg}\,(\nu(R_{\varphi(A)}))^\perp = \{0\}.$$

Since ν^{-1} preserves exactness, also $R_{\varphi(A)}$ is injective and since

$$\varphi_*(R_A) = 1 \otimes R_A = \alpha^{-1} \circ R_{\varphi(A)} \circ \alpha,$$

we see that $\varphi_*(f) = \varphi_*(R_A) = 1 \otimes R_A$ is injective.
We therefore have an exact sequence

$$0 \longrightarrow \varphi_*(\mathscr{N}^n) \stackrel{\varphi_*(f)}{\longrightarrow} \varphi_*(\mathscr{N}^n) \stackrel{\varphi_*(q)}{\longrightarrow} \varphi_*(TX) \longrightarrow 0, \tag{‡}$$

and with this exact sequence at our disposal, we are now able to prove the rest of Claim 2.
By additivity of the dimension function, applied to (‡), we get

$$\dim_{\mathscr{M}}(\varphi_*(TX)) = n - n = 0,$$

and hence

$$\begin{aligned}
\dim_{\mathscr{N}}(X) &= \dim_{\mathscr{N}}(PX) + \dim_{\mathscr{N}}(TX) && \text{(Additivity)}\\
&= \dim_{\mathscr{N}}(PX) + 0 \\
&= \dim_{\mathscr{M}} \varphi_*(PX) + 0 && \text{(by Part 1)}\\
&= \dim_{\mathscr{M}} \varphi_*(PX) + \dim_{\mathscr{M}} \varphi_*(TX) \\
&= \dim_{\mathscr{M}}(\varphi_*(X)). && \text{(Additivity)}
\end{aligned}$$

We now prove that $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X) = 0$.
Since $\mathrm{Tor}_*^{\mathscr{N}}(\mathscr{M}, -)$ respects direct sums (see e.g. [CE], Chp.VI, Proposition 1.3) we get

$$\begin{aligned}
\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X) &= \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, PX \oplus TX) \\
&\simeq \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, PX) \oplus \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, TX) \\
&\simeq \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, TX). && \text{(since } PX \text{ is } \mathscr{N}\text{-projective)}
\end{aligned}$$

It is therefore sufficient to prove that $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, TX) = 0$.
To see this, we note that (†) is a free $\mathscr{N}$-resolution of TX and hence $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, TX)$ can be computed as the homology in degree one, of the complex

$$0 \longrightarrow \underbrace{\varphi_*(\mathscr{N}^n)}_{\text{degree 1}} \stackrel{\varphi_*(f)}{\longrightarrow} \underbrace{\varphi_*(\mathscr{N}^n)}_{\text{degree 0}} \longrightarrow 0,$$

which vanishes by injectivity of $\varphi_*(f) = 1 \otimes R_A$.
This completes the proof of Part 2.

Part 3: *If X is a finitely generated $\mathscr{N}$-module, then* $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X) = 0$.

Since X is finitely generated, we may choose a finitely generated free module F and an epimorphism $p : F \to X$. Denote by K the kernel of p and consider the associated short-exact sequence

$$0 \longrightarrow K \stackrel{j}{\longrightarrow} F \stackrel{p}{\longrightarrow} X \longrightarrow 0,$$

(here j denotes the inclusion) and the induced long-exact sequence of Tor-groups:

$$\begin{array}{l}
\hookrightarrow \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, K) \longrightarrow \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, F) \longrightarrow \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X) \\
\longrightarrow \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, K) \longrightarrow \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, F) \longrightarrow \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, X) \longrightarrow 0
\end{array}$$

Since F is free (and hence projective) $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, F) = 0$ and it therefore suffices to show that the induced map

$$\mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, K) = \mathscr{M} \otimes_{\mathscr{N}} K \xrightarrow{\quad\varphi_*(j)\quad} \mathscr{M} \otimes_{\mathscr{N}} F = \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, F),$$

is injective. So, assume that $\varphi_*(j)(x) = (1 \otimes j)x = 0$ for some $x = \sum_{i=1}^n T_i \otimes k_i \in \mathscr{M} \otimes_{\mathscr{N}} K$ and let K' be the submodule of F generated by $k_1, \ldots, k_n$. As above, we have a long-exact sequence of Tor-groups

$$\begin{array}{l}
\hookrightarrow \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, K') \longrightarrow \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, F) \longrightarrow \mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, F/K') \\
\longrightarrow \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, K') \longrightarrow \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, F) \longrightarrow \mathrm{Tor}_0^{\mathscr{N}}(\mathscr{M}, F/K') \longrightarrow 0,
\end{array}$$

and since F/K' is finitely presented (Lemma 1.4.16 (ii)), Part 2 implies that $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, F/K') = 0$. This means that $1 \otimes j = \varphi_*(j) : \mathscr{M} \otimes_{\mathscr{N}} K' \to \mathscr{M} \otimes_{\mathscr{N}} F$ is injective and hence that x is zero in $\mathscr{M} \otimes_{\mathscr{N}} K'$. Then, in particular, x is zero in $\mathscr{M} \otimes_{\mathscr{N}} K$ and thus $\varphi_*(j) : \mathscr{M} \otimes_{\mathscr{N}} K \to \mathscr{M} \otimes_{\mathscr{N}} F$ is injective.

Part 4: *Induction with φ preserves exactness*

Since $\varphi_* := \mathscr{M} \otimes_{\mathscr{N}} -$ is right-exact by construction, it suffices to show that $\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X) = 0$ for all $\mathscr{N}$-modules X. (see e.g. [Wei] Ex. 3.2.1)
Let $(X_i)_{i \in I}$ denote the cofinal system of finitely generated submodules of X. (ordered by inclusion) Since $\mathrm{Tor}_*^{\mathscr{N}}(\mathscr{M}, -)$ commutes with inductive limits (see e.g. [CE], Chp. VI, Prop. 1.3), it suffices to show that

$$\mathrm{Tor}_1^{\mathscr{N}}(\mathscr{M}, X_i) = 0, \quad \text{for every } i \in I.$$

But this follows from Part 3; and the proof of *(ii)* is complete.

Proof of (iii): Assume X to be any $\mathscr{N}$-module and let $(X_i)_{i \in I}$ denote the system of finitely generated submodules of X, ordered by inclusion. This is a cofinal system of submodules and by Theorem 1.4.7 we have

$$\dim_{\mathscr{N}}(X) = \sup_{i \in I} \dim_{\mathscr{N}}(X_i).$$

Since φ_* preserves exactness, each $\varphi_*(X_i)$ is (isomorphic to) a submodule in $\varphi_*(X)$ and since each X_i is finitely generated over $\mathscr{N}$, each $\varphi_*(X_i)$ is finitely generated over $\mathscr{M}$. The system $(\varphi_*(X_i))_{i \in I}$ is therefore a cofinal system (of finitely generated submodules) in $\varphi_*(X)$ and hence

$$\dim_{\mathscr{M}} \varphi_*(X) = \sup_{i \in I} \dim_{\mathscr{M}}(\varphi_*(X_i)).$$

Because of this, it suffices to consider the case where X is finitely generated.
Choose a finitely generated free module F, a surjective homomorphism $f : F \to X$ and denote by K the kernel of f. By exactness of φ_*, the short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow X \longrightarrow 0,$$

induces a short-exact sequence

$$0 \longrightarrow \varphi_*(K) \to \varphi_*(F) \longrightarrow \varphi_*(X) \longrightarrow 0.$$

Applying additivity of the dimension function to the two short-exact sequences, gives the following equations

$$\dim_{\mathscr{N}}(X) = \dim_{\mathscr{N}}(F) - \dim_{\mathscr{N}}(K)$$
$$\dim_{\mathscr{M}}(\varphi_*(X)) = \dim_{\mathscr{M}}(\varphi_*(F)) - \dim_{\mathscr{M}}(\varphi_*(K)).$$

Since free modules are projective, Part 1 implies that it suffices to prove

$$\dim_{\mathscr{N}}(K) = \dim_{\mathscr{M}}(\varphi_*(K)).$$

By repeating the above cofinality-argument (with X replaced by K), we may assume that K is finitely generated. But then K is a finitely generated submodule of the projective $\mathscr{N}$-module F and therefore projective since $\mathscr{N}$ is semi-hereditary. (Corollary 1.3.21) Thus Part 1 applies and we conclude that

$$\dim_{\mathscr{N}}(X) = \dim_{\mathscr{M}}(\varphi_*(X)) = \dim_{\mathscr{M}}(\mathscr{M} \otimes_{\mathscr{N}} X),$$

as desired. □

Remark 1.5.2. *We place ourselves under the hypotheses of Theorem 1.5.1 and assume for simplicity that* $\mathscr{N}$ *is a subalgebra of* $\mathscr{M}$ *and that* φ *is the inclusion.*
Consider a ***right*** $\mathscr{N}$*-module* Y*. Then* Y *may be considered a left* $\mathscr{N}^{\mathrm{op}}$*-module via the action* $n^{\mathrm{op}} \cdot y := yn$ *and we can therefore form the induced module* $\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}^{\mathrm{op}}} Y$*.*
By construction, this is a left $\mathscr{M}^{\mathrm{op}}$*-module and may therefore be considered as a right* $\mathscr{M}$*-module via the action*

$$(a^{\mathrm{op}} \otimes y) \cdot m := m^{\mathrm{op}}(a^{\mathrm{op}} \otimes y) = (m^{\mathrm{op}} a^{\mathrm{op}}) \otimes y.$$

The map

$$\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}^{\mathrm{op}}} Y \ni m^{\mathrm{op}} \otimes y \longmapsto y \otimes m \in Y \otimes_{\mathscr{N}} \mathscr{M},$$

is an isomorphism of right $\mathscr{M}$*-modules and using this (and Theorem 1.5.1) one easily checks that the functor* $Y \mapsto Y \otimes_{\mathscr{N}} \mathscr{M}$ *is exact; from the category of right* $\mathscr{N}$*-modules to the the category of right* $\mathscr{M}$*-modules.*

Consider now a left $\mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}$*-module* Z*. Then* Z *can be considered as an* $\mathscr{N}$*-bimodule, (see e.g. section 2.1) with respect to the action*

$$a \cdot z := (a \otimes 1)z \quad \text{and} \quad z \cdot b := (1 \otimes b^{\mathrm{op}})z.$$

We may therefore form the $\mathscr{M}$*-bimodule* $\mathscr{M} \otimes_{\mathscr{N}} Z \otimes_{\mathscr{N}} \mathscr{M}$ *and we note that the functor* $Z \mapsto \mathscr{M} \otimes_{\mathscr{N}} Z \otimes_{\mathscr{N}} \mathscr{M}$ *is exact, as the composition of two exact functors.*
When $\mathscr{M} \otimes_{\mathscr{N}} Z \otimes_{\mathscr{N}} \mathscr{M}$ *is considered as a left* $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$*-module, with respect to the action*

$$(a \otimes b^{\mathrm{op}}) \cdot (m \otimes z \otimes n) := (am) \otimes z \otimes (nb),$$

$\mathscr{M} \otimes_{\mathscr{N}} Z \otimes_{\mathscr{N}} \mathscr{M}$ *is isomorphic to*

$$(\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}} Z,$$

with the standard action of multiplication on the first factor.
From this it follows, that also $(\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}} -$ *is exact; as functor from the category of left* $\mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}$*-modules, to the category of left* $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$*-modules.*

Chapter 2

Some Homology Theory

In this chapter, we recapitulate some basic facts on Hochschild homology and the bar-resolution. These subjects are central in the definition of L^2-homology for von Neumann algebras and to set up notation we give a short introduction here. The last half of the chapter consists of a short introduction to group homology and we explain a link between the homology of a group and the Hochschild homology of its associated group algebra.
This is not, in any way, a comprehensive treatment, but to avoid to many unnecessary digressions we just introduce what is needed for the sequel. For more details we refer to [Lod].

2.1 Hochschild homology

Consider a unital (associative) $\mathbb{C}$-algebra A. A *bimodule* over A (or an A-bimodule) is a (complex) vector space M, with structure as both a left- and a right A-module, such that

$$(am)b = a(mb) \quad \text{and} \quad (\lambda 1)m = m(\lambda 1) = \lambda m,$$

for all $a, b \in A$, $m \in M$ and $\lambda \in \mathbb{C}$.
Note, that A itself is an A-bimodule, with respect to multiplication from left and right.
An A-bimodule M may also be considered a left module over $A \otimes A^{\mathrm{op}}$, with respect to the action

$$(a \otimes b^{\mathrm{op}})m := amb,$$

and, similarly, as a right $A \otimes A^{\mathrm{op}}$-module with respect to the action $m(a \otimes b^{\mathrm{op}}) := bma$.
Conversely, given a left $A \otimes A^{\mathrm{op}}$-module M, we can turn M into an A-bimodule by setting

$$am := (a \otimes 1)m \quad \text{and} \quad mb := (1 \otimes b^{\mathrm{op}})m.$$

Similarly, if M is a right $A \otimes A^{\mathrm{op}}$-module, we can turn M into an A-bimodule, be setting

$$am := m(1 \otimes a^{\mathrm{op}}) \quad \text{and} \quad mb := m(b \otimes 1)$$

We will switch hence and forth between these different module-structures and unless otherwise mentioned, an A-bimodule M is always considered a left (resp. right) $A \otimes A^{\mathrm{op}}$-module with respect to the actions introduced above.
The algebraic tensor product $A \otimes A^{\mathrm{op}}$ is called the *enveloping algebra* and to simplify notation we will some times denote it by A^{e}. We now give the definition of Hochschild homology.

Definition 2.1.1. *Let A be a unital $\mathbb{C}$-algebra and let M be a bimodule over A. We then define $C_0(A, M) := M$ and*

$$C_n(A, M) := M \otimes A^{\otimes n} \quad \text{for} \quad n \in \mathbb{N}.$$

Here $A^{\otimes n} := A \otimes A \otimes \cdots \otimes A$. *(n copies.) We define the* n'th Hochschild boundary map $b_n : C_n(A, M) \to C_{n-1}(A, M)$ *as*

$$\begin{aligned} b_n(m, a_1, \ldots, a_n) := (ma_1, a_2, \ldots, a_n) \\ + \sum_{i=1}^{n-1} (-1)^i (m, a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n) \\ + (-1)^n (a_n m, a_1, \ldots, a_{n-1}), \end{aligned}$$

where we for notational convenience write $(m, a_1, \ldots, a_n)$ *in stead of* $m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n$.

A straight forward computation shows, that b_n as defined above is indeed a boundary map, in the sense that $b_n b_{n+1} = 0$. In stead of doing the calculation on b_n directly, we will prove the following slightly more general lemma, which has $b_n b_{n+1} = 0$ as an easy consequence.

Lemma 2.1.2. [Lod] *Let* $(K_n)_{n=0}^{\infty}$ *be a sequence of modules over a unital ring* R*, such that, for each* $n \geq 1$*, we have a family of maps*

$$d_i^n : K_n \to K_{n-1} \quad \text{for } i \in \{0, \ldots, n\}.$$

Assume moreover, that $d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1}$ *for* $0 \leq i < j \leq n$.
Setting $\partial_n := \sum_{i=0}^n (-1)^i d_i$ *we have* $\partial_n \partial_{n+1} = 0$ *and hence that* (K_*, ∂_*) *is a (differential) complex.*

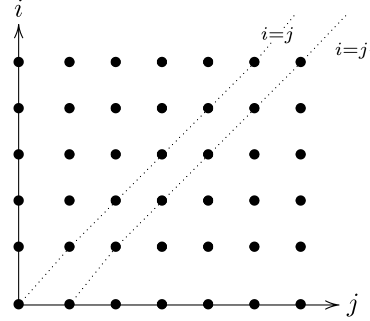
Proof. We have

$$\begin{aligned} \partial_n \partial_{n+1} &= (\sum_{i=0}^{n} (-1)^i d_i^n)(\sum_{j=0}^{n+1} (-1)^j d_j^{n+1}) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} d_i^n d_j^{n+1} \\ &= \sum_{\substack{i;j=0 \\ i<j}}^{n;n+1} (-1)^{i+j} d_i^n d_j^{n+1} + \sum_{\substack{i;j=0 \\ i\geq j}}^{n;n+1} (-1)^{i+j} d_i^n d_j^{n+1} \\ &= \sum_{\substack{i;j=0 \\ i<j}}^{n;n+1} (-1)^{i+j} d_{j-1}^n d_i^{n+1} + \sum_{\substack{i;j=0 \\ i\geq j}}^{n;n+1} (-1)^{i+j} d_i^n d_j^{n+1} \qquad \text{(by assumption)} \end{aligned}$$

For each term $(-1)^{i+j} d_{j-1}^n d_i^{n+1}$ in the left-most sum, the pair $(i', j') := (j-1, i)$ satisfies $i' \geq j'$ and we see that

$$(-1)^{i+j} d_{j-1}^n d_i^{n+1} + (-1)^{i'+j'} d_{i'}^n d_{j'}^{n+1} = (-1)^{i+j} d_{j-1}^n d_i^{n+1} + (-1)^{j-1+i} d_{j-1}^n d_i^{n+1} = 0.$$

Hence every term in the first sum cancels with one in the second and we just need to check that the number of terms in to sums are equal.
This can, for instance, be seen geometrically by considering a lattice of the following form:



□

Example 2.1.3. *Consider again the Hochschild complex* $C_n(A, M) := M \otimes A^{\otimes n}$ *and define* $d_i^n : C_n(A, M) \to C_{n-1}(A, M)$ *as*

$$\begin{aligned} d_0^n(m, a_1, \dots, a_n) &:= (ma_1, a_2, \dots, a_n) \\ d_i^n(m, a_1, \dots, a_n) &:= (m, a_1, \dots, a_i a_{i+1}, \dots, a_n) \quad \text{for } i \in \{1, \dots, n-1\} \\ d_n^n(m, a_1, \dots, a_n) &:= (a_n m, a_1, \dots, a_{n-1}). \end{aligned}$$

A direct computation shows that $d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1}$ *when* $0 \le i < j \le n$ *and since* $b_n = \sum_{i=0}^n (-1)^i d_i^n$ *by definition, we have* $b_n b_{n+1} = 0$ *by Lemma 2.1.2.*

Definition 2.1.4. *The homology of the complex* $(C_*(A, M), b_*)$ *is called the* Hochschild homology *of the algebra A* with coefficients in M *and is denoted* $H_*(A, M)$.

2.1.1 The Bar-resolution

In this section we give an equivalent description of the Hochschild homology of a unital algebra, in the language of homological algebra.
Assume, as before, that A is a unital $\mathbb{C}$-algebra and consider A as a left-module over $A \otimes A^{\text{op}}$ under the action $(a \otimes b^{\text{op}})x := axb$. We set

$$C_n^{\text{bar}}(A) := A^{\otimes n+2} \text{ for } n \in \mathbb{N}_0,$$

and define (for $n \ge 1$) $b'_n : C_n^{\text{bar}}(A) \to C_{n-1}^{\text{bar}}(A)$ by

$$b'_n(a_0, \dots, a_{n+1}) = \sum_{i=0}^n (-1)^i (a_0, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}).$$

Notice, that by putting $M = A$ in the definition of the Hochschild complex, we have $C_n^{\text{bar}}(A) = C_{n+1}(A, A)$ and $b'_n = \sum_{i=0}^n (-1)^i d_i^{n+1}$, where the facemaps d_i^n are the ones from Example 2.1.3. Hence $b'_n b'_{n+1} = 0$ by Lemma 2.1.2.
The complex $(C_*^{\text{bar}}(A), b'_*)$ is called the *bar-complex* of A.

Remark 2.1.5. *Each* $C_n^{\text{bar}}(A)$ *is also an* $A \otimes A^{\text{op}}$*-module under the action*

$$(\alpha \otimes \beta^{\text{op}})(a_0, \dots, a_{n+1}) = (\alpha a_0, a_1, \dots, a_n, a_{n+1}\beta).$$

Since A is a vector space over $\mathbb{C}$ *it has a basis* $(e_i)_{i \in I}$. *Then one easily verifies, that elements of the form* $1 \otimes e_{i_1} \otimes \dots \otimes e_{i_n} \otimes 1$ *forms a basis of* $C_n^{\text{bar}}(A)$*, when considered as an* $A \otimes A^{\text{op}}$*-module.*
Hence each $C_n^{\text{bar}}(A)$ *is a free* $A \otimes A^{\text{op}}$*-module.*
Note also, that the boundary-maps b'_n *commutes with the action of* $A \otimes A^{\text{op}}$.

Proposition 2.1.6. [Lod] *The bar-complex* $(C_*^{\text{bar}}(A), b'_*)$ *is acyclic and the multiplication map* $\mu : A \otimes A \to A$ *given by* $\mu(a' \otimes a'') = a'a''$ *is an augmentation of this complex. In this way,* (C_*^{bar}, b'_*) *becomes a free resolution of the* $A \otimes A^{\text{op}}$*-module A.*

Proof. Since b'_1 is defined by : $a_0 \otimes a_1 \otimes a_2 \mapsto (a_0 a_1) \otimes a_2 - a_0 \otimes (a_1 a_2)$, the associativity of A implies that $\mu b'_1 = 0$.
Conversely, if $\sum_i a_i \otimes b_i \in A \otimes A$ with $\sum_i a_i b_i = 0$, then

$$\begin{aligned} \sum_i a_i \otimes b_i &= \sum_i a_i \otimes (b_i 1) - (\sum_i a_i b_i) \otimes 1 \\ &= \sum_i a_i \otimes (b_i 1) - (a_i b_i) \otimes 1 \\ &= \sum_i -b'_1(a_i \otimes b_i \otimes 1) \in \text{rg}\,(b'_1). \end{aligned}$$

This shows, that the zero'th homology of the augmented complex vanishes and we now have to prove that this is also the case for the higher homology-groups.
Consider the map $s_n : A^{\otimes n+2} \to A^{\otimes n+3}$ given by

$$a_0 \otimes \cdots \otimes a_{n+1} \longmapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}.$$

We want to show that $(s_n)_{n=0}^{\infty}$ is a contracting homotopy; i.e. that

$$b'_{n+1}s_n + s_{n-1}b'_n = \mathrm{id}_{C_n^{\mathrm{bar}}(A)}.$$

For $i \in \{1, \ldots, n\}$ we have

$$s_{n-1}d_i^{n+1}(a_0, \ldots, a_{n+1}) = (1, a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) = d_{i+1}^{n+2}(1, a_0, \ldots, a_{n+2}) = d_{i+1}^{n+1}s_n(a_0, \ldots, a_{n+1}),$$

and a similar computation shows that $d_0^{n+2}s_n = \mathrm{id}_{C_n^{\mathrm{bar}}(A)}$. From this it follows[1] that

$$\begin{aligned} b'_{n+1}s_n + s_{n-1}b'_n &= \sum_{i=0}^{n+1}(-1)^i d_i^{n+2}s_n + \sum_{i=0}^{n}(-1)^i s_{n-1}d_i^{n+1} \\ &= d_0^{n+2}s_n + \sum_{i=1}^{n+1}(-1)^i s_{n-1}d_{i-1}^{n+1} + \sum_{i=0}^{n}(-1)^i s_{n-1}d_i^{n+1} \\ &= \mathrm{id}_{C_n^{\mathrm{bar}}(A)}. \end{aligned}$$

□

Consider the enveloping algebra $A \otimes A^{\mathrm{op}}$. As explained in the beginning of this section, $A \otimes A^{\mathrm{op}}$ can be considered as an A-bimodule with respect to the action

$$a(x \otimes y^{\mathrm{op}}) := x \otimes (y^{\mathrm{op}}a^{\mathrm{op}}) \quad \text{and} \quad (x \otimes y^{\mathrm{op}})b := (xb) \otimes y^{\mathrm{op}}.$$

We can therefore form the Hochschild complex $(C_*(A, A \otimes A^{\mathrm{op}}), b_*)$. Note, that each of the vector spaces $C_n(A, A \otimes A^{\mathrm{op}}) = (A \otimes A^{\mathrm{op}}) \otimes A^{\otimes n}$ is a left $A \otimes A^{\mathrm{op}}$-module, with respect to the action of multiplication on the first factor. By construction of the bimodule-structure on $A \otimes A^{\mathrm{op}}$, it follows that the Hochschild boundary maps commutes with the left action of $A \otimes A^{\mathrm{op}}$.
The following holds.

Proposition 2.1.7. *The family of maps* $\varphi_n : C_n^{\mathrm{bar}}(A) \to C_n(A, A \otimes A^{\mathrm{op}})$ *given by*

$$(a_0, a_1, \ldots, a_n, a_{n+1}) \longmapsto ((a_0 \otimes a_{n+1}^{\mathrm{op}}), a_1, \ldots, a_n),$$

is an isomorphism of complexes of left $A \otimes A^{\mathrm{op}}$*-modules. Here* $C_n^{\mathrm{bar}}(A)$ *is considered a left* $A \otimes A^{\mathrm{op}}$*-module with respect to the action from Remark 2.1.5.*

Proof. It is clear that φ_n is an isomorphism of left $A \otimes A^{\mathrm{op}}$-modules in each degree and a direct computation shows that φ_n commutes with the differentials. □

Corollary 2.1.8. *The complex* $(C_*(A, A \otimes A^{\mathrm{op}}), b_*)$ *is a free resolution of* A *as an* $A \otimes A^{\mathrm{op}}$*-module.*

Proof. This follows from Proposition 2.1.1 and Proposition 2.1.6 in conjunction. □

Corollary 2.1.9. [Lod] *For any* A*-bimodule* M*, the* n*'th Hochschild homology* $H_n(A, M)$ *is isomorphic to the* n*'th* Tor*-group* $\mathrm{Tor}_n^{A \otimes A^{\mathrm{op}}}(M, A)$*, where* M *is considered a right* $A \otimes A^{\mathrm{op}}$*-module with respect to the action* $m(a \otimes b^{\mathrm{op}}) := bma$*.*

[1] Recall that $b'_n = \sum_{i=1}^{n}(-1)^i d_i^{n+1}$

Proof. By Remark 2.1.5 and Corollary 2.1.8 the complex $(C_*(A, A^{\mathrm{e}}), b_*)$ is a free resolution of A as a left module over $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$.
The n'th homology of the induced complex $(M \otimes_{A^{\mathrm{e}}} C_*(A, A^{\mathrm{e}}), 1 \otimes b_*)$ is therefore $\mathrm{Tor}_n^{A^{\mathrm{e}}}(M, A)$. Consider the natural isomorphism

$$M \otimes_{A^{\mathrm{e}}} C_n(A, A^{\mathrm{e}}) = M \otimes_{A^{\mathrm{e}}} (A^{\mathrm{e}}) \otimes A^{\otimes n} \xrightarrow{\varphi_n} M \otimes A^{\otimes n} = C_n(A, M),$$

given by

$$\varphi_n : x \otimes (a \otimes b^{\mathrm{op}}) \otimes (a_1 \otimes \cdots \otimes a_n) \longmapsto (bxa) \otimes (a_1 \otimes \cdots \otimes a_n).$$

A direct computation shows that $b_n^M \circ \varphi_n = \varphi_{n-1} \circ (1 \otimes b_n^{A^{\mathrm{e}}})$ (the superscript denotes the ring of coefficients) and hence $(\varphi_n)_{n\in\mathbb{N}}$ is an isomorphism of complexes. Since the n'th homology of the target-complex is exactly $H_n(A, M)$, the claim follows. □

2.2 Group homology

In this section we briefly recall the notion of group von Neumann algebras and that of homology of a discrete group. Using this, we define the L^2-homology and the L^2-Betti numbers of a discrete group. Finally, we discuss a special case of the induction functor from Theorem 1.5.1 in this context.

Let G be is a countable discrete group with neutral element e. For each $g \in G$ we define a unitary operator $\lambda_g \in \mathscr{B}(l^2(G))$ by

$$\lambda_g(x)(\gamma) := x(g^{-1}\gamma) \quad \text{for } x \in l^2(G) \text{ and } \gamma \in G.$$

Then the map $\lambda : g \mapsto \lambda_g$ is a unitary representation of G, called the *left-regular representation.* Similarly we define a unitary $\rho_g \in \mathscr{B}(l^2(G))$ by setting

$$\rho_g(x)(\gamma) := x(\gamma g),$$

and the map $\rho : g \mapsto \rho_g$ is also a unitary representation, called the *right-regular representation* of G. Associated with these two representation are two von Neumann algebras, namely

$$\{\lambda_g | g \in G\}'' \quad \text{and} \quad \{\rho_g | g \in G\}''.$$

The von Neumann algebra $\{\lambda_g | g \in G\}''$ is called *the group von Neumann algebra of* G and will be denoted $\mathscr{N}(G)$. In the following, we will denote by $\delta_g \in l^2(G)$ the characteristic function of the singleton-set $\{g\}$ and we notice that $\{\delta_g | g \in G\}$ is an orthonormal basis of $l^2(G)$. We state the following basic facts without proofs.

1. The commutant, relative to $\mathscr{B}(l^2(G))$, of $\mathscr{N}(G)$ is $\{\rho(g) | g \in G\}''$.

2. $\mathscr{N}(G)$ is a finite von Neumann algebra.

3. The vector state $T \mapsto \langle T\delta_e | \delta_e \rangle_{l^2(G)}$ is a normal, faithful, tracial state on $\mathscr{N}(G)$ and is called the *von Neumann trace* on $\mathscr{N}(G)$.

4. If G is an i.c.c.-group[2], then $\mathscr{N}(G)$ is a factor of type $\mathbf{II}_1$.

[2] i.e. $|\{h^{-1}gh | h \in G\}| = \infty$ for all $g \neq e$.

For a more detailed description of group von Neumann algebras, and proofs of the the above statements, we refer to [KR2] section 6.7.

Proposition 2.2.1. *If G is a discrete group, then $\mathscr{N}(G \times G^{\text{op}}) \simeq \mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\text{op}}$.*

Proof. First note, that $l^2(G)\bar{\otimes}l^2(G^{\text{op}})$ is isomorphic to $l^2(G \times G^{\text{op}})$ via the unitary operator U defined as the extension of $\delta_g \otimes \delta_{h^{\text{op}}} \mapsto \delta_{(g,h^{\text{op}})}$. Hence

$$\text{Ad}_{U^*} : \mathscr{B}(l^2(G)\bar{\otimes}l^2(G^{\text{op}})) \longrightarrow \mathscr{B}(l^2(G \times G^{\text{op}})) \qquad (T \mapsto UTU^*)$$

takes $\lambda_g \otimes \lambda_{h^{\text{op}}}$ to $\lambda_{(g,h^{\text{op}})}$. Since Ad_{U^*} is normal, it is an isomorphism from $\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G^{\text{op}})$ to $\mathscr{N}(G \times G^{\text{op}})$. Since $\mathscr{N}(G^{\text{op}}) \simeq \mathscr{N}(G)^{\text{op}}$ via extension of $\lambda_{g^{\text{op}}} \mapsto \lambda_g^{\text{op}}$, the result follows. ☐

We now define the notion of group homology.

Definition 2.2.2. *Let G be a discrete group and let M be a right module over $\mathbb{C}G$. Define $C_0(G, M) := M$ and*

$$C_n(G, M) := M \otimes (\mathbb{C}G)^{\otimes n} \quad \text{for } n \in \mathbb{N}.$$

We define $\partial_n : C_n(G, M) \longrightarrow C_{n-1}(G, M)$ by

$$\begin{aligned} \partial_n : m \otimes g_1 \otimes \cdots \otimes g_n \longmapsto \; & (mg_1) \otimes g_2 \otimes \cdots \otimes g_n \\ & + \sum_{i=1}^{n-1} (-1)^i m \otimes g_1 \otimes \cdots \otimes g_{i-1} \otimes g_i g_{i+1} \otimes g_{i+2} \otimes \cdots \otimes g_n \\ & + (-1)^n m \otimes g_1 \otimes \cdots \otimes g_{n-1}. \end{aligned}$$

Then $(C_*(G, M))$ is a complex (Lemma 2.1.2) and we define the group homology *of G with coefficients in M, to be the homology of this complex.*

The complex $(C_*(G, M), \partial_*)$ is called the *Eilenberg- Mac Lane complex.*

Lemma 2.2.3. [Lod] *Let M be a $\mathbb{C}G$-bimodule. We then give M a new structure of a right $\mathbb{C}G$-module, by setting*

$$m \cdot g := g^{-1} mg.$$

When M is considered as a right $\mathbb{C}G$-module with respect this action, we denote it by $\tilde{M}$ to avoid confusion. Then the Hochschild homology $H_*(\mathbb{C}G, M)$ is isomorphic to the group homology $H_*(G, \tilde{M})$.

Proof. Consider the Hochschild complex $(C_*(\mathbb{C}G, M), b_*)$ and define $\varphi_n : C_n(\mathbb{C}G, M) \to C_n(G, \tilde{M})$ by

$$\varphi_n : m \otimes g_1 \otimes \cdots \otimes g_n \longmapsto (g_1 \cdots g_n m) \otimes g_1 \otimes \cdots \otimes g_n,$$

where we used the bimodule structure on M to define the product $g_1 \cdots g_n m$.
We now claim that this is an isomorphism of complexes. Each φ_n is clearly bijective, so we only have to prove that φ_n is compatible with the boundary-maps. Consider again the face-maps $d_0^n, \dots, d_n^n$ from Example 2.1.3, used to define the Hochschild boundary map, and let $\delta_0^n, \dots, \delta_n^n$ denote the obvious face-maps corresponding to ∂_n. For $i \in \{1, \dots, n-1\}$ we then get

$$\begin{aligned} \varphi_{n-1} \circ d_i^n(m \otimes g_1 \otimes \cdots \otimes g_n) &= \varphi_{n-1}(m \otimes g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n) \\ &= (g_1 \cdots g_n m) \otimes g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n) \\ &= \delta_i^n \varphi_n(m \otimes g_1 \otimes \cdots \otimes g_n). \end{aligned}$$

In the case $i = 0$, we get

$$\begin{aligned} \varphi_{n-1} \circ d_0^n(m \otimes g_1 \otimes \cdots \otimes g_n) &= \varphi_{n-1}(mg_1 \otimes g_2 \otimes \cdots \otimes g_n \\ &= (g_2 \cdots g_n mg_1) \otimes g_2 \otimes \cdots \otimes g_n \\ &= (g_1^{-1} g_1 \cdots g_n mg_1) \otimes g_2 \otimes \cdots \otimes g_n \\ &= \delta_0^n (g_1 \cdots g_n m) \otimes g_1 \otimes \cdots \otimes g_n \\ &= \delta_0^n \circ \varphi_n(m \otimes g_1 \otimes \cdots \otimes g_n) \end{aligned}$$

A similar computation shows that $\varphi_{n-1}d_n^n = \delta_n^n\varphi_n$. In particular, $(\varphi_n)_{n\in\mathbb{N}}$ is a morphism of complexes. □

In the language of homological algebra, the group homology has the following description.

Proposition 2.2.4. [Lod] *Let M be a right $\mathbb{C}G$-module. If $\mathbb{C}$ is considered as a left $\mathbb{C}G$-module with respect to the trivial action*[3]*, then*

$$H_*(G,M) \simeq \operatorname{Tor}_*^{\mathbb{C}G}(M,\mathbb{C}).$$

Proof. We view $\mathbb{C}G$ as a right module over itself and consider the complex $(C(G,\mathbb{C}G),\partial_*)$. This complex is acyclic, (which can bee seen by an argument similar to the one used to prove acyclicity of the bar-complex) and the map

$$\mathbb{C}G \ni \sum_{i=1}^{n}\lambda_i g_i \longmapsto \sum_{i=1}^{n}\lambda_i \in \mathbb{C},$$

is an augmentation of the complex. Each $C_n(G,\mathbb{C}G) = (\mathbb{C}G)^{\otimes(n+1)}$ is a free left $\mathbb{C}G$-module, when endowed with the action of multiplication on the first factor. Hence, $(C_*(G,\mathbb{C}G),\partial_*)$ is a free resolution of $\mathbb{C}$ in the category of left $\mathbb{C}G$-modules.
If we apply the functor $M\otimes_{\mathbb{C}G}-$ to this resolution, the homology of the induced complex is

$$\operatorname{Tor}_*^{\mathbb{C}G}(M,\mathbb{C}).$$

On the other hand, the induced complex is (in degree n) $M\otimes_{\mathbb{C}G}\otimes\mathbb{C}G^{\otimes(n+1)}$ and applying the natural isomorphism

$$M\otimes_{\mathbb{C}G}\otimes\mathbb{C}G^{\otimes(n+1)} \ni m\otimes g_0\otimes g_1\otimes\cdots\otimes g_n \longmapsto mg_0\otimes g_1\otimes\cdots\otimes g_n \in M\otimes\mathbb{C}G^{\otimes n},$$

yields an isomorphism of complexes from $(M\otimes_{\mathbb{C}G}C_*(G,\mathbb{C}G),1\otimes\partial_*)$ to $(C_*(G,M),\partial_*)$. Since the latter complex (by definition) computes the group homology, the claim follows. □

Recall that $\mathbb{C}G$ is a subalgebra of $\mathscr{N}(G)$ via the inclusion $\sum_i \alpha_i g_i \mapsto \sum_i \alpha_i\lambda_{g_i}$ and we may therefore use $M=\mathscr{N}(G)$ in the definition of group homology.
Note also, that in this case, each $H_p(G,\mathscr{N}(G))$ has structure of a left $\mathscr{N}(G)$-module in the following way: Each group in the Eilenberg- Mac Lane complex $\mathscr{N}(G)\otimes(\mathbb{C}G)^n$ is an $\mathscr{N}(G)$-module, with respect to the action of multiplication on the first factor. This action commutes with the boundary maps and hence pass down to an action on the homology groups $H_*(G,\mathscr{N}(G))$.
This allows us to make the following definition.

Definition 2.2.5. *For any $p\in\mathbb{N}_0$, the p'th L^2-*homology *of G is defined as*

$$H_p^{(2)}(G) := H_p(G,\mathscr{N}(G)),$$

*and we define the p'th L^2-*Betti number *of G as*

$$\beta_p^{(2)}(G) := \dim_{\mathscr{N}(G)}(H_p^{(2)}(G)),$$

where the dimension is taken with respect to the von Neumann trace on $\mathscr{N}(G)$.

2.3 Induction for group von Neumann algebras

Let H be a discrete group and let G be a subgroup of H. We denote by ι the inclusion $G\subseteq H$ and want to prove that ι extends to an injective $*$-algebra-homomorphism $\iota:\mathscr{N}(G)\to\mathscr{N}(H)$.
More precisely, we want to show the following.

[3] I.e. $gz=z$ for each $g\in G$ and $z\in\mathbb{C}$.

Proposition 2.3.1. [KR2] *The map* $\lambda_g \mapsto \lambda_{\iota(g)}$ $(g \in G)$ *extends to a* $*$*-algebra-isomorphism from* $\mathscr{N}(G)$ *to* $\mathscr{N}_0 := W^*(\{\lambda_{\iota(g)} | g \in G\}) \subseteq \mathscr{N}(H)$.

Proof. Put $\mathscr{H} := \overline{\{\delta_{\iota(g)} | g \in G\}}^{l^2(H)} \subseteq l^2(H)$ and let π denote the orthogonal projection in $\mathscr{B}(l^2(H))$ onto $\mathscr{H}$. Since the range of π is invariant under the action of each $\lambda_{\iota(g)}$ it is also invariant under the action of $\mathscr{N}_0$ and hence $\pi \in \mathscr{N}_0'$, where the commutatant is taken relative to $\mathscr{B}(l^2(H))$. Let C_π denote the central carrier of π, [4] and recall that ([KR1] Prop. 5.5.2)

$$\mathrm{rg}\,(C_\pi) = \overline{\mathrm{span}\{\mathscr{N}_0'\pi(l^2(H))\}}.$$

Since $\mathscr{N}_0' \supseteq \mathscr{N}(H)' = W^*(\rho(H))$ (see e.g [KR2] Thm. 6.7.2) and $\pi(\delta_{\iota(e)}) = \delta_{\iota(e)}$ we get

$$\mathrm{rg}\,(C_\pi) = \overline{\mathrm{span}\{\mathscr{N}_0'\pi(l^2(H))\}} \supseteq \overline{\{\rho_h(\delta_{\iota(e)}) | h \in H\}} = l^2(H),$$

and therefore $C_\pi = 1$. The map

$$\mathscr{N}_0 = \mathscr{N}_0 C_\pi \ni AC_\pi \xrightarrow{\quad\psi\quad} A\pi \in \mathscr{N}_0\pi \subseteq \mathscr{B}(\mathscr{H}),$$

is a $*$-algebra-isomorphism ([KR1] Prop. 5.5.6) and we now show that $\mathscr{N}(G)$ is isomorphic to $\mathscr{N}_0\pi$, where the latter is considered a subalgebra of $\mathscr{B}(\mathscr{H})$.
The map $\delta_g \mapsto \delta_{\iota(g)}$ extends to a unitary $U : l^2(G) \to \mathscr{H}$ and for $T := \sum_{i=1}^n z_i \lambda_{\iota(g_i)} \in \mathscr{N}_0$ we get

$$U^*(T\pi)U = \sum_{i=1}^n z_i \lambda_{g_i} \in \mathscr{N}(G).$$

By normality of $\mathrm{Ad}_U : S \mapsto U^*SU$, we see that U intertwines the action of $\mathscr{N}_0\pi$ on $\mathscr{H}$ with the action of $\mathscr{N}(G)$ on $l^2(G)$. Then $\psi^{-1} \circ \mathrm{Ad}_{U^*} : \mathscr{N}(G) \to \mathscr{N}_0$ is a $*$-algebra-isomorphism and for any $g \in G$ we have

$$\psi^{-1} \circ \mathrm{Ad}_{U^*}(\lambda_g) = \psi^{-1}(U\lambda_g U^*) = \psi^{-1}(\lambda_{\iota(g)}\pi) = \lambda_{\iota(g)}.$$

Hence $\psi^{-1} \circ \mathrm{Ad}_{U^*}$ has the desired properties. □

Remark 2.3.2. *With the notation from the above proof, we have that*

$$\varphi := \psi^{-1} \circ \mathrm{Ad}_{U^*} : \mathscr{N}(G) \longrightarrow \mathscr{N}(H),$$

is an injective $*$*-algebra-homomorphism. Moreover,* φ *preserves the von Neumann trace, since for* $T \in \mathscr{N}_0 = \mathrm{rg}\,(\varphi)$ *we have*

$$\langle T\delta_{\iota(e)} | \delta_{\iota(e)} \rangle_{l^2(H)} = \langle T\pi U\delta_e | U\delta_e \rangle_{l^2(H)} = \langle U^*(T\pi)U\delta_e | \delta_e \rangle_{l^2(G)} = \langle \varphi^{-1}(T)\delta_e | \delta_e \rangle_{l^2(G)}$$

Note in particular, that Theorem 1.5.1 applies and gives rise to an exact, dimension-preserving, functor

$$\mathrm{Mod}(\mathscr{N}(G)) \ni X \longmapsto \mathscr{N}(H) \otimes_{\mathscr{N}(G)} X \in \mathrm{Mod}(\mathscr{N}(H)).$$

[4] I.e. the smallest central projection in $\mathscr{N}_0'$ that contains π.

Chapter 3

L^2-Homology for finite von Neumann Algebras

3.1 Definitions and basic results

The aim of this section, is to introduce the notion of L^2-homology and L^2-Betti numbers for von Neumann algebras, as defined by Connes and Shlyakhtenko in [CS03]. As it turns out, the chosen way to do this works equally well in a slightly more general set-up and we therefore start with the following definition.

Definition 3.1.1. *Let $\mathscr{A}$ be a unital (complex) $*$-algebra and assume moreover that there exists a functional $\tau : \mathscr{A} \to \mathbb{C}$ with the following properties.*

- *τ is a faithful state, in the sense that $\tau(1) = 1$ and for all $a \in \mathscr{A}$ we have $\tau(a^*a) \geq 0$ and $\tau(a^*a) = 0$ only if $a = 0$.*
- *τ is a trace, in the sense that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathscr{A}$.*
- *For all $a \in \mathscr{A}$ there exists $C_a > 0$ such that*

$$\tau(x^*a^*ax) \leq C_a \tau(x^*x) \quad \text{for all } x \in \mathscr{A}.$$

We will refer to such an algebra as a (unital) tracial $*$-algebra.

The reason for the above definition is, that it is exactly what is needed to get a representation of $\mathscr{A}$ as bounded operators on the Hilbert space provided by the GNS-construction with respect to τ. This is explained in more details in the following.
If we consider a tracial $*$-algebra $(\mathscr{A}, \tau)$, the relation

$$\langle a | b \rangle := \tau(b^*a),$$

defines a faithful, positive definite, sesquilinear form on $\mathscr{A}$, where faithful and positive definite here means that $\langle a|a\rangle \geq 0$ and $\langle a|a\rangle = 0$ only if $a = 0$. The pair $(\mathscr{A}, \langle \cdot | \cdot \rangle)$ is therefore a pre-Hilbert space and we denote by $L^2(\mathscr{A}, \tau)$ the Hilbert space completion of $(\mathscr{A}, \langle \cdot | \cdot \rangle)$. Denote by η the inclusion of $\mathscr{A}$ into $L^2(\mathscr{A}, \tau)$ and put $\xi_0 := \eta(1)$. Note, that the map $\eta(a) \mapsto \eta(a^*)$ extends to an anti-unitary $J : L^2(\mathscr{A}, \tau) \to L^2(\mathscr{A}, \tau)$ in exactly the same way as in Section 1.2.
For each $a \in \mathscr{A}$, a linear map $\pi(a) : \eta(\mathscr{A}) \to \eta(\mathscr{A})$ is defined by $\pi(a)\eta(x) := \eta(ax)$ and the last

requirement on τ (from Definition 3.1.1) implies that $\pi(a)$ is bounded with respect to the norm $\|\cdot\|_2$ induced by $\langle\cdot|\cdot\rangle$. This is because

$$\|\pi(a)\eta(x)\|_2^2 = \|\eta(ax)\|_2^2 = \tau(x^*a^*ax) \leq C_a\tau(x^*x) = C_a\|\eta(x)\|_2^2.$$

Thus, $\pi(a)$ extends to a bounded operator (also denoted $\pi(a)$) on $L^2(\mathscr{A},\tau)$.
One easily checks, that $\pi : \mathscr{A} \to \mathscr{B}(L^2(\mathscr{A},\tau))$ (mapping a to $\pi(a)$) is an injective unital $*$-algebra-homomorphism. Denote by $\mathscr{M}$ the von Neumann algebra in $\mathscr{B}(L^2(\mathscr{A},\tau))$ generated by $\pi(\mathscr{A})$.
From now on we identify $\mathscr{A}$ with its isomorphic image under π and to simplify notation we put $L^2(\mathscr{A}) := L^2(\mathscr{A},\tau)$.

Lemma 3.1.2. *The tracial state $\tau : \mathscr{A} \to \mathbb{C}$ extends to a faithful, normal, tracial state on $\mathscr{M}$.*

Proof. We first note, that $\tau(a) = \langle a\xi_0|\xi_0\rangle$ and since $\omega_{\xi_0} : x \mapsto \langle x\xi_0|\xi_0\rangle$ is a vector-state (in particular normal) on $\mathscr{M}$, we only need to see that ω_{ξ_0} is faithful and tracial.
Let $x, y \in \mathscr{M}$ be given and choose nets $(x_i), (y_j)$ in $\mathscr{A}$, converging weakly to x and y respectively. We then get

$$\begin{aligned}
\omega_{\xi_0}(xy) &:= \langle xy\xi_0|\xi_0\rangle \\
&= \lim_i \langle x_i y\xi_0|\xi_0\rangle \\
&= \lim_i \lim_j \langle y_j\xi_0|x_i^*\xi_0\rangle \\
&= \lim_i \lim_j \tau(x_iy_j) \\
&= \lim_i \lim_j \tau(y_jx_i) \\
&= \lim_i \lim_j \langle y_jx_i\xi_0|\xi_0\rangle \\
&= \lim_i \langle x_i\xi_0|y^*\xi_0\rangle \\
&= \omega_{\xi_0}(yx),
\end{aligned}$$

and hence that ω_{ξ_0} is tracial.
To see that ω_{ξ_0} is faithful on $\mathscr{M}$, consider any $x \in \mathscr{M}$ and assume that

$$0 = \omega_{\xi_0}(x^*x) = \langle x\xi_0|x\xi_0\rangle = \|x\xi_0\|_2^2,$$

and hence that $x\xi_0 = 0$. Since ξ_0 is cyclic for $\mathscr{A}$, it is in particular cyclic for $\mathscr{M}$ and hence separating for $\mathscr{M}'$. By copying the proof of Proposition 1.2.2 we see that $JxJ \in \mathscr{M}'$ and by the choice of x we get

$$(JxJ)\xi_0 = Jx\xi_0 = 0.$$

Therefore $JxJ = 0$ and hence $x = J(JxJ)J = 0$.

□

In the sequel we will often also use the symbol τ, to denote the extension $\omega_{\xi_0}|_{\mathscr{M}}$ of τ to $\mathscr{M}$.
Note, that because of Lemma 3.1.2, the von Neumann algebra tensor-product $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ is also endowed with a faithful, normal, tracial state $\tau \otimes \tau^{\rm op}$, with the property that

$$\tau \otimes \tau^{\rm op}(x \otimes y^{\rm op}) := \tau(x)\tau(y) \text{ for all } x \in \mathscr{M}, y \in \mathscr{M}^{\rm op}.$$

(This was also proven in Proposition 1.2.9) Note also, that $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ has structure of an $\mathscr{A}$-bimodule, with respect to the action

$$aT := T(1 \otimes a^{\rm op}) \quad \text{and} \quad Ta := T(a \otimes 1),$$

where we think of $\mathscr{A} \otimes \mathscr{A}^{\rm op}$ as a subalgebra of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$.

Definition 3.1.3. *With the above notation, the* p'th L^2-homology *of* $\mathscr{A}$ *is defined as the Hochschild homology*

$$H_p^{(2)}(\mathscr{A},\tau) := H_p(\mathscr{A},\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}). \qquad (p\in\mathbb{N}_0)$$

Each $C_p(\mathscr{A},\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})$ *in the Hochschild complex is a left* $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$*-module with respect to the action of multiplication on the first factor and since the Hochschild boundary maps commutes with this action, each* $H_p^{(2)}(\mathscr{A},\tau)$ *inherits the structure of a left* $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$*-module.*
This allows us to define the p'th L^2-Betti number *of* $\mathscr{A}$ *as*

$$\beta_p^{(2)}(\mathscr{A},\tau) := \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(H_p^{(2)}(\mathscr{A},\tau)),$$

where the dimension is the extended dimension function from Chapter 1, arising from the trace $\tau\otimes\tau^{\text{op}}$ *on* $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$*.*

The results of Chapter 2 provides us with the following alternative description of the L^2-homology:

Lemma 3.1.4. *The p'th* L^2*-homology* $H_p^{(2)}(\mathscr{A},\tau)$ *is equal to* $\text{Tor}_p^{\mathscr{A}\otimes\mathscr{A}^{\text{op}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}},\mathscr{A})$*, where* $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ *is considered a right* $\mathscr{A}\otimes\mathscr{A}^{\text{op}}$*-module via the inclusion* $\mathscr{A}\otimes\mathscr{A}^{\text{op}}\subseteq\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$*.*

Proof. This follows directly from Corollary 2.1.9. □

3.2 Computational results

The results of Chapter 2 provides us with two (isomorphic) projective resolutions, with which we can compute the L^2-homology of a tracial $*$-algebra $\mathscr{A}$; namely the bar-resolution $(C_*^{\text{bar}}(\mathscr{A}),b'_*)$ and the Hochschild complex $(C_*(\mathscr{A},\mathscr{A}\otimes\mathscr{A}^{\text{op}},b_*)$. We shall primarily use the latter resolution and will refer also to this, as the bar-resolution of $\mathscr{A}$.
Although these resolutions tells us how to compute the L^2-homology, this is not a very explicit description. In this section we will give more explicit discriptions of the L^2-homology in some special cases and develop formulas to compute the corresponding L^2-Betti numbers.
We first relate the L^2-homology of groups, to the L^2-homology of their corresponding group-algebras.

Let G be a countable discrete group and consider the diagonal embedding $\iota: G\to G\times G^{\text{op}}$ given by $g\mapsto(g,(g^{-1})^{\text{op}})$. Recall from Proposition 2.3.1, that ι extends to a $*$-algebra-homomorphism from $\mathscr{N}(G)$ into $\mathscr{N}(G\times G^{\text{op}})$ and that $\mathscr{N}(G\times G^{\text{op}})$ can be considered a right $\mathscr{N}(G)$-module via the homomorphism ι. If we consider $\mathbb{C}G$ as a (dense) subalgebra of $\mathscr{N}(G)$, and τ denote the von Neumann trace $T\mapsto\langle T\delta_e|\delta_e\rangle$, then the pair $(\mathbb{C}G,\tau)$ fulfills the requirements in Definition 3.1.1 and the following holds.

Proposition 3.2.1. [CS03] *The* L^2*-homology of* G *is related to the* L^2*-homology of* $(\mathbb{C}G,\tau)$ *in the following way:*

$$H_k^{(2)}(\mathbb{C}G,\tau) = \mathscr{N}(G\times G^{\text{op}})\otimes_{\mathscr{N}(G)}H_k^{(2)}(G) = \iota_*(H_k^{(2)}(G))$$
$$\beta_k^{(2)}(\mathbb{C}G,\tau) = \beta_k^{(2)}(G)$$

Proof First note, that $\mathscr{N}(G\times G^{\text{op}}) = \mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\text{op}}$ by Proposition 2.2.1.
Also note, that the restriction of ι to $\mathbb{C}G$ gives rise to a right $\mathbb{C}G$-module structure on $\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\text{op}}$, which coincides with the right $\mathbb{C}G$-module structure introduced in Lemma 2.2.3. Applying the result of Lemma 2.2.3 yields

$$H_k^{(2)}(\mathbb{C}G,\tau) := H_k(\mathbb{C}G,\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\text{op}}) = H_k(G,\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\text{op}}). \qquad (*)$$

Next we consider the group L^2-homology $H_k^{(2)}(G)$. To compute this, we choose a resolution $(F_i, f_i)_{i=0}^{\infty}$ of the trivial $\mathbb{C}G$-module $\mathbb{C}$ by free $\mathbb{C}G$-modules and the L^2-homology of G is then the homology of the complex

$$\dots \xrightarrow{1\otimes f_{n+1}} \mathscr{N}(G)\otimes_{\mathbb{C}G} F_n \xrightarrow{1\otimes f_n} \dots\dots \xrightarrow{1\otimes f_2} \mathscr{N}(G)\otimes_{\mathbb{C}G} F_1 \xrightarrow{1\otimes f_1} \mathscr{N}(G)\otimes_{\mathbb{C}G} F_0 \longrightarrow 0.$$

We now apply the induction-functor $\iota_* := (\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}})\otimes_{\mathscr{N}(G)} -$ to this complex. Since ι_* is an exact (additive) functor, the homology of the induced complex is just

$$\iota_*(H_k^{(2)}(G)) = (\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}})\otimes_{\mathscr{N}(G)} H_k^{(2)}(G).$$

(see e.g. [CE] Chp. IV, Thm. 7.2) On the other hand; the module in degree k in the induced complex is

$$(\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}})\otimes_{\mathscr{N}(G)}\mathscr{N}(G)\otimes_{\mathbb{C}G} F_k,$$

with boundary-map $(1\otimes 1)\otimes 1\otimes f_k$ and if we apply the canonical isomorphism

$$\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}}\otimes_{\mathscr{N}(G)}\mathscr{N}(G)\otimes_{\mathbb{C}G} F_k \xrightarrow{\sim} \mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}}\otimes_{\mathbb{C}G} F_k$$

in each degree, the resulting complex is (in degree k)

$$(\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}})\otimes_{\mathbb{C}G} F_k,$$

with boundary map $(1\otimes 1)\otimes f_k$.
By definition, this complex computes $H_k(G, \mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}})$ and combining this with the equation $(*)$ now gives

$$H_k^{(2)}(\mathbb{C}G, \tau) = \iota_*(H_k^{(2)}(G)).$$

The equality of the corresponding L^2-Betti numbers now follows from Theorem 1.5.1, since

$$\begin{aligned}\beta_k^{(2)}(G) &:= \dim_{\mathscr{N}(G)} H_k^{(2)}(G)\\ &= \dim_{\mathscr{N}(G\times G^{\mathrm{op}})} \iota_*(H_k^{(2)}(G))\\ &= \dim_{\mathscr{N}(G)\bar{\otimes}\mathscr{N}(G)^{\mathrm{op}}} H_k^{(2)}(\mathbb{C}G, \tau)\\ &=: \beta_k^{(2)}(\mathbb{C}G, \tau).\end{aligned}$$

□

Consider any von Neumann algebra $\mathscr{M}$, acting on a Hilbert space $\mathscr{H}$, and a non-zero projection $p\in\mathscr{M}$. Then $p\mathscr{M}p$ is a von Neumann algebra in $\mathscr{B}(p\mathscr{H})$ (see e.g. [KR1] Prop. 5.5.6) and if $\mathscr{M}$ has a normal, faithful, tracial state τ, then $\frac{1}{\tau(p)}\tau|_{p\mathscr{M}p}$ is a normal, faithful, tracial state on $p\mathscr{M}p$. In the following section we prove a formula relating the L^2-homology of $\mathscr{M}$ with that of the compressed algebra $p\mathscr{M}p$, in the case where $\mathscr{M}$ is a (finite) factor.

3.2.1 Compression formula for finite factors

Throughout this section, $\mathscr{M}$ denotes a finite **factor** with (unique, normal, faithful) tracial state τ, and p denotes a non-trivial projection in $\mathscr{M}$ of trace $\alpha\in]0,1]$. Before stating and proving the compression formula (Theorem 3.2.8) we need some results, concerning the relationship between the projective modules over $\mathscr{M}$ and the projective modules over $p\mathscr{M}p$. To this end, we first note, that if V is module over $\mathscr{M}$, then the set pV is module over $p\mathscr{M}p$ with respect to the restricted action.

Lemma 3.2.2. *The module $p\mathscr{M}$ is finitely generated and projective as a module over $p\mathscr{M}p$. Therefore, for any projective $\mathscr{M}$-module V the $p\mathscr{M}p$-module pV is also projective.*

Proof. We have $p\mathscr{M} = p\mathscr{M}1$. Since $\mathscr{M}$ is a factor, any two projections can be compared and hence we can split 1 as an orthogonal sum $r + \sum_{i=1}^{n} p_i$ where $p_i \sim p$ for all i and $r \sim r' \leq p$. (Note, that since $\tau(1) = 1$ and $\tau(p_i) = \tau(p) = \alpha$ the sum has to be finite)
Thus, the $p\mathscr{M}p$-module $p\mathscr{M}$ decomposes as

$$p\mathscr{M} = p\mathscr{M}r \oplus \Big(\bigoplus_{i=1}^{n} p\mathscr{M}p_i\Big).$$

By construction of the p_i's, we can find partial isometries $v_i \in \mathscr{M}$ with $v_i^* v_i = p_i$ and $v_i v_i^* = p$. Because of this, the map

$$p\mathscr{M}p_i \ni pxp_i \overset{f_i}{\longmapsto} pxp_i v_i^* = pxv_i^* p \in p\mathscr{M}p$$

is bijective and clearly compatible with the left actions of $p\mathscr{M}p$. That is, f_i is an isomorphism of $p\mathscr{M}p$-modules. Hence

$$p\mathscr{M} \simeq p\mathscr{M}r \oplus (\bigoplus_{i=1}^{n} p\mathscr{M}p), \qquad \text{(as } p\mathscr{M}p\text{-modules)}$$

Since $\oplus_{i=1}^{n} p\mathscr{M}p$ is free over $p\mathscr{M}p$, it suffices to see that $p\mathscr{M}r$ is projective.
By replacing r with a suitable equivalent projection (the r' above), we may assume that $r \leq p$ and hence we get

$$p\mathscr{M}r = p\mathscr{M}(pr) = (p\mathscr{M}p)r,$$

Thus $p\mathscr{M}r$ is projective and the first part of the lemma is proven.

To prove the last statement, we let V be any projective $\mathscr{M}$-module. Then $\mathscr{M}^{(X)} \simeq V \oplus W$ for a suitable set X and $\mathscr{M}$-module W.
Since $p(\mathscr{M}^{(X)}) = (p\mathscr{M})^{(X)}$ and $p\mathscr{M}^{(X)} \simeq pV \oplus pW$ (as $p\mathscr{M}p$-modules), the $p\mathscr{M}p$-module pV is a direct summand in $(p\mathscr{M})^{(X)}$. Because $p\mathscr{M}$ is projective over $p\mathscr{M}p$, $(p\mathscr{M})^{(X)}$ is also projective over $p\mathscr{M}p$.
Thus, pV is a direct summand in a projective $p\mathscr{M}p$-module and hence projective.

□

The following lemma will be needed in the proof of Proposition 3.2.4 below, but is also of interest in it self, since it shows that the finitely generated projective modules over a finite factor, can be described in a particular simple form.

Lemma 3.2.3. *Let V be a finitely generated projective module over $\mathscr{M}$. Then V is isomorphic to $\oplus_{i=1}^{k}\mathscr{M}q_i$, for some $k \in \mathbb{N}$ and some projections $q_1, \ldots, q_k \in \mathscr{M}$. Moreover, for any non-zero projection $p \in \mathscr{M}$ we can choose $k \in \mathbb{N}$ and $q_1, \ldots, q_k$ such that $\tau(q_i) \leq \tau(p)$ for all $i \in \{1, \ldots, k\}$.*

In the proof, τ denotes also the induced trace-state on $M_k(\mathscr{M})$, given by

$$\tau(\{a_{ij}\}_{i,j=1}^{k}) := \frac{1}{k}\sum_{i=1}^{k} \tau(a_{ii}).$$

Proof. Since V is finitely generated and projective we can find a projection $p \in M_k(\mathscr{M})$ such that $V \simeq \mathscr{M}^k p$. Because $M_k(\mathscr{M})$ is also a factor, we can split p as a sum of projections $p_1 + \cdots + p_k$, such that $\tau(p_i) \leq \frac{1}{k}$ for all $i \in \{1, \ldots, k\}$.
Let e_i denote the $k \times k$-matrix over $\mathscr{M}$ given by

$$(e_i)_{st} = \begin{cases} 1, & s = t = i; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\tau(p_i) \leq \tau(e_i) = \frac{1}{k}$ we have $p_i \sim f_i \leq e_i$ for a suitable subprojection f_i of e_i. By construction of e_i, the subprojection f_i must have all but the (i,i)'th entrance equal zero and some projection $q_i \in \mathscr{M}$ in the (i,i)'th position. Then

$$V \simeq \mathscr{M}^k p \simeq \bigoplus_{i=1}^{k} \mathscr{M}^k p_i \simeq \bigoplus_{i=1}^{k} \mathscr{M}^k f_i \simeq \bigoplus_{i=1}^{k} \mathscr{M} q_i.$$

To prove the last statement, let $p \in \mathscr{M}$ be given. If $\mathscr{M}$ is a type **I** factor, we can split each q_i into a sum of minimal projections which of course all have minimal trace. If $\mathscr{M}$ is type $\mathbf{II}_1$ we can halve each q_i in the sense that there exist q_i', q_i'' such that $q_i = q_i' + q_i''$ and $q_i' \sim q_i''$. In particular

$$\tau(q_i') = \tau(q_i'') = \frac{1}{2}\tau(q_i),$$

so by successively halving the q_i's we can achieve that their traces are all smaller than $\tau(p)$.

□

Recall that $p \in \mathscr{M}$ is a projection of trace α. The following holds.

Proposition 3.2.4. *For any $\mathscr{M}$-module V we have*

$$\dim_{p\mathscr{M}p}(pV) = \frac{1}{\alpha}\dim_{\mathscr{M}}(V).$$

Proof. Assume first that V is finitely generated and projective over $\mathscr{M}$. By Lemma 3.2.3, the module V is then (up to isomorphism) of the form

$$\bigoplus_{i=1}^{k} \mathscr{M} q_i,$$

for some projections $q_i \in \mathscr{M}$ and (also by Lemma 3.2.3) we may assume that $\tau(q_i) \leq \tau(p)$ for each $i \in \{1, \dots k\}$. Since τ is faithful, we have $q_i \sim p_i \leq p$ for some suitable subprojections p_i of p and therefore

$$pV \simeq \bigoplus_{i=1}^{k} p\mathscr{M} q_i \simeq \bigoplus_{i=1}^{k} p\mathscr{M} p_i = \bigoplus_{i=1}^{k} (p\mathscr{M}p) p_i.$$

Hence

$$\dim_{p\mathscr{M}p}(pV) = \sum_{i=1}^{k} \frac{1}{\tau(p)}\tau(p_i) = \frac{1}{\tau(p)}\sum_{i=1}^{k}\tau(p_i) = \frac{1}{\tau(p)}\sum_{i=1}^{k}\tau(q_i) = \frac{1}{\alpha}\dim_{\mathscr{M}}(V).$$

We now consider the general case, where V is any $\mathscr{M}$-module, and choose a finitely generated projective submodule $W \subseteq V$. By what is already proven, pW is finitely generated an projective over $p\mathscr{M}p$ and $\alpha \dim_{p\mathscr{M}p}(pW) = \dim_{\mathscr{M}}(W)$. Hence

$$\dim_{p\mathscr{M}p}(pV) \geq \frac{1}{\alpha}\dim_{\mathscr{M}}(V).$$

For the opposite inequality, it suffices to see that if $T \subseteq pV$ is finitely generated and projective over $p\mathscr{M}p$, then T has the form pW for some finitely generated projective $\mathscr{M}$-module W.
By Lemma 3.2.3, there exist projections $s_1, \dots, s_l \in p\mathscr{M}p \subseteq \mathscr{M}$ such that

$$T \simeq \bigoplus_{j=1}^{l} (p\mathscr{M}p) s_j = p(\bigoplus_{j=1}^{l} \mathscr{M}(ps_j)).$$

Since $s_j \in p\mathscr{M}p$ it commutes with p and hence ps_j is a projection in $\mathscr{M}$. Therefore $\oplus_{j=1}^{l}\mathscr{M}(ps_j)$ is finitely generated and projective over $\mathscr{M}$ and hence T has the desired form.

□

The next lemma is a technical detail, which will be needed in the proof of Theorem 3.2.8.

Lemma 3.2.5. *Put $N := \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$, $\mathscr{N} := \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ and $q := p \otimes p^{\mathrm{op}}$. Then q is a projection in $N \subseteq \mathscr{N}$ and the multiplication map*

$$\begin{aligned}\varphi : q\mathscr{N}q \otimes_{qNq} qN &\longrightarrow q\mathscr{N}\\ qxq \otimes qy &\longmapsto qxqy,\end{aligned}$$

is an isomorphism of left $q\mathscr{N}q$-modules. Here $q\mathscr{N}q \otimes_{qNq} qN$ is considered as a left $q\mathscr{N}q$-module, with respect to the action of multiplication on the first factor.

For the proof we need a few observations.

Observation 3.2.6. *Let R and S be unital $*$-rings, and let X be an (S,R)-bimodule. If $p, q \in R$ are self-adjoint idempotents, and there exists $v \in R$ with $v^*v = p$ and $vv^* = q$, then Xp is isomorphic (as a left S-module) to Xq.*

Proof. It is straightforward to show that $Xp \ni xp \longmapsto xpv^* = xv^*q \in Xq$ is an isomorphism. □

We now assume to be under the hypothesis of Lemma 3.2.5. Then the following holds

Observation 3.2.7. *For any projection $e \in N$ with $e \leq q$, the multiplication map*

$$\varphi : q\mathscr{N}q \otimes_{qNq} qNe \longrightarrow q\mathscr{N}e,$$

is an isomorphism of left $q\mathscr{N}q$-modules.

Proof. Given $qxe \in q\mathscr{N}e$ we have $qxe = qxqe = \varphi(qxq \otimes q1e)$, and thus φ is surjective. To prove injectivity, assume that

$$0 = \varphi(\sum_i qx_iq \otimes qy_ie) = \sum_i qx_iqy_ie.$$

Then

$$\begin{aligned}\sum_i qx_iq \otimes qy_ie &= \sum_i qx_iq \otimes qy_iqeq\\ &= \sum_i qx_iqy_ie \otimes e\\ &= (\sum_i qx_iqqy_ie) \otimes e\\ &= 0.\end{aligned}$$

□

Proof of Lemma 3.2.5 First note that the following diagram commutes

$$\begin{array}{ccc} q\mathscr{N}q \otimes_{qNq} qN & \longrightarrow & q\mathscr{N}q \otimes_{qNq} qNq \bigoplus q\mathscr{N}q \otimes_{qNq} qN(1-q)\\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle(\varphi,\varphi)}\\ q\mathscr{N} & \longrightarrow & q\mathscr{N}q \bigoplus q\mathscr{N}(1-q)\end{array}$$

On the right-hand side of this diagram, the map φ is clearly an isomorphism on the first coordinate, so the proof is finished once we prove that this is also the case on the second coordinate. Write

$$1-q=(1-p)\otimes p^{\mathrm{op}}+p\otimes(1-p)^{\mathrm{op}}+(1-p)\otimes(1-p)^{\mathrm{op}}.$$

Since $\mathscr{M}$ is a $\mathbf{II}_1$-factor, we can split $1-p$ as an orthogonal sum of projections, all of which are sub-equivalent to p. Thus $1-q$ can be written as $\sum_n r_n\otimes s_n^{\mathrm{op}}$ where

$$\forall n\exists v_n\in\mathscr{M}, r_n'\leq p: v_n^*v_n=r_n \text{ and } v_nv_n^*=r_n'$$
$$\forall n\exists u_n^{\mathrm{op}}\in\mathscr{M}^{\mathrm{op}}, s_n'^{\mathrm{op}}\leq p^{\mathrm{op}}: (u_n^{\mathrm{op}})^*u_n^{\mathrm{op}}=s_n^{\mathrm{op}} \text{ and } u_n^{\mathrm{op}}(u_n^{\mathrm{op}})^*=(s_n^{\mathrm{op}})'$$

A direct computation shows that $f_n:=r_n\otimes s_n^{\mathrm{op}}\sim r_n'\otimes s_n^{\mathrm{op}\prime}=:g_n$ **inside** N, via the partial isometry $z_n:=v_n\otimes u_n^{\mathrm{op}}$. We thus get

$$\begin{aligned}
q\mathscr{N}q\otimes_{qNq}qN(1-q) &\simeq \bigoplus_n q\mathscr{N}q\otimes_{qNq}qNf_n\\
&\simeq \bigoplus_n q\mathscr{N}q\otimes_{qNq}qNg_n && \text{(by Obs. 3.2.6)}\\
&\simeq \bigoplus_n q\mathscr{N}g_n && \text{(by Obs. 3.2.7)}\\
&\simeq \bigoplus_n q\mathscr{N}f_n && \text{(by Obs. 3.2.6)}\\
&\simeq q\mathscr{N}(\sum_n f_n)\\
&= q\mathscr{N}(1-q)
\end{aligned}$$

By tracing a vector through the composition of these isomorphisms, it is easy to see that the total composition equals φ. □

We are now able to prove the main theorem of this section. Recall that $\mathscr{M}$ denotes a finite factor with tracial state τ and $p\in\mathscr{M}$ is a projection of trace α. Then the following holds.

Theorem 3.2.8 (Compression Formula). [CS03] *Let τ_p denote the restriction of τ to $p\mathscr{M}p$. Then the L^2-homology of $\mathscr{M}$ and that of the compressed factor $p\mathscr{M}p$ are related in the following way.*

$$H_n^{(2)}(p\mathscr{M}p,\frac{1}{\alpha}\tau_p)=(p\otimes p^{\mathrm{op}})H_n^{(2)}(\mathscr{M},\tau) \qquad \textit{and} \qquad \beta_n^{(2)}(p\mathscr{M}p,\frac{1}{\alpha}\tau_p)=\frac{1}{\alpha^2}\beta_n^{(2)}(\mathscr{M},\tau),$$

for all $n\in\mathbb{N}_0$.

Proof. Consider the bar-resolution $(C(\mathscr{M},\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}),b_n)_{n=0}^\infty$ of $\mathscr{M}$ and the projection $q:=p\otimes p^{\mathrm{op}}\in\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$. Define $N:=\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$, $\mathscr{N}:=\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ and put $F(V):=qV\in\mathrm{Mod}(qNq)$, for $V\in\mathrm{Mod}(N)$. For any $\mathrm{Mod}(N)$-morphism $f:V\to V'$ we have $f(qx)=qf(x)$ for all $x\in V$ and hence such a morphism restricts to a $\mathrm{Mod}(qNq)$-morphism

$$F(f):=f|_{qV}:F(V)\to F(V').$$

In this way, F is turned into a covariant functor from $\mathrm{Mod}(N)$ to $\mathrm{Mod}(qNq)$, with the following properties:

- F is exact. To see this, we consider an exact sequence $V'\xrightarrow{f}V\xrightarrow{g}V''$ of N-modules. The induced sequence then is
$$qV'\xrightarrow{F(f)}qV\xrightarrow{F(g)}qV''$$

Obviously the restrictions $F(f)$ and $F(g)$ of f and g has $F(g)F(f) = 0$. Conversely, if $x \in \ker(F(g)) = qV \cap \ker(g)$, then, by exactness of the original sequence, $x = fx'$ for some $x' \in V'$ and since $x \in qV$ we have

$$x = qx = qfx' = f(qx') \in \mathrm{rg}\,(f|_{qV'}) = \mathrm{rg}\,(F(f)).$$

- $F(\mathscr{M}) = q\mathscr{M} = p\mathscr{M}p$.

- F preserves projectivity. To see this, we first note that $p\mathscr{M}$ is finitely generated and projective over $p\mathscr{M}p$, by Lemma 3.2.2. Similarly $p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}$ is finitely generated and projective over $p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}p^{\mathrm{op}} = (p\mathscr{M}p)^{\mathrm{op}}$.
  Thus, for suitable $n, m \in \mathbb{N}$ and $R \in \mathrm{Mod}(p\mathscr{M}p), S \in \mathrm{Mod}(p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}p^{\mathrm{op}})$ we have

  $$(p\mathscr{M}p)^n \simeq p\mathscr{M} \oplus R \quad \text{and} \quad (p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}p^{\mathrm{op}})^m \simeq p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}} \oplus S,$$

  and therefore

  $$\begin{aligned} (qNq)^{nm} &= (p \otimes p^{\mathrm{op}}(\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})p \otimes p^{\mathrm{op}})^{nm} \\ &\simeq (p\mathscr{M}p)^n \otimes (p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}p^{\mathrm{op}})^m \\ &\simeq (p\mathscr{M} \oplus R) \otimes (p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}} \oplus S) \\ &\simeq (p\mathscr{M} \otimes p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}) \oplus \Big((p\mathscr{M} \otimes S) \oplus (R \otimes p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}) \oplus (R \otimes S)\Big). \end{aligned}$$

  From this it follows, that $qN := p\mathscr{M} \otimes p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}$ is projective over $qNq = p\mathscr{M}p \otimes (p\mathscr{M}p)^{\mathrm{op}}$.
  Continuing with an argument similar to the one given in Lemma 3.2.2 (the last 5 lines), we see that F maps projective N-modules to projective qNq-modules.

Hence, by applying F to the (free) resolution $(C_*(\mathscr{M}, N), b_*)$ of $\mathscr{M}$ we get a resolution of $p\mathscr{M}p$ by projective qNq-modules, with which we can compute the relevant Tor-groups.
Since $(p\mathscr{M}p)\bar{\otimes}(p^{\mathrm{op}}\mathscr{M}^{\mathrm{op}}p^{\mathrm{op}}) = q\mathscr{N}q$, this shows that $H_*^{(2)}(p\mathscr{M}p, \frac{1}{\alpha}\tau_p)$ can be computed as the homology of the complex

$$((q\mathscr{N}q) \otimes_{qNq} (qN \otimes \mathscr{M}^{\otimes n})\ ,\ \mathrm{id} \otimes b_n)_{n=0}^{\infty}$$

The $q\mathscr{N}q$-isomorphism $\varphi : q\mathscr{N}q \otimes_{qNq} qN \to q\mathscr{N}$, (from Lemma 3.2.5) given by

$$qxq \otimes qy \longmapsto qxqy,$$

gives rise to an isomorphism of complexes

$$\varphi \otimes \mathrm{id}_{\mathscr{M}^{\otimes *}} : ((q\mathscr{N}q) \otimes_{qNq} qN \otimes \mathscr{M}^{\otimes *}, \mathrm{id} \otimes b_*) \xrightarrow{\sim} (q\mathscr{N} \otimes \mathscr{M}^{\otimes *}, b_*).$$

Note, that the right-most complex is exactly $(F(C_*(\mathscr{M}, \mathscr{N})), F(b_*))$.
Since F is an exact (additive) functor, the homology of the right-most complex above is therefore $F(H_*^{(2)}(\mathscr{M}, \tau)) = qH_*^{(2)}(\mathscr{M}, \tau)$. (see e.g. [CE] Ch. IV, Thm. 7.2)
Hence $H_*^{(2)}(p\mathscr{M}p, \frac{1}{\alpha}\tau_p) = qH_*^{(2)}(\mathscr{M}, \tau)$.
The claimed identity for the Betti numbers now follows from Proposition 3.2.4, since

$$\tau \otimes \tau(q) = \tau \otimes \tau(p \otimes p^{\mathrm{op}}) = \tau(p)^2 = \alpha^2.$$

□

3.2.2 Direct sums

We now want to see how L^2-homology and L^2-Betti numbers behaves with respect to direct sums. For this, we return to the general set-up, of unital tracial $*$-algebras, as defined in the beginning of this chapter.

Proposition 3.2.9 (Sum-formula). [CS03] *Let $(\mathscr{A}_i,\tau_i)_{i=1}^n$ be a finite family of unital tracial $*$-algebras and put $\mathscr{A} := \oplus_{i=1}^n \mathscr{A}_i$. Choose a family $\alpha_1,\dots,\alpha_n \in]0,1[$ with $\sum_{i=1}^n \alpha_i = 1$ and endow $\mathscr{A}$ with the normalized faithful trace τ, given by*

$$\tau(a_1,\dots,a_n) := \sum_{i=1}^n \alpha_i \tau_i(a_i)$$

Then, for any $p\in\mathbb{N}_0$, we have

$$H_p^{(2)}(\mathscr{A},\tau) \simeq \bigoplus_{i=1}^n H_p^{(2)}(\mathscr{A}_i,\tau_i) \quad \textit{and} \quad \beta_p^{(2)}(\mathscr{A},\tau) = \sum_{i=1}^n \alpha_i^2 \beta_p^{(2)}(\mathscr{A}_i,\tau_i)$$

Proof. Define $K_p(\mathscr{A}) := \oplus_{i=1}^n C_p(\mathscr{A}_i,\mathscr{A}_i^{\mathrm{e}})$ and $d_p := (b_p^{(1)},\dots,b_p^{(n)})$, where $b_p^{(i)}$ denotes the p'th boundary map in the Bar-resolution of $\mathscr{A}_i$.
Since each $(C_*(\mathscr{A}_i,\mathscr{A}_i^{\mathrm{e}}), b_*^{(i)})$ is acyclic, the complex $(K_*(\mathscr{A}),d_*)$ is also acyclic.
Each $\mathscr{A}_i\otimes\mathscr{A}_j^{\mathrm{op}}$ can be considered a left module over $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$ via the action

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \otimes \begin{pmatrix} y_1^{\mathrm{op}} \\ \vdots \\ y_n^{\mathrm{op}} \end{pmatrix} a\otimes b^{\mathrm{op}} := (x_i\otimes y_j^{\mathrm{op}})(a\otimes b^{\mathrm{op}}) = x_i a \otimes y_j^{\mathrm{op}} b^{\mathrm{op}}, \qquad (a\otimes b^{\mathrm{op}} \in \mathscr{A}_i\otimes\mathscr{A}_j^{\mathrm{op}})$$

and in particular each $C_p(\mathscr{A}_i,\mathscr{A}_i^{\mathrm{e}})$ becomes a left $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$-module, with respect to this action on the first factor. Being the direct sum of the $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$-modules $C_p(\mathscr{A}_1,\mathscr{A}_1^{\mathrm{e}}),\dots,C_p(\mathscr{A}_n,\mathscr{A}_n^{\mathrm{e}})$; $K_p(\mathscr{A})$ also has structure as a left $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$-module. Explicitly, the $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$-action on $K_p(\mathscr{A})$ is given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \otimes \begin{pmatrix} y_1^{\mathrm{op}} \\ \vdots \\ y_n^{\mathrm{op}} \end{pmatrix} (c_1,\dots,c_n) := ((x_1\otimes y_1^{\mathrm{op}})c_1,\dots,(x_n\otimes y_n^{\mathrm{op}})c_n), \qquad (*)$$

for $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathscr{A}$ and $(c_1,\dots,c_n)\in K_p(\mathscr{A})$.
We now claim, that $K_p(\mathscr{A})$ is projective over $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$. Each $C_p(\mathscr{A}_i,\mathscr{A}_i^{\mathrm{e}})$ is free over $\mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}}$ and therefore isomorphic to $(\mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}})^{(X_i)}$ for a suitable set X_i. Putting $X := X_1\coprod\cdots\coprod X_n$ we have

$$\oplus_{i=1}^n(\mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}})^{(X_i)} \simeq (\oplus_{i=1}^n\mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}})^{(X)},$$

(where the isomorphism is isomorphism between $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$-modules) and hence

$$\bigoplus_{i=1}^n C_p(\mathscr{A}_i,\mathscr{A}_i^{\mathrm{e}}) \simeq \Big(\bigoplus_{i=1}^n \mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}}\Big)^{(X)}.$$

It now suffices to see that $(\oplus_{i=1}^n\mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}})^{(X)}$ is projective over $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$.
Since $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}} \simeq \oplus_{i,j=1}^n \mathscr{A}_i\otimes\mathscr{A}_j^{\mathrm{op}}$, the $\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}$-module $(\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}})^{(X)}$ splits as

$$(\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}})^{(X)} = (\bigoplus_{i=1}^n \mathscr{A}_i\otimes\mathscr{A}_i^{\mathrm{op}})^{(X)} \oplus (\bigoplus_{\substack{i=1\\ i\neq j}}^n \mathscr{A}_i\otimes\mathscr{A}_j^{\mathrm{op}})^{(X)},$$

and, being a direct summand in a free $\mathscr{A}\otimes\mathscr{A}^{\rm op}$-module, $(\oplus_{i=1}^n\mathscr{A}_i\otimes\mathscr{A}_i^{\rm op})^{(X)}$ is therefore projective as an $\mathscr{A}\otimes\mathscr{A}^{\rm op}$-module. Thus, $K_p(\mathscr{A})$ is projective as an $\mathscr{A}\otimes\mathscr{A}^{\rm op}$-module.

By what was just proven, the complex $(K_*(\mathscr{A}), b_*)$ is a resolution of $\oplus_{i=1}^n\mathscr{A}_i =: \mathscr{A}$ by projective $\mathscr{A}\otimes\mathscr{A}^{\rm op}$-modules and we now use this resolution to compute the L^2-homology of $\mathscr{A}$.
Denote by e_i the element

$$(0,\dots,0,1_i,0,\dots,0)\in\mathscr{A},$$

where 1_i is the unit in $\mathscr{A}_i$ and is placed in the i'th coordinate.
Let $\mathscr{M}_i$ be the von Neumann algebra generated by $\mathscr{A}_i$ in the GNS-representation with respect to τ_i and let $\mathscr{M}$ be the von Neumann algebra generated by $\mathscr{A}$, in the GNS-representation with respect to τ. The map

$$\eta_{\tau_i}(\mathscr{A}_i)\ni\eta_{\tau_i}(x)\longmapsto\frac{1}{\sqrt{\alpha_i}}\eta_\tau(0,\dots,0,x,0\dots,0)\in\eta_\tau(\mathscr{A}),$$

extends to an isometry from $L^2(\mathscr{A}_i,\tau_i)$ to $e_iL^2(\mathscr{A},\tau)$, which intertwines the action of $\mathscr{A}_i$ with the action of $\mathscr{A}$ on the subspace $e_iL^2(\mathscr{A})$. It therefore gives rise to an isomorphism φ_i from $\mathscr{M}_i$ to $e_i\mathscr{M}$. If we identify $e_iL^2(\mathscr{A},\tau)\bar{\otimes}e_j^{\rm op}L^2(\mathscr{A},\tau)^c$ with $e_i\otimes e_j^{\rm op}L^2(\mathscr{A},\tau)\bar{\otimes}L^2(\mathscr{A},\tau)^c$, we get an isomorphism

$$\varphi_i\otimes\varphi_j^{\rm op}:\mathscr{M}_i\bar{\otimes}\mathscr{M}_j^{\rm op}\longrightarrow(e_i\otimes e_j^{\rm op})\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op},$$

such that

$$\tau_i\otimes\tau_j^{\rm op}(T)=\frac{1}{\alpha_i\alpha_j}\tau\otimes\tau(\varphi_i\otimes\varphi_j^{\rm op}(T)).$$

In the following we will suppress the isomorphisms $\varphi_1,\dots,\varphi_n$ and simply identify $\mathscr{M}_i$ with $e_i\mathscr{M}$ and $\mathscr{M}_i\bar{\otimes}\mathscr{M}_j^{\rm op}$ with $e_i\otimes e_j^{\rm op}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})$.

Claim 1 *The two complexes*

$$\Big((\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})\otimes_{\mathscr{A}^{\rm e}}K_*(\mathscr{A}),\quad 1\otimes(\oplus_{i=1}^n b_*^{(i)})\Big)$$

and

$$\Big(\bigoplus_{i=1}^n\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}\otimes_{\mathscr{A}_i^{\rm e}}C_*(\mathscr{A}_i,\mathscr{A}_i^{\rm e}),\quad\oplus_{i=1}^n 1\otimes b_*^{(i)}\Big),$$

are isomorphic as complexes of left $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-modules.
Here $\bigoplus_{i=1}^n\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}\otimes_{\mathscr{A}_i^{\rm e}}C_*(\mathscr{A}_i,\mathscr{A}_i^{\rm e})$ is considered as an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-module, with respect to the action given by the analogue of the formula $(*)$.

Proof of Claim 1: Consider any $p\in\mathbb{N}_0$ and an elementary tensor
$T\otimes(c_1,\dots,c_n)\in\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}\otimes_{\mathscr{A}^{\rm e}}K_p(\mathscr{A})$. We note that $\sum_{i,j=1}^n e_i\otimes e_j^{\rm op}$ is the unit in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ and from this we get

$$\begin{aligned}T\otimes(c_1,\dots,c_n)&=\sum_{i,j=1}^n(Te_i\otimes e_j^{\rm op})\otimes(c_1,\dots,c_n)\\&=\sum_{i,j=1}^n(Te_i\otimes e_j^{\rm op})\otimes(e_i\otimes e_j^{\rm op}(c_1,\dots,c_n))\qquad(e_i\otimes e_j^{\rm op}\text{ projection in }\mathscr{A}\otimes\mathscr{A}^{\rm op})\\&=\sum_{i=1}^n(Te_i\otimes e_i^{\rm op})\otimes(0,\dots,0,c_i,0,\dots,0)\end{aligned}$$

We now define

$$\varphi_p : (\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}) \otimes_{\mathscr{A}^{\text{e}}} K_p(\mathscr{A}) \longrightarrow \bigoplus_{i=1}^{n} \mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}} \otimes_{\mathscr{A}_i^{\text{e}}} C_p(\mathscr{A}_i, \mathscr{A}_i^{\text{e}}),$$

on the elementary tensor $T \otimes (c_1, \dots, c_n)$, by setting

$$\varphi_p(T \otimes (c_1, \dots, c_n)) = \Big((Te_1 \otimes e_1^{\text{op}}) \otimes c_1, \dots, (Te_n \otimes e_n^{\text{op}}) \otimes c_n\Big),$$

and extend by additivity.
A direct computation reveals that φ_p is well-defined, $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-linear and commutes with the differentials. From the above calculation it follows that φ_k is bijective and hence an isomorphism of complexes.

From Claim 1 we get

$$H_p^{(2)}(\mathscr{A}, \tau) = \operatorname{Tor}_p^{\mathscr{A}^{\text{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}, \mathscr{A}) \simeq \bigoplus_{i=1}^{n} \operatorname{Tor}_p^{\mathscr{A}_i^{\text{e}}}(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}}, \mathscr{A}_i) = \bigoplus_{i=1}^{n} H_p^{(2)}(\mathscr{A}_i, \tau_i).$$

This finishes the proof of the sum-formula for the L^2-homology of $\mathscr{A}$.

To see the claimed identity for the L^2-Betti numbers, we prove the following (slightly) more general fact.

Claim 2: *If $W_1, \dots, W_n$ are modules over $\mathscr{M}_1\bar{\otimes}\mathscr{M}_1^{\text{op}}, \dots, \mathscr{M}_n\bar{\otimes}\mathscr{M}_n$ respectively, then each W_i can be considered as an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-module, with respect to the action*

$$Tw := (T(e_i \otimes e_i^{\text{op}}))w, \tag{†}$$

for $T \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ and $w \in W_i$. Let $W := \oplus_{i=1}^{n} W_i$ be the direct sum of these $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-modules. Then

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(W) = \sum_{i=1}^{n} \alpha_i^2 \dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}}}(W_i).$$

Proof of Claim 2: By additivity of the extended dimension function, (Theorem 1.4.7) we get

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(W) = \sum_{i=1}^{n} \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(W_i),$$

so it suffices to show that $\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(W_i) = \alpha_i^2 \dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i}(W_i)$ for each $i \in \{1, \dots, n\}$. We now fix an arbitrary $i \in \{1, \dots, n\}$. Assume first that W_i is finitely generated and projective over $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}}$ and hence (up to isomorphism) of the form $(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}})^m p$ for some idempotent matrix

$$p := \{p_{st}\}_{s,t=1}^{m} \in M_m(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}}).$$

For $z \in \mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}}$, we will denote by zp the matrix $\{zp_{st}\}_{s,t=1}^{n}$. Since $p_{st} = (e_i \otimes e_i^{\text{op}})p_{st}$ for all $s, t \in \{1, \dots, n\}$ we get

$$(x_1, \dots, x_m)p = (x_1, \dots, x_m)((e_i \otimes e_i^{\text{op}})p) = ((e_i \otimes e_i^{\text{op}})x_1, \dots, (e_i \otimes e_i^{\text{op}})x_m)p,$$

for all $(x_1, \dots, x_n) \in (\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})^m$. The identity-mapping on $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})p$ may therefore be considered as a map from $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})^m p$ to $(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\text{op}})^m p$ and written as

$$\text{id} : (x_1, \dots, x_m)p \longmapsto (e_i \otimes e_i^{\text{op}})(x_1, \dots, x_m)p.$$

This is clearly an isomorphism from $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m p$ with standard left $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-action to $(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op})p$ with the $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-action given by the formula (†) in Claim 2.
So, as a module over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$, W_i is isomorphic to $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m p$ and we conclude that

$$\begin{aligned}\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(W_i) &= \sum_{k=1}^m \tau\otimes\tau^{\rm op}(p_{kk})\\ &= \sum_{k=1}^m \alpha_i^2\tau_i\otimes\tau_i^{\rm op}(p_{kk}) \qquad\qquad (\text{since } p_{kk}\in\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op})\\ &= \alpha_i^2\dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}}(W_i).\end{aligned}$$

This shows, that if W_i is finitely generated and projective over $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$, then the same is true when W_i is considered as a module over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ and we have $\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(W_i)=\alpha_i^2\dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i}(W_i)$. Thus, for **any** $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$-module W_i, we have

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(W_i)\geq\alpha_i^2\dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}}(W_i).$$

To see the opposite inequality, it suffices to prove the following:

Let W_i be any $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$-module. If S is an $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i$-submodule of W_i, which is finitely generated and projective when considered as a module over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$, then S is also finitely generated and projective as an $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$-module and

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(S)=\alpha_i^2\dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}}(S).$$

Assume that S is projective as an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-module and choose a suitable idempotent $p\in M_m(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})$ and an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-isomorphism

$$\varphi: S\longrightarrow(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m p.$$

Since $e_i\otimes e_i^{\rm op}$ is the unit in $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$, it follows that

$$\varphi(x)=\varphi(e_i\otimes e_i^{\rm op}x)=e_i\otimes e_i^{\rm op}\varphi(x),$$

for all $x\in S$. Since φ is surjective, we conclude that

$$(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m p=(e_i\otimes e_i^{\rm op})(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m p=(e_i\otimes e_i^{\rm op}\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m(e_i\otimes e_i^{\rm op}p)=(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i)^m(e_i\otimes e_i^{\rm op}p).$$

Since φ in particular is $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i$-linear, it provides us with an isomorphism of $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$-modules, from S to $(\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op})(e_i\otimes e_i^{\rm op}p)$. In particular S is projective as an $\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}$-module. Since we have

$$(\mathscr{M}\otimes\mathscr{M}^{\rm op})^m p=(e_i\otimes e_i^{\rm op})(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m p=(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op})^m(e_i\otimes e_i^{\rm op}p),$$

we may use $e_i\otimes e_i^{\rm op}p$ to compute the dimension of S over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ and we get

$$\begin{aligned}\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(S) &= \sum_{k=1}^m\tau\otimes\tau^{\rm op}(e_i\otimes e_i^{\rm op}p_{kk})\\ &= \sum_{k=1}^m\alpha_i^2\tau_i\otimes\tau_i^{\rm op}(e_i\otimes e_i^{\rm op}p_{kk})\\ &= \alpha_i^2\dim_{\mathscr{M}_i\bar{\otimes}\mathscr{M}_i^{\rm op}}S.\end{aligned}$$

This finishes the proof of Claim 2.

The formula for the L^2-Betti numbers of $\mathscr{A}$ now follows from Claim 2, once we note that the $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ action on $\oplus_{i=1}^n H_p^{(2)}(\mathscr{A}_i,\tau_i)$ given by Claim 2, is intertwined with the natural $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-action on $H_p^{(2)}(\mathscr{A},\tau)$ via the isomorphism arising from Claim 1.

□

With both the compression formula and the sum formula at our disposal, we are able to compute all Betti numbers of finite von Neumann algebras, with finite linear dimension over $\mathbb{C}$.
So, let $\mathscr{M}$ be a finite von Neumann algebra with center $\mathscr{C}$, endowed with a normal, faithful, tracial state τ. Assume moreover, that $\mathscr{M}$ has finite linear dimension over $\mathbb{C}$. Because of this, the center of $\mathscr{M}$ is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Let e_i be the projection in $\mathscr{C}$ corresponding to the i'th standard basis-vector of $\mathbb{C}^n$. Then $e_1 + \cdots + e_n = 1$ and $\mathscr{M} = \oplus_{i=1}^n e_i\mathscr{M}$ and since the center of $e_i\mathscr{M}$ is $e_i\mathscr{C}$, each summand $e_i\mathscr{M}$ is a factor. Of coarse $e_i\mathscr{M}$ is also of finite linear dimension over $\mathbb{C}$ and hence isomorphic to $M_{n_i}(\mathbb{C})$ for suitable $n_i \in \mathbb{N}$.
So, up to isomorphism, we now have $\mathscr{M} = \oplus_{i=1}^n M_{n_i}(\mathbb{C})$. Let 1_{n_i} denote the identity-matrix in $M_{n_i}(\mathbb{C})$ and note that $e_i = (0, \ldots, 0, 1_{n_i}, 0, \ldots, 0) \in \mathscr{M}$, where the 1_{n_i} is in the i'th coordinate.

Proposition 3.2.10. [CS03] *With $\mathscr{M} = \oplus_{i=1}^n M_{n_i}(\mathbb{C})$ as above and $\alpha_i := \tau(e_i)$, we have*

$$\beta_p^{(2)}(\mathscr{M}, \tau) = \begin{cases} \sum_{i=1}^n \frac{\alpha_i^2}{n_i^2}, & \text{when } p = 0; \\ 0, & \text{when } p \geq 1. \end{cases}$$

We let tr_k denote the normalized standard trace on $M_k(\mathbb{C})$.

Proof. Since $\mathscr{M} = \oplus_{i=1}^n M_{n_i}(\mathbb{C})$ we have $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} = \mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ and all L^2-homology in dimensions higher than zero vanishes, since those are (isomorphic to) the homology-groups of the acyclic Bar-complex.
To prove the formula for the zero'th Betti number, we first note that

$$H_0^{(2)}(\mathbb{C}, \mathrm{id}) := \mathbb{C} \otimes \mathbb{C}^{\mathrm{op}} / \langle \lambda \otimes 1 - 1 \otimes \lambda^{\mathrm{op}} | \lambda \in \mathbb{C} \rangle \simeq \mathbb{C} \otimes \mathbb{C}^{\mathrm{op}},$$

and hence $\beta_0^{(2)}(\mathbb{C}, \mathrm{id}) = 1$. We now consider the i'th summand $M_{n_i}(\mathbb{C})$ and the projection

$$p = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{n_i}(\mathbb{C}),$$

and note that $\mathrm{tr}_{n_i}(p) = \frac{1}{n_i}$.
Since $pM_{n_i}(\mathbb{C})p \simeq \mathbb{C}$, the compression-formula gives

$$1 = \beta_0^{(2)}(pM_{n_i}(\mathbb{C})p, \frac{1}{1/n_i}\mathrm{tr}_{n_i}|_{pM_{n_i}p}) = \frac{1}{(1/n_i)^2}\beta_0^{(2)}(M_{n_i}(\mathbb{C}), \mathrm{tr}_{n_i}) = n_i^2\beta_0^{(2)}(M_{n_i}(\mathbb{C}), \mathrm{tr}_{n_i}).$$

Under the isomorphism of $\mathscr{M}$ with $\oplus_{i=1}^n M_{n_i}(\mathbb{C})$, the trace τ corresponds to the trace

$$(X_1, \ldots, X_n) \longmapsto \sum_{i=1}^n \alpha_i \mathrm{tr}_{n_i}(X_i),$$

and the sum-formula now implies

$$\beta_0^{(2)}(\mathscr{M}, \tau) = \sum_{i=1}^n \alpha_i^2 \beta_0^{(2)}(M_{n_i}(\mathbb{C}), \mathrm{tr}_{n_i}) = \sum_{i=1}^n \frac{\alpha_i^2}{n_i^2}.$$

□

3.2.3 L^2-homology as an inductive limit

One of the difficulties, connected with the computation of Hochschild homology, is the size of the modules appearing in the Hochschild complex. These are typically not finitely generated and this makes it difficult to determine the homology-groups in concrete cases.
We now describe a family of (finitely generated) subcomplexes of the Hochshild complex and show that the L^2-homology can be computed as the inductive limit of the finitely generated homology-groups, associated with this family of subcomplexes. A similar technique will be taken to use in Section 3.5, where we will take a closer look at the first Betti number.

Proposition 3.2.11. [CS03] *Let $(\mathscr{A},\tau)$ be a unital tracial $*$-algebra and let $\mathscr{M}$ denote the von Neumann algebra generated by $\mathscr{A}$ in the GNS-representation with respect to τ.*
There exists a directed family of subcomplexes, $(C_*^{(i)}(\mathscr{A}), b_*^{(i)})_{i\in I}$, of the Hochschild complex $(C_*(\mathscr{A},\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}), b_*)$, with the following properties.

- *For each $i\in I$ and $n\in\mathbb{N}$, $C_n^{(i)}(\mathscr{A})$ is finitely generated and free as an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-module.*
- *If φ_{ji} denotes the homomorphism of complexes $C_*^{(i)}(\mathscr{A}) \longrightarrow C_*^{(j)}(\mathscr{A})$ when $j\geq i$ and $H_*^{(i)}(\mathscr{A})$ denotes the homology of the complex $(C_*^{(i)}(\mathscr{A}), {b_*}^{(i)})$, then*
$$H_n^{(2)}(\mathscr{A},\tau) = \varinjlim (H_n^{(i)}(\mathscr{A}), \varphi_{ji_*}),$$
where φ_{ji_*} denotes the map in homology induced by φ_{ji}.

Furthermore, $\beta_n^{(2)}(\mathscr{A},\tau) = \sup_i \inf_{j\geq i} \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\varphi_{ji_*} H_n^{(i)}(\mathscr{A})\Big)$.

Proof. We first construct the family of subcomplexes. Take any finite set $E\subseteq\mathscr{A}$ and any $n\in\mathbb{N}$ and define
$$V^{E,n} := \mathrm{span}_{\mathbb{C}}(E) \qquad \text{and} \qquad V_0^{E,n} := (V^{E,n})^{\otimes n}.$$
Let $\mu : \mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ be the multiplication map and define recursively
$$V_m^{E,n} := \mathrm{span}_{\mathbb{C}}\Big(\{(\mu\otimes 1^{\otimes(n-1-m)})V_{m-1}^{E,n}, (1\otimes\mu\otimes 1^{\otimes(n-2-m)})V_{m-1}^{E,n}, \ldots, (1^{\otimes(n-1-m)}\otimes\mu)V_{m-1}^{E,n}\}\Big),$$
for $m\in\{1,\ldots,n-1\}$. We then define
$$C_k^{(E,n)}(\mathscr{A}) = \begin{cases} \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, & \text{when } k=0;\\ \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}\otimes V_{n-k}^{E,n}, & \text{when } 1\leq k\leq n;\\ 0, & \text{when } k>n.\end{cases}$$

Since each $V_m^{(E,n)}$ has finite linear dimension over $\mathbb{C}$, each $C_k^{(E,n)}(\mathscr{A})$ is finitely generated as a module over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$. Note also, that the Hochschild boundary map b_k maps $C_k^{(E,n)}(\mathscr{A})$ into $C_{k-1}^{(E,n)}(\mathscr{A})$, such that $(C_*^{(E,n)}(\mathscr{A}), b_*)$ is a subcomplex of $(C_*(\mathscr{A},\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}), b_*)$. Denote by $H_*^{(E,n)}(\mathscr{A})$, the homology of the complex $(C_*^{(E,n)}(\mathscr{A}), b_*)$.
Let I denote the set of such pairs (E,n) and put an ordering on I by requiring that
$$(E,n)\preceq(E',n') \quad \text{iff} \quad E\subseteq E' \text{ and } n\leq n'.$$

If $i=(E,n)\preceq j=(E',n')$ we have that $(C_*^{(E,n)}(\mathscr{A}), b_*)$ is a subcomplex of $(C_*^{(E',n')}(\mathscr{A}), b_*)$. Let φ_{ji} denote the inclusion-map and denote by φ_{ji_*} the morphism induced by φ_{ji} on the level of homology. Then, since each φ_{ji} is an inclusion, we have $\varphi_{kj_*}\circ\varphi_{ji_*} = \varphi_{ki_*}$ when $i\preceq j\preceq k$. So, the system $(H_n^{(i)}(\mathscr{A}))_{i\in I}$ has an inductive limit $H_n(\mathscr{A})$ and we now have to prove that this inductive limit is isomorphic to $H_n^{(2)}(\mathscr{A},\tau)$.
For each $i\in I$ we have an inclusion of complexes $\psi_i : (C_*^{(i)}(\mathscr{A}), b_*)\to(C_*(\mathscr{A},\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}), b_*)$ and

hence an induced morphism $\psi_{i_*}: H_n^{(i)}(\mathscr{A}) \to H_n^{(2)}(\mathscr{A}, \tau)$.
Clearly the ψ_{i_*}'s are compatible with the φ_{ji_*}'s, in the sense that $\psi_{j_*} \circ \varphi_{ji_*} = \psi_{i_*}$, and by Proposition 1.4.12 the proof is complete if we can show that $H_n^{(2)}(\mathscr{A}, \tau) = \cup_{i\in I}\psi_{i_*}(H_n^{(i)}(\mathscr{A}))$.
But this follows, since $C_n(\mathscr{A}, \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) = \cup_{i\in I} C_n^{(i)}(\mathscr{A})$.

Since each $C_n^{(E,n)}(\mathscr{A})$ is finitely generated over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ the same is true for the homology groups and by Theorem 1.4.7 we therefore have

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}} \left(\varphi_{ji*}(H_n^{(i)}(\mathscr{A}))\right) < \infty.$$

By applying Theorem 1.4.13 the formula for the Betti numbers now follows. □

3.3 The zero'th Betti number

In this section we want to give a more explicit descriptions of the zero'th L^2-homology and Betti number. This description arises from the fact that we have good access to the zero'th L^2-homology, because of the simplicity of the last boundary map in the Hochschild complex.
Let $\mathscr{M}$ be a finite von Neumann algebra, endowed with a normal, faithful, tracial state τ and recall that $H_0^{(2)}(\mathscr{M}, \tau) := \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}/\mathrm{rg}\,(b_1)$, where b_1 is given by

$$\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes \mathscr{M} \ni T\otimes c \overset{b_1}{\longmapsto} T(c\otimes 1 - 1\otimes c^{\mathrm{op}}) \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}.$$

Lemma 3.3.1. [CS03] *The zero'th L^2-homology $H_0^{(2)}(\mathscr{M}, \tau)$ is isomorphic, as an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-module, to*

$$\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^e} \mathscr{M}.$$

Proof. Consider the $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-morphism

$$\varphi : \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^e} \mathscr{M},$$

given by $T \mapsto T\otimes 1$. One easily checks that φ vanishes on $\mathrm{rg}\,(b_1)$ and hence φ factorizes through a morphism $\tilde{\varphi} : H_0^{(2)}(\mathscr{M}, \tau) \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^e} \mathscr{M}$. A direct computation now shows, that the map $\psi : \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^e} \mathscr{M} \to H_0^{(2)}(\mathscr{M}, \tau)$ given by

$$\psi(T\otimes a) = [T(a\otimes 1)],$$

is well-defined and inverse to $\tilde{\varphi}$. Here $[T(a\otimes 1)]$ denotes the coset in $H_0^{(2)}(\mathscr{M}, \tau)$ represented by $T(a\otimes 1)$. □

Lemma 3.3.2. [KR2] *Let $(\mathscr{H}, \langle\cdot|\cdot\rangle)$ be a Hilbert space and consider a unit vector x in $\mathscr{H}$. Assume moreover that $\alpha_1, \ldots, \alpha_n \in]0,1[$ has sum 1 and that $x_1, \ldots, x_n \in (\mathscr{H})_1 := \{\xi \in \mathscr{H} \mid \|\xi\| \leq 1\}$ such that*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = x.$$

Then $x = x_i$ for each $i \in \{1, \ldots, n\}$.

Proof. We prove the Lemma by induction on n, starting with the case $n = 2$.
So assume $\alpha_1, \alpha_2 \in]0,1[$ with sum 1 and that $x = \alpha_1 x_1 + \alpha_2 x_2$. Then

$$1 = \langle x|x\rangle = \langle x|\alpha_1 x_1 + \alpha_2 x_2\rangle = \alpha_1\langle x|x_1\rangle + \alpha_2\langle x|x_2\rangle.$$

Since 1 is an extremal point in $\{z \in \mathbb{C} : |z| \leq 1\}$ this implies that $1 = \langle x | x_1 \rangle = \langle x | x_2 \rangle$. Therefore

$$\langle x | x_1 \rangle = 1 \geq \|x_1\| = \|x_1\|\|x\| \geq \langle x | x_1 \rangle. \qquad (\dagger)$$

From (†) it follows that we have equality in the Cauchy-Schwartz inequality applied to x and x_1 and this can only happen if $x = \lambda x_1$ for some $\lambda \in [0, \infty[$. (see e.g. [KR1] Theorem 2.1.3) But (†) also implies that $\|x_1\| = 1 = \|x\|$ and we conclude that $x = x_1$.
By a symmetric argument, we also have $x_2 = x$ and the proof is complete in the case $n = 2$.
Assume now, that the result holds for some fixed $n \in \mathbb{N}$ and consider $\alpha_1, \dots, \alpha_{n+1} \in]0, 1[$ with sum 1 and $x_1, \dots, x_{n+1} \in (\mathscr{H})_1$ with $\sum_{i=1}^{n+1} \alpha_i x_i = x$. We rewrite x as

$$x = (1 - \alpha_{n+1}) \underbrace{\left(\frac{\alpha_1}{1 - \alpha_{n+1}} x_1 + \cdots + \frac{\alpha_n}{1 - \alpha_{n+1}} x_n \right)}_{=:\xi} + \alpha_{n+1} x_{n+1},$$

and since $\sum_{i=1}^{n} \frac{\alpha_i}{1-\alpha_{n+1}} = \frac{1-\alpha_{n+1}}{1-\alpha_{n+1}} = 1$, we have $\xi \in (\mathscr{H})_1$.
The result now follows from the induction hypothesis and the case $n = 2$. □

Recall, that a von Neumann algebra is called *hyper-finite*, if it is the weak closure of the union of an increasing family of finite dimensional subalgebras.
As the following theorem shows, the zero'th L^2-homology detects hyper-finiteness. (Compare with Theorem 3.4.5)

Theorem 3.3.3. [CS03] *Let $\mathscr{M}$ be a factor of type $\mathbf{II}_1$ and let τ be the (unique, normal, faithful) tracial state on $\mathscr{M}$. Then $H_0^{(2)}(\mathscr{M}, \tau) \neq 0$ if, and only if, $\mathscr{M}$ is hyperfinite.*

For the proof we will need the following characterization if hyper-finiteness.

Proposition 3.3.4. [Con76] *The factor $\mathscr{M}$ is hyperfinite if, and only if, there exists a state*[1] *$\theta : \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \to \mathbb{C}$ such that $\theta(x \otimes y^{\mathrm{op}}) = \tau(xy)$ for all $x, y \in \mathscr{M}$.*

We omit the proof, since it would give rise to a rather big digression from the present subject of L^2-homology and L^2-Betti numbers. A proof can be found in [Con76] Theorem 5.1. (the equivalence of 1. and 5.)

Proof of Theorem 3.3.3. Assume first that $\mathscr{M}$ is hyperfinite and choose $\theta : \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \to \mathbb{C}$ as in Proposition 3.3.4. Let $\mathscr{J}$ be the left ideal in $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ generated by elements of the form $m \otimes 1 - 1 \otimes m^{\mathrm{op}}$ for $m \in \mathscr{M}$, such that $H_0^{(2)}(\mathscr{M}, \tau) = \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} / \mathscr{J}$.
Consider any $m \in \mathscr{M}$ and put $X := m \otimes 1 - 1 \otimes m^{\mathrm{op}}$. Then

$$\begin{aligned}
\theta(X^*X) &= \theta((m^* \otimes 1 - 1 \otimes m^{\mathrm{op}*})(m \otimes 1 - 1 \otimes m^{\mathrm{op}})) \\
&= \theta(m^*m \otimes 1 - m^* \otimes m^{\mathrm{op}} - m \otimes m^{\mathrm{op}*} + 1 \otimes m^{\mathrm{op}*} m^{\mathrm{op}}) \\
&= \tau(m^*m - m^*m - mm^* + mm^*) \\
&= 0.
\end{aligned}$$

Since θ is a state, we have (see e.g. [KR1] Proposition 4.3.1)

$$|\theta(Y^*X)| \leq \theta(X^*X)\theta(Y^*Y) \text{ for all } Y \in \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}},$$

and hence $\theta(YX) = 0$ for all $Y \in \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$. Since $\mathscr{J}$ is the left ideal generated by elements of the form $X = m \otimes 1 - 1 \otimes m^{\mathrm{op}}$ it follows that $\theta|_{\mathscr{J}} = 0$. Hence θ factorizes through a functional $\tilde{\theta} : H_0^{(2)}(\mathscr{M}, \tau) \to \mathbb{C}$. Since $\theta \neq 0$ we have $\tilde{\theta} \neq 0$ and this of course implies $H_0^{(2)}(\mathscr{M}, \tau) \neq 0$.

Assume conversely, that $H_0^{(2)}(\mathscr{M}, \tau) \neq 0$. We want to construct state θ on $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ as in Proposition 3.3.4. Consider n unitaries $u_1, \dots, u_n \in \mathscr{M}$ and the element

$$X := \frac{1}{n} \sum_{i=1}^{n} (u_i \otimes u_i^{*\mathrm{op}} - 1 \otimes 1) = \frac{1}{n} \sum_{i=1}^{n} (1 \otimes u_i^{*\mathrm{op}})(u_i \otimes 1 - 1 \otimes u_i^{\mathrm{op}}) \in \mathscr{J}.$$

[1] Note, that there is no normality-condition on θ

Claim 1: *There exists a state φ on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ with $\varphi(X^*X)=0$.*

Proof of claim. Since $\mathscr{J}$ is a left ideal we have $X^*X\in\mathscr{J}$ and by assumption

$$0\neq H_0^{(2)}(\mathscr{M},\tau):=\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}/\mathscr{J}.$$

Therefore X^*X can not be invertible in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ and hence $0\in\sigma(X^*X)\subseteq[0,\infty[$.
By continuous functional calculus (see e.g. [Arv] Theorem 2.3.1) we have $C^*(X^*X,1)\simeq C(\sigma(X^*X))$ and on the latter C^*-algebra we can consider the functional ev_0 given by

$$C(\sigma(X^*X))\ni f\longmapsto f(0)\in\mathbb{C}.$$

Note, that ev_0 is a state with $\mathrm{ev}_0(\mathrm{id}_{\sigma(X^*X)})=0$. Let φ be the corresponding state on $C^*(X^*X,1)$, vanishing on X^*X. By the Hahn-Banach Theorem, φ extends to a functional, also denoted φ, on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ with the same norm. Thus, $\|\varphi\|=\varphi(1)=1$ and hence φ is a state on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$. (See e.g. [KR1] Theorem 4.3.2) This extension has the desired property; and Claim 1 follows.

Let $\mathscr{U}(\mathscr{M})$ denote the group of unitaries in $\mathscr{M}$ and let $\mathcal{P}_e(\mathscr{U}(\mathscr{M}))$ denote the system of finite subsets of $\mathscr{U}(\mathscr{M})$. Then, by what we have just proven, for each $I:=\{u_1,\dots,u_n\}\in\mathcal{P}_e(\mathscr{U}(\mathscr{M}))$ we get an element $X_I:=\frac{1}{n}\sum_{i=1}^n(u_i\otimes u_i^{*\mathrm{op}}-1\otimes 1)\in\mathscr{J}$ and a state φ_I on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ vanishing on $X_I^*X_I$.

Claim 2: *There exists a state φ on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ vanishing on*

$$\{(u\otimes u^{*\mathrm{op}}-1\otimes 1)^*(u\otimes u^{*\mathrm{op}}-1\otimes 1)\mid u\in\mathscr{U}(\mathscr{M})\}.$$

Proof of claim. Ordering $\mathcal{P}_e(\mathscr{U}(\mathscr{M})$ by inclusion, we get a net $(\varphi_I)_{I\in\mathcal{P}_e(\mathscr{U}(\mathscr{M}))}$ in the unit ball of $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^*$ and by the Alouglu-Bourbaki Theorem (see e.g. [KR1] Thm. 1.6.5) this unit ball is weak-$*$-compact. Hence $(\varphi_I)_{I\in\mathcal{P}_e(\mathscr{U}(\mathscr{M}))}$ contains a convergent sub-net $(\varphi_{I\alpha})_{\alpha\in\mathbb{A}}$. Let φ denote the weak-$*$-limit of this sub-net and note that φ is state since each φ_{I_α} is a state. We now aim to prove that φ has the desired property.
Let $u\in\mathscr{U}(\mathscr{M})$ be given and consider any $I:=\{u_1,\dots,u_n\}\in\mathcal{P}_e(\mathscr{U}(\mathscr{M}))$ with $u\in I$. By doing the GNS-constuction with respect to φ_I we get a Hilbert space $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}},\varphi_I)$.
Let $\eta_I:\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}\to L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}},\varphi_I)$ denote the map sending x to the vector represented by x in $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}},\varphi_I)$. Since φ_I vanishes on $X_I^*X_I$, we have $\eta_I(X_I)=0$ and hence

$$\sum_{i=1}^n\frac{1}{n}\eta_I(u_i\otimes u_i^{*\mathrm{op}})=\eta(1\otimes 1).\tag{†}$$

Because $\eta_I(1\otimes 1),\eta_I(u_1\otimes u_1^{*\mathrm{op}}),\dots,\eta_I(u_n\otimes u_n^{*\mathrm{op}})$ are all unit vectors in $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}},\varphi_I)$ and the left-hand side of (†) is a convex combination, it follows from Lemma 3.3.2 that

$$\eta_I(u_i\otimes u_i^{*\mathrm{op}})=\eta_I(1\otimes 1)\ \text{ for all } i\in\{1,\dots,n\}$$

In particular $\eta_I(u\otimes u^{*\mathrm{op}})=\eta_I(1\otimes 1)$ and hence

$$0=\|\eta_I(u\otimes u^{*\mathrm{op}}-1\otimes 1)\|_2^2=\varphi_I((u\otimes u^{*\mathrm{op}}-1\otimes 1)^*(u\otimes u^{*\mathrm{op}}-1\otimes 1)).$$

This proves that $(\varphi_I)_{I\in\mathcal{P}_e(\mathscr{U}(\mathscr{M}))}$ is zero on $(u\otimes u^{*\mathrm{op}}-1\otimes 1)^*(u\otimes u^{*\mathrm{op}}-1\otimes 1)$ from a certain point ($\{u\}$) and hence the same is true for the subnet $(\varphi_{I\alpha})_{\alpha\in\mathbb{A}}$. Since φ is the weak-$*$-limit of $(\varphi_{I\alpha})_{\alpha\in\mathbb{A}}$, we conclude that

$$\varphi((u\otimes u^{*\mathrm{op}}-1\otimes 1)^*(u\otimes u^{*\mathrm{op}}-1\otimes 1))=0,$$

and Claim 2 is proven.

We now aim to prove, that the state φ from Claim 2 has the property from Proposition 3.3.4; i.e. that

$$\varphi(x \otimes y^{\text{op}}) = \tau(xy) \text{ for all } x \in \mathscr{M}, y^{\text{op}} \in \mathscr{M}^{\text{op}}.$$

Consider the GNS-construction $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}, \varphi)$ with respect to φ and let $\eta : \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \to L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}, \varphi)$ denote the map sending x to the vector represented by x. Put $\xi := \eta(1 \otimes 1)$ and denote by $\langle \cdot | \cdot \rangle$ the inner product on $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}, \varphi)$. To simplify notation, we will suppress the (generally non-faithful!) GNS-representation of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ on $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}, \varphi)$.
By construction of φ, we have $u \otimes u^{*\text{op}}\xi = \xi$ for any $u \in \mathscr{U}(\mathscr{M})$ and from this we get:

$$\begin{gathered} u \otimes u^{*\text{op}}\xi = \xi \\ \Downarrow \\ (1 \otimes u^{\text{op}})(u \otimes u^{*\text{op}}) = (1 \otimes u^{\text{op}})\xi \\ \Downarrow \\ (u \otimes 1)\xi = (1 \otimes u^{\text{op}})\xi. \end{gathered}$$

Since every element in $\mathscr{M}$ can be written as a linear combination of four unitaries (see e.g. [KR1] Theorem 4.1.7) we have

$$(m \otimes 1)\xi = (1 \otimes m^{\text{op}})\xi \quad \text{for all } m \in \mathscr{M}. \tag{$*$}$$

For any $m, n \in \mathscr{M}$, we get from $(*)$ that

$$(mn \otimes 1)\xi = (m \otimes 1)(n \otimes 1)\xi = (m \otimes 1)(1 \otimes n^{\text{op}})\xi = (m \otimes n^{\text{op}})\xi. \tag{$**$}$$

With these two identities established we are able to prove, that the state φ has the desired property. For arbitrary $m, n \in \mathscr{M}$ we have

$$\varphi(m \otimes n^{\text{op}}) = \langle m \otimes n^{\text{op}}\xi | \xi \rangle = \langle (mn \otimes 1)\xi | \xi \rangle = \varphi(mn \otimes 1). \tag{$***$}$$

Moreover,

$$\begin{aligned} \varphi(nm \otimes 1) &= \langle (nm \otimes 1)\xi | \xi \rangle \\ &= \langle n \otimes m^{\text{op}}\xi | \xi \rangle && \text{(by } (**)) \\ &= \langle (1 \otimes m^{\text{op}})(n \otimes 1)\xi | \xi \rangle \\ &= \langle (n \otimes 1)\xi | (1 \otimes m^{*\text{op}})\xi \rangle \\ &= \langle (n \otimes 1)\xi | (m^* \otimes 1)\xi \rangle && \text{(by } (*)) \\ &= \langle (mn \otimes 1)\xi | \xi \rangle \\ &= \varphi(mn \otimes 1). \end{aligned}$$

Let $\mathscr{M} \otimes 1$ denote the subalgebra $\{m \otimes 1 \mid m \in \mathscr{M}\}$ in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ and note that this subalgebra is isomorphic to $\mathscr{M}$.
The above calculation shows, that the restriction of φ to $\mathscr{M} \otimes 1$ is a trace and since φ is a state, we must have that $\varphi(mn \otimes 1) = \tau(mn)$, since the trace-state τ is unique.
Combining this with the equation $(***)$ we conclude that $\varphi(m \otimes n^{\text{op}}) = \tau(mn)$ for all $m \in \mathscr{M}$ and $n^{\text{op}} \in \mathscr{M}^{\text{op}}$ and hence that $\mathscr{M}$ is hyper-finite. □

Consider again an arbitrary finite von Neumann algebra $\mathscr{M}$, endowed with a faithful, normal, tracial state τ.

Theorem 3.3.5. [CS03] *Assume that $\mathscr{M}$ contains a normal element x, with the property that associated spectral measure $E : \mathcal{B}(\sigma(x)) \to \mathscr{B}(L^2(\mathscr{M}))$ has $E(\{t\}) = 0$ for all $t \in \sigma(x)$. Then $\beta_0^{(2)}(\mathscr{M}, \tau) = 0$.*

Here $\mathcal{B}(\sigma(x))$ denotes the Borel σ-algebra on the spectrum of x.
Note, that the requirement $E(\{t\}) = 0$ for all $t \in \sigma(x)$ is equivalent to saying that x has no eigenvalues.

Proof. Since τ is faithful, the GNS-construction provides us with a faithful normal representation of $\mathscr{M}$ on $L^2(\mathscr{M}) := L^2(\mathscr{M}, \tau)$ and (as usual) we identify $\mathscr{M}$ with its isomorphic image in $\mathscr{B}(L^2(\mathscr{M}))$.
We proof the claim by contradiction. So, assume that $\beta_0^{(2)}(\mathscr{M}, \tau) \neq 0$. Since $H_0^{(2)}(\mathscr{M}, \tau)$ is finitely (singly) generated as a module over $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$, Lemma 1.4.8 gives

$$\beta_0^{(2)}(\mathscr{M}, \tau) := \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(H_0^{(2)}(\mathscr{M}, \tau)) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\mathrm{Hom}_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(H_0^{(2)}(\mathscr{M}, \tau), \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})\Big)$$

Thus, $\beta_0^{(2)}(\mathscr{M}, \tau) \neq 0$ implies that there exists a non-zero $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-linear map

$$\varphi : H_0^{(2)}(\mathscr{M}, \tau) \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}.$$

Let $\mathscr{J}$ denote the left ideal in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ generated by elements of the form $m\otimes 1 - 1\otimes m^{\mathrm{op}}$ where $m \in \mathscr{M}$. As an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-module, $H_0^{(2)}(\mathscr{M}, \tau) := \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}/\mathscr{J}$ is generated by $[1\otimes 1]_{\mathscr{J}}$ and since $\varphi \neq 0$ we must have $\varphi([(1\otimes 1)]_{\mathscr{J}}) \neq 0$.
Consider the Hilbert space $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \tau\otimes\tau) =: L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$ and the non-zero vector $\xi \in L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$, corresponding to the element $\varphi([1\otimes 1]_{\mathscr{J}}) \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$. For any $m \in \mathscr{M}$ we have

$$\begin{aligned}(m\otimes 1 - 1\otimes m^{\mathrm{op}})\xi &= (m\otimes 1 - 1\otimes m^{\mathrm{op}})\eta\Big(\varphi([(1\otimes 1]_{\mathscr{J}})\Big) \\ &= \eta\Big(\varphi([m\otimes 1 - 1\otimes m^{\mathrm{op}}]_{\mathscr{J}})\Big) && (\varphi \text{ is } \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}\text{-linear}) \\ &= 0, && (m\otimes 1 - 1\otimes m^{\mathrm{op}} \in \mathscr{J})\end{aligned}$$

which gives

$$(m\otimes 1)\xi = (1\otimes m^{\mathrm{op}})\xi \quad \text{for all } m \in \mathscr{M}. \tag{†}$$

The isomorphism $\Psi : L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \tau\otimes\tau) \simeq \mathscr{HS}(L^2(\mathscr{M}))$, from Proposition 1.2.12, maps ξ onto a non-zero Hilbert-Schmidt operator T and (by Proposition 1.2.15) applying Ψ to the identity (†) yields

$$mT = Tm \;\text{ for all } m \in \mathscr{M}. \tag{‡}$$

Especially T commutes with x. By taking adjoints on both sides of (‡), we see that also T^* commutes with x.
Since Hilbert-Schmidt operators in particular are compact, T^*T is a non-zero, positive, compact operator commuting with x. Since T^*T is compact, its spectrum is a discrete subset of $[0, \infty[$ and may be ordered as a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0 and since $T^*T \neq 0$, $\lambda_n \neq 0$ for at least one $n \in \mathbb{N}$.
Let $\mathscr{H}_n$ denote the eigenspace corresponding to the eigenvalue λ_n and let P_n denote the orthogonal projection onto $\mathscr{H}_n$. Since each P_n is a spectral projection of T^*T, it commutes with x and hence $x_n := x|_{\mathscr{H}_n}$ is a bounded normal (since x is normal) operator on $\mathscr{H}_n$.
For $\lambda_n \neq 0$ the space $\mathscr{H}_n$ is a non-trivial finite-dimensional space (see e.g. [MV] Prop. 15.12) and hence $\mathscr{H}_n$ has a basis consisting of eigenvectors of x_n.
These, in particular, are non-trivial eigenvectors for x, contradicting the choice of x.

□

Definition 3.3.6. *A normal element $x \in \mathscr{M}$, satisfying the properties in Theorem 3.3.5, is called an element with* diffuse spectrum. *So, x has diffuse spectrum exactly when the associated spectral measure*

$$E : \mathcal{B}(\sigma(x)) \to \mathscr{B}(L^2(\mathscr{M})),$$

has $E(\{t\}) = 0$ for all $t \in \sigma(x)$.

As we already noted, $E(\{\lambda\}) \neq 0$ if, and only if, λ is an eigenvalue of x, so having diffuse spectrum is equivalent to having no eigenvalues.

Remark 3.3.7. *In the last part of the proof of Theorem 3.3.5 we actually proved the following fact: If $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ contains an element with diffuse spectrum, then $\mathscr{M}'$ intersects trivially with the compact operators on $\mathscr{H}$. This observation will turn out useful later.*

Corollary 3.3.8. [CS03] *If $\mathscr{M}$ is a factor of type $\mathbf{II}_1$ then $\beta_0^{(2)}(\mathscr{M}, \tau) = 0$.*

One way of proving the Corollary, is to note that each $\mathbf{II}_1$-factor contains an element with diffuse spectrum. Such an element can be constructed explicitly, as a weak limit of a sequence consisting of weighted averages of projections.
We choose a slightly shorter proof, adapting the ideas from the proof of Theorem 3.3.5.

Proof. Assume, towards a contradiction, that $\beta_0^{(2)}(\mathscr{M}, \tau) \neq 0$. Then, as in the proof of Theorem 3.3.5, there exists a non-zero $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-linear map $\varphi : \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$, vanishing at elements of the form

$$m \otimes 1 - 1 \otimes m^{\mathrm{op}} \quad \text{for } m \in \mathscr{M}.$$

Since $\varphi \neq 0$ and $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-linear, we must have $x := \varphi(1 \otimes 1) \neq 0$ and

$$(m \otimes 1)x = (1 \otimes m^{\mathrm{op}})x,$$

for all $m \in \mathscr{M}$. Consider any $n \in \mathbb{N}$. Since $\mathscr{M}$ is a $\mathbf{II}_1$-factor, we can find n equivalent, orthogonal, projections $p_1, \ldots, p_n$ with sum 1. We then get

$$x = \sum_{i=1}^{n} (p_i \otimes 1)x = \sum_{i=1}^{n} (p_i \otimes 1)^2 x = \sum_{i=1}^{n} (p_i \otimes 1)(1 \otimes p_i^{\mathrm{op}})x = \sum_{i=1}^{n} (p_i \otimes p_i^{\mathrm{op}})x.$$

From this we see that

$$\begin{aligned}
\tau \otimes \tau^{\mathrm{op}}(x^*x) &= \tau \otimes \tau^{\mathrm{op}}\Big(\sum_{i,j=1}^{n} (p_i \otimes p_i^{\mathrm{op}} x)^* (p_j \otimes p_j^{\mathrm{op}})x \Big) \\
&= \sum_{i=1}^{n} \tau \otimes \tau^{\mathrm{op}} \Big(x^* (p_i \otimes p_i^{\mathrm{op}})x \Big) \\
&= \sum_{i=1}^{n} \tau \otimes \tau^{\mathrm{op}} \Big((p_i \otimes p_i^{\mathrm{op}})^* x^* x (p_i \otimes p_i^{\mathrm{op}}) \Big) && (\tau \otimes \tau^{\mathrm{op}} \text{ trace}) \\
&\leq \sum_{i=1}^{n} \|x\|_\infty^2 \tau \otimes \tau \Big((p_i \otimes p_i^{\mathrm{op}})^* (p_i \otimes p_i^{\mathrm{op}}) \Big) && (\tau \otimes \tau \text{ state}) \\
&= \sum_{i=1}^{n} \|x\|_\infty^2 \tau(p_i) \tau^{\mathrm{op}}(p_i^{\mathrm{op}}) \\
&= \frac{\|x\|_\infty^2}{n}. && (\tau(p_i) = \tfrac{1}{n} = \tau^{\mathrm{op}}(p_i^{\mathrm{op}}))
\end{aligned}$$

Since this holds for any n, we conclude that $\tau \otimes \tau^{\mathrm{op}}(x^*x) = 0$. But since $\tau \otimes \tau^{\mathrm{op}}$ is faithful and $x \neq 0$, this is a contradiction.

□

As a consequence of Corollary 3.3.8 we see that if $\mathscr{M}$ is a factor of type $\mathbf{II}_1$, then $H_0^{(2)}(\mathscr{M},\tau)$ contains no non-trivial, finitely generated, projective, $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-submodules.
Note however, that since the extended dimension function is not faithful, we can not conclude that the zero'th L^2-homology vanishes. (Compare with Theorem 3.3.3)

Remark 3.3.9. *Corollary 3.3.8 together with Proposition 3.2.10 gives us a formula for the zero'th Betti number of any finite factor. More precisely, if $\mathscr{M}$ is a finite factor with (unique) tracial state τ, then*

$$\beta_0^{(2)}(\mathscr{M},\tau)=\begin{cases}\frac{1}{n^2}, & \text{if } \mathscr{M} \text{ is of type } \mathbf{I}_n;\\ 0, & \text{when } \mathscr{M} \text{ is of type } \mathbf{II}_1.\end{cases}$$

3.4 Betti numbers of the hyper-finite factor

In this section we introduce the so-called *hyper-finite factor* and use the the compression formula to give an estimate of its L^2-Betti numbers. The presented results (except Theorem 3.4.5) can all be found, in more or less the same form, in [KR2] Chapter 10 and 11.

Put $A_n := M_2(\mathbb{C})^{\otimes n}$ and define $\varphi_n : A_n \to A_{n+1}$ by

$$\varphi_n : x_1\otimes\cdots\otimes x_n \longmapsto x_1\otimes\cdots\otimes x_n\otimes 1,$$

where 1 denotes the unit matrix in $M_2(\mathbb{C})$. Note, that each φ_n is an injective and unital $*$-algebra-homomorphism and let φ_{mn} denote the composition $\varphi_m\circ\varphi_{m-1}\circ\cdots\circ\varphi_n$ for $m\geq n$. The system (A_n,φ_n) defines a directed system of C^*-algebras and we denote by $\mathscr{A}$ its inductive limit and by $(\alpha_n)_{n\in\mathbb{N}}$ the corresponding sequence of (injective, unital) $*$-algebra-homomorphisms from A_n into $\mathscr{A}$. Recall, that $\mathscr{A}$ is the norm-closure of $\cup_n \mathrm{rg}\,(\alpha_n)$ (see e.g. [KR2] Proposition 11.4.1) and denote by $\mathscr{A}_n$ the range $\mathrm{rg}\,(\alpha_n)$. Let tr denote the (unique) trace-state on $M_2(\mathbb{C})$. Then

$$\mathrm{tr}^{\otimes n} := \mathrm{tr}\otimes\cdots\otimes\mathrm{tr}$$

is a trace-state on A_n and since each A_n is a finite factor, this trace-state is unique. Hence each $\mathscr{A}_n$ has a unique trace-state, which we will also denote by $\mathrm{tr}^{\otimes n}$.
For $m\geq n$ we have

$$\begin{aligned}\mathrm{tr}^{\otimes m}\circ\varphi_{mn}(x_1\otimes\cdots\otimes x_n) &= \mathrm{tr}^{\otimes m}(x_1\otimes\cdots\otimes x_n\otimes 1\otimes\cdots\otimes 1)\\ &= \mathrm{tr}(x_1)\cdots\mathrm{tr}(x_n)\mathrm{tr}(1)\cdots\mathrm{tr}(1)\\ &= \mathrm{tr}(x_1)\cdots\mathrm{tr}(x_n)\\ &= \mathrm{tr}^{\otimes n}(x_1\otimes\cdots\otimes x_n).\end{aligned}$$

Because of this, we can define a functional τ on the algebra $\cup_n\mathscr{A}_n$ by setting

$$\tau(\alpha_n(a)) := \mathrm{tr}^{\otimes n}(\alpha_n(a)).$$

Since each $\mathrm{tr}^{\otimes n}$ is a state, and hence of norm 1, τ is a bounded linear functional and has therefore a bounded extension (also denoted by τ) to all of $\mathscr{A}$, of norm at most 1. Since $\tau(1)=\mathrm{tr}(1)=1$, we have that

$$\|\tau\| = 1 = \tau(1),$$

and this implies that τ is a state. (see e.g. [KR1] Theorem 4.3.2) Any other trace-state on $\mathscr{A}$ restricts to a trace-state on $\mathscr{A}_n$ and must therefore coincide with $\mathrm{tr}^{\otimes n}$ here. Thus, τ is the only trace-state on $\mathscr{A}$.

Proposition 3.4.1. *The inductive limit $\mathscr{A}$ is a simple C^*-algebra.*

Proof. Assume $\mathscr{J}$ to be a proper, closed, two-sided ideal in $\mathscr{A}$ and let π denote the quotient-mapping $\mathscr{A} \to \mathscr{A}/\mathscr{J}$. Since any matrix-algebra $M_k(\mathbb{C})$ is simple, each $\mathscr{A}_n$ is simple and hence we have either $\mathscr{A}_n \cap \mathscr{J} = \{0\}$ or $\mathscr{A}_n \cap \mathscr{J} = \mathscr{A}_n$. Since $\mathscr{A}_n$ contains the unit of $\mathscr{A}$, and $\mathscr{J}$ is a proper ideal, the latter case is impossible and we therefore have $\mathscr{A}_n \cap \mathscr{J} = \{0\}$.
This implies, that the restriction $\pi|_{\mathscr{A}_n} : \mathscr{A}_n \longrightarrow \mathscr{A}/\mathscr{J}$ is an injective $*$-algebra-homomorphism and hence an isometry. Hence $\pi|_{\cup_n \mathscr{A}_n}$ is an isometry and since $\cup_n \mathscr{A}_n$ is dense in $\mathscr{A}$, π is an isometry — in particular injective. Thus $\mathscr{J} = \{0\}$. □

Corollary 3.4.2. *The trace-state τ on $\mathscr{A}$ is faithful.*

Proof. Put $J = \{a \in \mathscr{A} | \tau(a^*a) = 0\}$. For any $a, b \in \mathscr{A}$ we have

$$(a+b)^*(a+b) + (a-b)^*(a-b) = 2(a^*a + b^*b),$$

and hence $(a+b)^*(a+b) \leq 2(a^*a + b^*b)$ and from this it follows that J is a subspace of $\mathscr{A}$.
Moreover, since $a^*b^*ba \leq \|b\|^2 a^*a$ it follows that J is a left ideal and since τ is tracial, the same relation implies that J is a right ideal.
So, if $J \neq 0$ its norm-closure $\mathscr{J}$ is a non-trivial, closed, two-sided ideal in $\mathscr{A}$, which then have to be all of $\mathscr{A}$. But this is not possible, since for instance $1 = \tau(1^*1)$, and we conclude that τ is faithful. □

Proposition 3.4.3. *The C^*-algebra $\mathscr{A}$ can be faithfully represented as a strongly dense subalgebra of a $\mathbf{II}_1$-factor.*

Proof. Let π_τ be the GNS-representation of $\mathscr{A}$ on $L^2(\mathscr{A}) := L^2(\mathscr{A}, \tau)$. Then, since $\mathscr{A}$ is simple, π_τ is a faithful representation and we denote by $\mathscr{M}$ the strong closure of $\pi_\tau(\mathscr{A})$.
We now wish to show that $\mathscr{M}$ is a factor of type $\mathbf{II}_1$.
By Lemma 3.1.2, τ extends to a faithful, normal, tracial state on $\mathscr{M}$ and thus $\mathscr{M}$ is finite.
Let ρ be the (unique, normal) center-valued trace on $\mathscr{M}$ and assume that p is a non-zero central projection in $\mathscr{M}$. Choose moreover a unit vector $x \in \operatorname{rg}(p)$ and an arbitrary unit vector $y \in L^2(\mathscr{A})$ and let ω_x and ω_y denote the corresponding (normal) vector-states.
Note, that $\omega_x \circ \rho$ and $\omega_y \circ \rho$ are both normal, tracial, states on $\mathscr{M}$. Since $\mathscr{A}$ admits only one trace-state, the restrictions of $\omega_x \circ \rho$ and $\omega_y \circ \rho$ to $\mathscr{A}$ must coincide and since $\mathscr{A}$ is ultra-weakly dense in $\mathscr{M}$, the normality of the two states implies that they must agree on all of $\mathscr{M}$.
We have

$$\begin{aligned}\omega_x \circ \rho(p) &= \langle \rho(p)x | x \rangle = \langle px | x \rangle = \langle x | x \rangle = 1 \\ \omega_y \circ \rho(p) &= \langle \rho(p)y | y \rangle = \langle py | y \rangle\end{aligned}$$

Since y was an arbitrary unit vector, we conclude from this that $\langle py | y \rangle = 1$ for every unit vector y and hence

$$\|(1-p)y\|_2^2 = \langle (1-p)y | (1-p)y \rangle = \langle (1-p)y | y \rangle = 1 - 1 = 0.$$

Thus, $p = 1$ and hence $\mathscr{M}$ is a factor. Since $\mathscr{A}$, and hence $\mathscr{M}$, contains matrix-algebras of arbitrarily large sizes, $\mathscr{M}$ can not be of type $\mathbf{I}_n$ for any $n \in \mathbb{N}$ and we conclude that $\mathscr{M}$ is type $\mathbf{II}_1$. □

The factor $\mathscr{M}$ constructed above, is called *the hyper finite factor*. One can show, that every other finite factor, containing an increasing family of matrix-algebras, are isomorphic to the one constructed here. Hence the word "the" in the name "the hyperfinite factor". This however, is not particular relevant at this moment. See e.g. [KR2] Theorem 12.2.1.

Theorem 3.4.4. *The hyperfinite factor $\mathscr{M}$ is isomorphic to $M_2(\mathscr{M})$.*

Proof. Consider again the directed system (A_n, φ_n) from the beginning of this section and put $B_n := A_{n+1}$ and $\psi_n = \varphi_{n+1}$, so that $B_n = M_2(\mathbb{C}) \otimes A_n$.
Then (B_n, ψ_n) forms another directed system of C^*-algebras and we denote by $\mathscr{B}$ its inductive limit and by (β_n) the associated family of (injective, unital) $*$-algebra-homomorphisms from B_n into $\mathscr{B}$.
A straight forward argument, using the uniqueness of inductive limits, reveals the following two facts.

- The inductive limit $\mathscr{B}$ is isomorphic to $\mathscr{A}$.
- The inductive limit $\mathscr{B}$ is isomorphic to $M_2(\mathbb{C}) \otimes \mathscr{A}$.

(Note, that there is no ambiguity in writing $M_2(\mathbb{C}) \otimes \mathscr{A}$, since $M_2(\mathbb{C})$ is nuclear.) Because of this, we get an isomorphism of $\mathscr{A}$ with $M_2(\mathbb{C}) \otimes \mathscr{A}$. Let $\mathscr{N}$ be the von Neumann algebra generated by $M_2(\mathbb{C}) \otimes \mathscr{A}$ in $\mathscr{B}(\mathbb{C}^2 \otimes L^2(\mathscr{A}))$. Clearly $\mathscr{N}$ contains $M_2(\mathbb{C}) \otimes \mathscr{M}$ and since the latter is a von Neumann algebra we get that $\mathscr{N} = M_2(\mathbb{C}) \otimes \mathscr{M}$. In particular, $\mathscr{N}$ is a factor of type $\mathbf{II}_1$.
We now need to see that $\mathscr{M}$ and $\mathscr{N}$ are isomorphic.
Let σ denote the unique tracial state on $\mathscr{N}$, let π_σ be the GNS-representation of $\mathscr{N}$ on $L^2(\mathscr{N}, \sigma)$ and let ξ_σ denote the cyclic trace-vector corresponding to the unit in $\mathscr{N}$.
We now prove, that the image $\pi_\sigma(M_2(\mathbb{C}) \otimes \mathscr{A})$ is strongly dense in $\pi_\sigma(\mathscr{N})$. To see this, let $\pi_\sigma(n) \in \pi_\sigma(\mathscr{N})$ be given. Since $M_2(\mathbb{C}) \otimes \mathscr{A}$ is strongly dense in $\mathscr{N}$, we can find a bounded net (x_α) in $M_2(\mathbb{C}) \otimes \mathscr{A}$ converging strongly to n. (the Kaplansky density Theorem)
Since π_σ is a $*$-algebra-isomorphism, it is strongly continuous on bounded sets and hence $(\pi_\sigma(x_\alpha))$ converges strongly to $\pi_\sigma(n)$.
From this it follows, that $\pi_\sigma(M_2(\mathbb{C}) \otimes \mathscr{A})\xi_\sigma$ is dense in $L^2(\mathscr{N}, \sigma)$. To see this, it suffices to check that $\pi_\sigma(M_2(\mathbb{C}) \otimes \mathscr{A})\xi_\sigma$ is dense in $\pi_\sigma(\mathscr{N})\xi_\sigma$. By what was just proven, for any $\pi_\sigma(n) \in \pi_\sigma(\mathscr{N})$ we can find a net (x_α) in $M_2(\mathbb{C}) \otimes \mathscr{A}$ such that $(\pi_\sigma(x_\alpha))$ converges strongly to $\pi_\sigma(n)$. In particular $(\pi_\sigma(x_\alpha)\xi_\sigma)$ converges to $\pi_\sigma(n)\xi_\sigma$.

Since $\mathscr{A} \simeq M_2(\mathbb{C}) \otimes \mathscr{A}$, we may consider the restrictions of π_τ and π_σ as two different representations π_1, π_2 of the C^*-algebra $\mathscr{A}$, each having a cyclic vector $\xi_1 := \xi_\tau$ and $\xi_2 := \xi_\sigma$ respectively. Since ξ_1 and ξ_2 are unit trace-vectors for $\mathscr{M} = W^*(\pi_1(\mathscr{A}))$ and $\pi_2(\mathscr{N})$ respectively, we must have

$$\omega_{\xi_1} \circ \pi_1 = \omega_{\xi_2} \circ \pi_2, \qquad (\dagger)$$

since the trace-state on $\mathscr{A}$ is unique. We now claim, that the map $\pi_1(a)\xi_1 \mapsto \pi_2(a)\xi_2$ $(a \in \mathscr{A})$ is well-defined and extends to a unitary from $L^2(\mathscr{A}, \tau)$ to $L^2(\mathscr{N}, \sigma)$.
To see this, we first note, that for any $a \in \mathscr{A}$ we have

$$\begin{aligned}
\|\pi_1(a)\xi_1\|_2^2 &= \langle \pi_1(a)\xi_1 \,|\, \pi_1(a)\xi_1 \rangle \\
&= \langle \pi_1(a^*a)\xi_1 \,|\, \xi_1 \rangle \\
&= \omega_{\xi_1} \circ \pi_1(a^*a) \\
&= \omega_{\xi_2} \circ \pi_2(a^*a) \qquad \text{(By } (\dagger)\text{)} \\
&= \|\pi_2(a)\xi_2\|_2^2.
\end{aligned}$$

From this it follows that $U : \pi_1(\mathscr{A})\xi_1 \to \pi_2(\mathscr{A})\xi_2$ given by $\pi(a)\xi_1 \mapsto \pi_2(a)\xi_2$ is well-defined and isometric. Since $\pi_1(\mathscr{A})\xi_1$ is dense in $L^2(\mathscr{A}, \tau)$ and $\pi_2(\mathscr{A})\xi_2$ is dense in $L^2(\mathscr{N}, \sigma)$, U extends to an isometri (and hence unitary) from $L^2(\mathscr{A}, \tau)$ onto $L^2(\mathscr{N}, \sigma)$.
For any $a, b \in \mathscr{A}$ we have

$$\begin{aligned}
U\pi_1(b)(\pi_1(a)\xi_1) &= U\pi_1(ba)\xi_1 \\
&= \pi_2(ba)\xi_2 \\
&= \pi_2(b)\pi_2(a)\xi_2 \\
&= \pi_2(b)U(\pi_1(a)\xi_1),
\end{aligned}$$

and, by density of $\pi_1(\mathscr{A})\xi_1$, we have $U\pi_1(b) = \pi_2(b)U$.
So, U is unitary which intertwines the two representations of $\mathscr{A}$. In particular it gives rise to a $*$-algebra-isomorphism (Ad_{U^*}) from $\mathscr{M} = W^*(\pi_1(\mathscr{A}))$ to $W^*(\pi_2(\mathscr{A})) = \pi_\sigma(\mathscr{N})$.
Thus $\mathscr{M}$ and $\mathscr{N}$ are $*$-algebra-isomorphic and since $\mathscr{N} = M_2(\mathbb{C}) \otimes \mathscr{M} \simeq M_2(\mathscr{M})$, the proof is complete. □

With this information about the hyper-finite factor at our disposal, we are now able to (partially) compute its L^2-Betti numbers. The claim is the following.

Theorem 3.4.5. *Each of the L^2-Betti numbers of the hyper-finite factor $\mathscr{M}$ is either zero or infinite.*

Proof. Let $n \in \mathbb{N}_0$ be given and let τ denote the unique tracial state on $\mathscr{M}$. We wish to prove that $\beta_n^{(2)}(\mathscr{M}, \tau) \in \{0, \infty\}$
By Theorem 3.4.4, there exists a $*$-algebra-isomorphism $\varphi : \mathscr{M} \to M_2(\mathscr{M})$. Let $\tilde{\tau}$ be the trace-state on $M_2(\mathscr{M})$ given by

$$\tilde{\tau} : \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \longmapsto \frac{1}{2}(\tau(m_{11})) + \tau(m_{22})),$$

and note that $\tilde{\tau} \circ \varphi = \tau$, since the trace-state on $\mathscr{M}$ is unique.
We now consider the projection $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathscr{M})$ of trace $\frac{1}{2}$.
By the compression formula (Theorem 3.2.8) we have

$$\beta_n^{(2)}(pM_2(\mathscr{M})p, \frac{1}{\tilde{\tau}(p)}\tilde{\tau}|_{pM_2(\mathscr{M})p}) = \frac{1}{(\tilde{\tau}(p))^2}\beta_n^{(2)}(M_2(\mathscr{M}), \tilde{\tau}) = 4\beta_n^{(2)}(M_2(\mathscr{M}), \tilde{\tau}). \qquad (*)$$

Since $\varphi : \mathscr{M} \to M_2(\mathscr{M})$ is an isomorphism with $\tau = \tilde{\tau} \circ \varphi$ we also get

$$\beta_n^{(2)}(\mathscr{M}, \tau) = \beta_n^{(2)}(M_2(\mathscr{M}), \tilde{\tau}). \qquad (**)$$

On the other hand, the compressed algebra $pM_2(\mathscr{M})p$ is isomorphic to $\mathscr{M}$ (via $\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \mapsto m$) and under this isomorphism $\frac{1}{\tilde{\tau}(p)}\tilde{\tau}|_{pM_2(\mathscr{M})p}$ corresponds to τ. From this we get

$$\beta_n^{(2)}(pM_2(\mathscr{M})p, \frac{1}{\tilde{\tau}(p)}\tilde{\tau}|_{pM_2(\mathscr{M})p}) = \beta_n^{(2)}(\mathscr{M}, \tau). \qquad (***)$$

Substituting with $(**)$ and $(***)$ in the equation $(*)$ we arrive at the formula

$$\beta_n^{(2)}(\mathscr{M}, \tau) = 4\beta_n^{(2)}(\mathscr{M}, \tau),$$

and conclude that $\beta_n^{(2)}(\mathscr{M}, \tau)$ is either zero or infinite. □

Note, that by Corollary 3.3.8 we must have $\beta_0^{(2)}(\mathscr{M}, \tau) = 0$.

Remark 3.4.6 (The Fundamental Group). *Consider a* $\mathbf{II}_1$*-factor $\mathscr{M}$ and let τ denote the trace-state on $\mathscr{M}$. We then define*

$$\mathcal{F}_0 := \{\tau(p) | p \in \mathscr{M} \text{ is a projection with } p\mathscr{M}p \simeq \mathscr{M}\}.$$

We now prove that $\mathcal{F}_0$ is a multiplicative sub-semi-group in]0,1]. Let $\lambda, \mu \in \mathcal{F}_0$ and choose projections $p, q \in \mathscr{M}$, satisfying the isomorphism-condition, with $\tau(p) = \lambda$ and $\tau(q) = \mu$.
Choose a $*$-algebra-isomorphism $\alpha : \mathscr{M} \to p\mathscr{M}p$. Then $\alpha(q)$ is a subprojection of p and

$$\lambda = \tau(q) = \frac{1}{\tau(p)}\tau(\alpha(q)) = \frac{1}{\mu}\tau(\alpha(q)).$$

To see that $\lambda\mu \in \mathcal{F}_0$, it is therefore sufficient to prove that $\alpha(q)\mathscr{M}\alpha(q) \simeq \mathscr{M}$.
But this follows, since

$$\mathscr{M} \simeq q\mathscr{M}q \simeq \alpha(q)\alpha(\mathscr{M})\alpha(q) = \alpha(q)p\mathscr{M}p\alpha(q) = \alpha(q)\mathscr{M}\alpha(q),$$

where the last equality comes from the fact that $\alpha(q) \leq p$.
Clearly $1 \in \mathcal{F}_0$ and hence $\mathcal{F}_0$ is a semi-group.
We denote by $\mathcal{F}$ the subgroup in $(\mathbb{R}_+, \cdot)$ generated by $\mathcal{F}_0$. The group $\mathcal{F}$ is called the fundamental group *of the factor $\mathscr{M}$.*
Note, that if $\mathscr{M}$ has non-trivial fundamental group, then each of the L^2-Betti numbers of $\mathscr{M}$ is either zero or infinite, since the proof of Theorem 3.4.5 applies in this case.
In more details; if $p \in \mathscr{M}$ is a projection of trace $\alpha \neq 1$ with $p\mathscr{M}p \simeq \mathscr{M}$, then the Compression formula yields

$$\beta_n^{(2)}(p\mathscr{M}p, \frac{1}{\alpha}\tau|_{p\mathscr{M}p}) = \frac{1}{\alpha^2}\beta_n^{(2)}(\mathscr{M}, \tau).$$

If $\varphi : \mathscr{M} \to p\mathscr{M}p$ is an isomorphism, then (since $\mathscr{M}$ is a factor) we must have $\frac{1}{\alpha}\tau|_{p\mathscr{M}p} = \tau \circ \varphi^{-1}$ and hence

$$\beta_n^{(2)}(p\mathscr{M}p, \frac{1}{\alpha}\tau|_{p\mathscr{M}p}) = \beta_n^{(2)}(\mathscr{M}, \tau).$$

Since $\alpha \neq 1$, we conclude from this that $\beta_n^{(2)}(\mathscr{M}, \tau) \in \{0, \infty\}$. However, by Corollary 3.3.8 we must have $\beta_0^{(2)}(\mathscr{M}, \tau) = 0$.

3.5 The first Betti number — Part I

In this section we focus on the first L^2-Betti number of a finite von Neumann algebra $\mathscr{M}$, with faithful, normal tracial state τ. We construct a family of "small" complexes, whose homology-groups forms an inductive system with $H_1^{(2)}(\mathscr{M}, \tau)$ as its inductive limit. Using this, we give some estimates concerning the first L^2-Betti number in some special cases.
Note, that we already gave one such description of $H_1^{(2)}(\mathscr{M}, \tau)$ in Proposition 3.2.11. Since we are only concerned with the first L^2-homology, we can (as we shall see in a moment) do with a simpler family of complexes, than the ones constructed in the proof of Proposition 3.2.11.

Consider a finite set $F := \{x_1, \ldots, x_n\} \subseteq \mathscr{M}$ and denote by A the algebra generated by F. Note, that we do not (yet) require A to be stable under the involution $*$.
Define $C_1(F) := \mathscr{M} \otimes \mathscr{M}^{\text{op}} \otimes \text{span}_{\mathbb{C}}(F)$ and $d_F : C_1(F) \to \mathscr{M} \otimes \mathscr{M}^{\text{op}}$ by

$$d_F(T \otimes a) = T(a \otimes 1 - 1 \otimes a^{\text{op}}).$$

Clearly d_F commutes with the left action of $\mathscr{M} \otimes \mathscr{M}^{\text{op}}$, so we get a complex of left $\mathscr{M} \otimes \mathscr{M}^{\text{op}}$-modules

$$0 \longrightarrow \ker(d_F) \stackrel{\iota}{\longrightarrow} C_1(F) \stackrel{d_F}{\longrightarrow} \mathscr{M} \otimes \mathscr{M}^{\text{op}} \longrightarrow 0,$$

where ι denote the inclusion map. We now apply the functor $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \otimes_{\mathscr{M}^e} -$ to this complex and get the following complex of left $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-modules.

$$0 \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \otimes_{\mathscr{M}^e} \ker(d_F) \stackrel{1\otimes\iota}{\longrightarrow} \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \otimes_{\mathscr{M}^e} C_1(F) \stackrel{1\otimes d_F}{\longrightarrow} \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \otimes_{\mathscr{M}^e} \mathscr{M} \otimes \mathscr{M}^{\text{op}} \longrightarrow 0.$$

We then define

$$H(F) := \frac{\ker(1 \otimes d_F)}{\text{rg}\,(1 \otimes \iota)} \qquad \text{and} \qquad \beta(F) := \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(H(F)),$$

where the dimension function $\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\cdot)$ is the one arising from the trace-state $\tau \otimes \tau^{\text{op}}$ on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$.

Lemma 3.5.1. [CS03] *If A (the algebra generated by F) is unital, then* $\operatorname{coker}(d_F) := \mathscr{M}^{\mathrm{e}}/\operatorname{rg}(d_F)$ *and $\mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A$ are isomorphic as left $\mathscr{M}^{\mathrm{e}}$-modules.*

Proof. We apply the technique from Lemma 3.3.1. A direct calculation shows, that the $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$-linear map $\varphi : \mathscr{M}^{\mathrm{e}} \longrightarrow \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A$ given by $T \mapsto T \otimes 1$ is well-defined and vanishes on the elements

$$x_i \otimes 1 - 1 \otimes x_i^{\mathrm{op}} \quad \text{for } i \in \{1, \dots, n\}.$$

Since $\operatorname{rg}(d_F)$ is the ideal in $\mathscr{M}^{\mathrm{e}}$ generated by these elements, φ factorizes through a morphism

$$\tilde{\varphi} : \mathscr{M}^{\mathrm{e}}/\operatorname{rg}(d_F) \longrightarrow \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A.$$

As in Lemma 3.3.1, we wish to define a map $\psi : \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A \longrightarrow \mathscr{M}^{\mathrm{e}}/\operatorname{rg}(d_F)$ by the relation

$$\psi : T \otimes a \longmapsto [T(a \otimes 1)],$$

where the bracket indicates the coset in $\mathscr{M}^{\mathrm{e}}/\operatorname{rg}(d_F)$ represented by $T(a \otimes 1)$.
If ψ is well-defined, it is clearly a two-sided inverse of $\tilde{\varphi}$ and in this case the proof is complete. So, we have to prove that ψ is well-defined. Consider the map $\psi_0 : \mathscr{M}^{\mathrm{e}} \times A \longrightarrow \mathscr{M}^{\mathrm{e}}/\operatorname{rg}(d_F)$ given by

$$(T, a) \longmapsto [T(a \otimes 1)].$$

This is clearly $\mathbb{C}$-bilinear and to see that ψ is well-defined we just need to prove that

$$\psi_0(T(\sum_i a_i \otimes b_i^{\mathrm{op}}), x) = \psi_0(T, \sum_i a_i x b_i),$$

for all $T \in \mathscr{M}^{\mathrm{e}}, \sum_i a_i \otimes b_i^{\mathrm{op}} \in A^{\mathrm{e}}$ and $x \in A$.
We have

$$\begin{aligned}
\psi_0(T(\sum_i a_i \otimes b_i^{\mathrm{op}}), x) - \psi_0(T, \sum_i a_i x b_i) &= \Big[T(\sum_i a_i \otimes b_i^{\mathrm{op}})(x \otimes 1) - T\Big((\sum_i a_i x b_i) \otimes 1\Big)\Big] \\
&= \sum_i \Big[T(a_i x \otimes b_i^{\mathrm{op}}) - T(a_i x b_i \otimes 1)\Big] \\
&= \sum_i \Big[T(a_i x \otimes 1)\Big(1 \otimes b_i^{\mathrm{op}} - b_i \otimes 1\Big)\Big],
\end{aligned}$$

and since d_F is $\mathscr{M}^{\mathrm{e}}$-linear, it suffices to show that $1 \otimes a^{\mathrm{op}} - a \otimes 1 \in \operatorname{rg}(d_F)$ for every $a \in A$.
By the bilinearity of the tensor-product and the additivity of d_F, it suffices to consider the case when a has the form $x_{i_1} \cdots x_{i_k}$, with $x_{i_1}, \dots, x_{i_k} \in F$.
In this case we have

$$\begin{aligned}
1 \otimes x_{i_k}^{\mathrm{op}} \cdots x_{i_1}^{\mathrm{op}} - x_{i_1} \cdots x_{i_k} \otimes 1 = {} & x_{i_1} \cdots x_{i_{k-1}} \otimes x_k^{\mathrm{op}} - x_{i_1} \cdots x_{i_k} \otimes 1 \\
& + x_{i_1} \cdots x_{i_{k-2}} \otimes x_{i_k}^{\mathrm{op}} x_{i_{k-1}}^{\mathrm{op}} - x_{i_1} \cdots x_{i_{k-1}} \otimes x_{i_k}^{\mathrm{op}} \\
& + x_{i_1} \cdots x_{i_{k-3}} \otimes x_{i_k}^{\mathrm{op}} x_{i_{k-1}}^{\mathrm{op}} x_{i_{k-2}}^{\mathrm{op}} - x_{i_1} \cdots x_{i_{k-2}} \otimes x_{i_k}^{\mathrm{op}} x_{i_{k-1}}^{\mathrm{op}} \\
& \vdots \\
& + 1 \otimes x_{i_k}^{\mathrm{op}} \cdots x_{i_1}^{\mathrm{op}} - x_{i_1} \otimes x_{i_k}^{\mathrm{op}} \cdots x_{i_2}^{\mathrm{op}}.
\end{aligned}$$

Each term in the above sum lies in $\operatorname{rg}(d_F)$ since

$$\begin{aligned}
& x_{i_1} \cdots x_{i_{k-j}} \otimes x_{i_k}^{\mathrm{op}} \cdots x_{k-j+1}^{\mathrm{op}} - x_{i_1} \dots x_{i_{k-j+1}} \otimes x_{i_k}^{\mathrm{op}} \cdots x_{i_{k-j+2}}^{\mathrm{op}} \\
&= x_{i_1} \cdots x_{i_{k-j}} \otimes x_k^{\mathrm{op}} \cdots x_{k-j+2}^{\mathrm{op}} \Big(1 \otimes x_{i_{k-j+1}}^{\mathrm{op}} - x_{i_{k-j+1}} \otimes 1\Big),
\end{aligned}$$

and the proof is complete. □

Let $\mathcal{P}_e(\mathscr{M})$ denote the system of finite subsets of $\mathscr{M}$ and order $\mathcal{P}_e(\mathscr{M})$ by inclusion. If $F, G \in \mathcal{P}_e(\mathscr{M})$ with $F \subseteq G$, this inclusion extends to an inclusion

$$C_1(F) := (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}) \otimes \operatorname{span}_{\mathbb{C}}(F) \xrightarrow{\varphi^{G,F}} (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}) \otimes \operatorname{span}_{\mathbb{C}}(G) =: C_1(G),$$

and since $\varphi^{G,F}(\ker(d_F)) \subseteq \ker(d_G)$ we get a morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(d_F) & \xrightarrow{\subseteq} & C_1(F) & \xrightarrow{d_F} & \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}} & \longrightarrow & 0 \\ & & \downarrow \varphi^{G,F} & & \downarrow \varphi^{G,F} & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \ker(d_G) & \xrightarrow{\subseteq} & C_1(G) & \xrightarrow{d_G} & \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}} & \longrightarrow & 0 \end{array}$$

We therefore have an induced morphism

$$\varphi_*^{G,F} : H(F) \to H(G).$$

Since each $\varphi^{G,F}$ is an inclusion, we obviously have $\varphi^{H,G} \circ \varphi^{G,F} = \varphi^{H,F}$ whenever $F \subseteq G \subseteq H$ and hence also $\varphi_*^{H,G} \circ \varphi_*^{G,F} = \varphi_*^{H,F}$. In this way $(H(F))_{F \in \mathcal{P}_e(\mathscr{M})}$ is turned into an inductive system of $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$-modules and the following holds.

Proposition 3.5.2. [CS03] *We have* $H_1^{(2)}(\mathscr{M}, \tau) = \varinjlim H(F)$ *and*

$$\beta_1^{(2)}(\mathscr{M}, \tau) = \sup_{F \in \mathcal{P}_e(\mathscr{M})} \inf_{\substack{G \in \mathcal{P}_e(\mathscr{M}) \\ G \supseteq F}} \dim_{\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}} \Big(\varphi_*^{G,F}(H(F)) \Big).$$

Proof. Consider the bar-resolution $(C_n(\mathscr{M}, \mathscr{M}^{\mathrm{e}}), b_n)_{n=0}^{\infty}$ of $\mathscr{M}$ and the maps

$$1 \otimes d_F : \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} C_1(F) \longrightarrow \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} \mathscr{M}^{\mathrm{e}}$$
$$1 \otimes b_1 : \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} C_1(\mathscr{M}, \mathscr{M}^{\mathrm{e}}) \longrightarrow \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} \mathscr{M}^{\mathrm{e}}$$

Since d_F and b_1 is given by the same formula, we get an inclusion $\ker(1 \otimes d_F) \subseteq \ker(1 \otimes b_1)$ and in particular a homomorphism

$$\psi^F : \ker(1 \otimes d_F) \longrightarrow \ker(1 \otimes b_1)/\mathrm{rg}\,(1 \otimes b_2) =: H_1^{(2)}(\mathscr{M}, \tau),$$

by composing the inclusion with the quotient-morphism.
Let $\sum_{i=1}^k T_i \otimes x_i \in (\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^{\mathrm{e}}} \ker(d_F)$ be given and consider the element

$$(1 \otimes \iota)(\sum_{i=1}^k T_i \otimes x_i) = \sum_{i=1}^k T_i \otimes \iota(x_i) \in \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} C_1(F) \subseteq \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} C_1(\mathscr{M}, \mathscr{M}^{\mathrm{e}}).$$

By exactness of the bar-resolution, each $\iota(x_i)$ has the form $b_2(y_i)$ for some $y_i \in C_2(\mathscr{M}, \mathscr{M}^{\mathrm{e}})$ and hence

$$(1 \otimes \iota)(\sum_{i=1}^k T_i \otimes x_i) = \sum_{i=1}^k T_i \otimes b_2(y_i) = 1 \otimes b_2(\sum_{i=1}^k T_i \otimes y_k).$$

Thus, ψ^F induces a homomorphism

$$\psi_*^F : H(F) := \ker(1 \otimes d_F)/\mathrm{rg}\,(1 \otimes \iota) \longrightarrow H_1^{(2)}(\mathscr{M}, \tau).$$

Since ψ_*^F is induced by the inclusion $\ker(1\otimes d_F)\subseteq \ker(1\otimes b_1)$, we see that

$$\psi_*^G\circ\varphi_*^{G,F}=\psi_*^F \quad \text{when } F\subseteq G\ .$$

Because $\cup_{F\subseteq\mathscr{M}}C_1(F)=C_1(\mathscr{M},\mathscr{M}^{\mathrm{e}})$ we get $\cup_{F\subseteq\mathscr{M}}\mathrm{rg}\,(\psi_*^F)=H_1^{(2)}(\mathscr{M},\tau)$ and by the uniqueness of inductive limits the first claim in the proposition follows.

To prove the formula for the first Betti number, we prove that $\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(H(F))<\infty$ for all $F\subseteq\mathscr{M}$ and the formula then follows from Theorem 1.4.13.
For this we first note that

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}\otimes_{\mathscr{M}^{\mathrm{e}}}C_1(F))=\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}((\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})\otimes\mathrm{span}_{\mathbb{C}}(F))\leq |F|<\infty.$$

By construction, $H(F)$ fits into the short-exact sequence

$$0\longrightarrow \mathrm{rg}\,(1\otimes\iota)\longrightarrow \ker(1\otimes d_F)\longrightarrow H(F)\longrightarrow 0,$$

and by additivity (Theorem 1.4.7) of the dimension function we see that

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(H(F))\leq\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\ker(1\otimes d_F)\leq\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}((\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})\otimes_{\mathscr{M}^{\mathrm{e}}}C_1(F))<\infty.$$

□

As the above proposition shows, one way of getting access to the first Betti number, is by computing the values of the family

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\varphi_*^{G,F}(H(F))),$$

where $F\subseteq G\subseteq\mathscr{M}$. Hence the following definition.

Definition 3.5.3. *We let $H(G:F)$ denote $\varphi_*^{G,F}(H(F))$ and define*

$$\beta(G:F):=\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\varphi_*^{G,F}(H(G:F))).$$

Remark 3.5.4. *Note, that for any pair $F,G\in\mathcal{P}_e(\mathscr{M})$ with $F\subseteq G$, we have $H(G:F)\subseteq H(G)$ and therefore $\beta(G:F)\leq\beta(G)$.*
Also note, that

$$\varphi_*^{G,F}(H(F))\simeq\frac{\varphi^{G,F}(\ker(1\otimes d_F))}{\varphi^{G,F}(\ker(1\otimes d_F))\cap(\mathrm{rg}\,(1\otimes\iota_G))},$$

where ι_G denotes the inclusion $\ker(d_G)\subseteq C_1(G)$.

In order to investigate the first Betti number further, it is convenient to introduce the notion of Betti numbers of bimodule-maps. This is done in the following section.

3.6 Betti numbers of maps

We still assume that $\mathscr{M}$ is a finite von Neumann algebra, endowed with a fixed normal, faithful, tracial state τ. Denote by $L^2(\mathscr{M}),L^2(\mathscr{M}^{\mathrm{op}})$ and $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$, the Hilbert spaces arising from $\mathscr{M}$, $\mathscr{M}^{\mathrm{op}}$ and $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ respectively, in the GNS-construction with respect to τ,τ^{op} and $\tau\otimes\tau^{\mathrm{op}}$. Let $n,m\in\mathbb{N}$ be given and consider an $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$-linear map

$$f:(\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^n\longrightarrow(\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^m,$$

and the induced map

$$1 \otimes f : \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^m$$

For each $k \in \mathbb{N}$ we have a canonical isomorphism of left $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-modules
$\alpha_k : \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^k \longrightarrow (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^k$, given by

$$\alpha_k : T \otimes (x_1, \dots, x_n) \longmapsto (Tx_1, \dots, Tx_n).$$

Let $f^{vN} : (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n \to (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n$ denote the map $\alpha_m \circ (1 \otimes f) \circ \alpha_n^{-1}$ and note that f^{vN} extends the map f. Since f^{vN} is $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-linear, it extends to a bounded $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-equivariant operator[2]

$$f^{(2)} : L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n \longrightarrow L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^m.$$

Consider $\ker(f)$ as a subspace of $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n$ via the natural inclusions

$$(\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n \subseteq (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n \subseteq L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n,$$

and let $\overline{\ker(f)}$ denote its closure in $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n$.
Then $\overline{\ker(f)}$ is an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-invariant subspace of $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n$ and hence a finitely generated Hilbert $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-module. Since $f^{(2)}$ is an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-equivariant operator its kernel is also a finitely generated Hilbert $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-module and since $\overline{\ker(f)} \subseteq \ker(f^{(2)})$, also the quotient

$$\frac{\ker(f^2)}{\overline{\ker(f)}},$$

is a finitely generated Hilbert $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-module.
This allows us (See e.g. Definition 1.3.18) to define *the Betti number of f* as

$$\beta(f) := \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\frac{\ker(f^{(2)})}{\overline{\ker(f)}}\Big) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(f^{(2)})) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\overline{\ker(f)}),$$

where the last identity follows from additivity of the dimension function and Corollary 1.4.6. As usual, the dimension $\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\cdot)$ is the dimension function arising from the tensor-trace $\tau \otimes \tau^{\mathrm{op}}$ on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$.
We now aim to give an algebraic description of the Betti number of f.
Let ι denote the inclusion $\ker(f) \subseteq (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n$ and consider the complex

$$0 \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} \ker(f) \xrightarrow{1\otimes\iota} \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^n \xrightarrow{1\otimes f} (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^m \longrightarrow 0.$$

Then the following holds.

Proposition 3.6.1. [CS03] *With the above notation we have*

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(f^{(2)})) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(1 \otimes f))$$

and

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\overline{\ker(f)}) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\mathrm{rg}\,(1 \otimes \iota)).$$

In particular

$$\beta(f) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\frac{\ker(1 \otimes f)}{\mathrm{rg}\,(1 \otimes \iota)}\Big) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(1 \otimes f)) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\mathrm{rg}\,(1 \otimes \iota)).$$

[2]In the language of Chapter 1, $f^{(2)}$ is the map $\nu(f^{vN})$.

For the proof, some notation will be convenient:
Put $N := \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ and $\mathscr{N} := \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$. In the proof of Proposition 3.6.1 we will make use of the functor ν from Theorem 1.3.17 and hence it is practical to distinguish between $\mathscr{N}^n$ as a finitely generated projective $\mathscr{N}$-module and $\mathscr{N}^n$ as a subspace of $L^2(\mathscr{N})^n$. We therefore let η denote the inclusion $\mathscr{N}^n \subseteq L^2(\mathscr{N})^n$ and put $K := \eta(\ker(f))$. In this notation, the second equality in Proposition 3.6.1 becomes

$$\dim_{\mathscr{N}}(\overline{K}) = \dim_{\mathscr{N}}(\mathrm{rg}\,(1 \otimes \iota)).$$

The proof of Proposition 3.6.1 uses ideas from the proof of [Lüc98] Theorem 5.1

Proof of Proposition 3.6.1. We aim to prove that

$$\dim_{\mathscr{N}}(\ker(f^{(2)})) = \dim_{\mathscr{N}}(\ker(1 \otimes f)) \quad \text{and} \quad \dim_{\mathscr{N}}(\overline{K}) = \dim_{\mathscr{N}}(\mathrm{rg}\,(1 \otimes \iota)). \tag{$*$}$$

We first note, that the inclusion $\iota : \ker(f) \subseteq N^n$ and the natural isomorphism $\alpha_n : \mathscr{N} \otimes_N N^n \xrightarrow{\sim} \mathscr{N}^n$, gives rise to an $\mathscr{N}$-linear map $g := \alpha_n \circ (1_{\mathscr{N}} \otimes \iota) : \mathscr{N} \otimes_N \ker(f) \to \mathscr{N}^n$. Let $p \in \mathscr{B}(L^2(\mathscr{N})^n)$ denote the orthogonal projection onto $\overline{K}$. Since $\overline{K}$ is $\mathscr{N}$-invariant, the projection p is $\mathscr{N}$-equivariant and we may therefore consider the map $\nu^{-1}(p) : \mathscr{N}^n \to \mathscr{N}^n$. We now prove that

$$\mathrm{rg}\,(\nu^{-1}(p))) = \overline{\mathrm{rg}\,(g)}^{\mathrm{alg}}, \tag{$\dagger$}$$

where closure is the algebraic closure relative to $\mathscr{N}^n$. (See e.g. Definition 1.1.8)

"$\supseteq$" Let $x \in \ker(f)$ and $T \in \mathscr{N}$ be given. We then have

$$(1_{\mathscr{N}^n} - \nu^{-1}(p)) \circ g(T \otimes x) = (1_{L^2} - p)(\eta(Tx)) = T\eta(x) - Tp\eta(x) = 0,$$

where the last two equalities come from the fact that p is $\mathscr{N}$-equivariant and $px = x$.
Since $\mathscr{N}^n$ (the target-space of $1_{\mathscr{N}^n} - \nu^{-1}(p)$) is $\mathscr{N}$-projective, Lemma 1.1.9 implies that $\ker(1_{\mathscr{N}^n} - \nu^{-1}(p))$ is closed. Thus,

$$\overline{\mathrm{rg}\,(g)}^{\mathrm{alg}} \subseteq \ker(1_{\mathscr{N}^n} - \nu^{-1}(p)) = \mathrm{rg}\,(\nu^{-1}(p)).$$

"$\subseteq$" Let $\varphi \in \mathrm{Hom}_{\mathscr{N}}(\mathscr{N}^n, \mathscr{N})$ be given and assume that φ vanishes on $\mathrm{rg}\,(g)$. We need to see that φ vanishes on $\mathrm{rg}\,(\nu^{-1}(p))$; or, equivalently, that $\nu(\varphi) \circ p = 0$.
Since $\ker(\nu(\varphi))$ is closed in $L^2(\mathscr{N})^n$ and p is the projection onto $\overline{K}$, it suffices to prove that $K \subseteq \ker(\nu(\varphi))$.
But since $K \subseteq \eta(\mathrm{rg}\,(g))$, this follows from the construction of φ. We now have

$$\begin{aligned}
\dim_{\mathscr{N}}(\overline{K}) &:= \dim_{\mathscr{N}}(\nu^{-1}(\overline{K})) \\
&= \dim_{\mathscr{N}}(\mathrm{rg}\,(\nu^{-1}(p))) \\
&= \dim_{\mathscr{N}}(\overline{\mathrm{rg}\,(g)}^{\mathrm{alg}}) && \text{(by ($\dagger$))} \\
&= \dim_{\mathscr{N}}(\mathrm{rg}\,(g)) && \text{(by Thm. 1.4.7)} \\
&= \dim_{\mathscr{N}}(\alpha_n(\mathrm{rg}\,(1 \otimes \iota))) \\
&= \dim_{\mathscr{N}}(\mathrm{rg}\,(1 \otimes \iota)), && \text{(α_n isomorphism)}
\end{aligned}$$

as desired.

To prove the first identity in ($*$), we consider the following exact complex of finitely generated Hilbert $\mathscr{N}$-modules:

$$0 \longrightarrow \ker(f^{(2)}) \xrightarrow{\subseteq} L^2(\mathscr{N})^n \xrightarrow{f^{(2)}} L^2(\mathscr{N})^n.$$

Since ν^{-1} preserves exactness, (Lemma 1.4.6) we get the following exact complex of finitely generated projective $\mathscr{N}$-modules:

$$0 \longrightarrow \nu^{-1}(\ker(f^{(2)})) \overset{\nu^{-1}(\subseteq)}{\longrightarrow} \mathscr{N}^n \overset{\nu^{-1}(f^{(2)})}{\longrightarrow} \mathscr{N}^n.$$

Recalling that $\nu^{-1}(f^{(2)}) = f^{vN}$ we now get

$$\begin{aligned}\dim_{\mathscr{N}}(\ker(f^{(2)})) &:= \dim_{\mathscr{N}}(\nu^{-1}(\ker(f^{(2)}))) \\ &= \dim_{\mathscr{N}}(\ker(\nu^{-1}(f^{(2)}))) \\ &= \dim_{\mathscr{N}}(\ker(f^{vN})) \\ &= \dim_{\mathscr{N}}(\ker(1 \otimes f)),\end{aligned}$$

and the proof is complete. □

Corollary 3.6.2. [CS03] *For the map* $f : (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n \to (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^m$ *we have*

$$\beta(f) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\mathrm{Tor}_1^{\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathrm{coker}\,(f))\Big).$$

Proof. As in the above proof, we put $N := \mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$ and $\mathscr{N} := \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$. Since f is a map between free $\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}}$-modules, the exact sequence

$$N^n \overset{f}{\longrightarrow} N^m \longrightarrow \mathrm{coker}\,(f) \longrightarrow 0,$$

may be extended to a resolution

$$\cdots \longrightarrow N^{(X_l)} \overset{f_l}{\longrightarrow} N^{(X_{l-1})} \overset{f_{l-1}}{\longrightarrow} \cdots\cdots \overset{f_3}{\longrightarrow} N^{(X_2)} \overset{f_2}{\longrightarrow} N^n \overset{f}{\longrightarrow} N^m,$$

of $\mathrm{coker}\,(f)$ by free N-modules. We now apply $\mathscr{N} \otimes_N -$ to this resolution and get

$$\cdots \longrightarrow \mathscr{N} \otimes_N N^{(X_l)} \overset{1\otimes f_l}{\longrightarrow} \cdots\cdots \overset{1\otimes f_3}{\longrightarrow} \mathscr{N} \otimes_N N^{(X_2)} \overset{1\otimes f_2}{\longrightarrow} \mathscr{N} \otimes_N N^n \overset{1\otimes f}{\longrightarrow} \mathscr{N} \otimes_N N^m.$$

By definition of Tor, we have

$$\mathrm{Tor}_1^N(\mathscr{N}, \mathrm{coker}\,(f)) = \frac{\ker(1 \otimes f)}{\mathrm{rg}\,(1 \otimes f_2)},$$

and by exactness of the free resolution we get

$$\begin{aligned}\mathrm{rg}\,(1 \otimes f_2) &= \{\sum_i T_i \otimes f_2(x_i) \mid T_i \in \mathscr{N}, x_i \in N^{(X_2)}\} \\ &= \{\sum_i T_i \otimes z_i \mid T_i \in \mathscr{N}, z_i \in \ker(f)\} \\ &= \mathrm{rg}\,(1 \otimes \iota),\end{aligned}$$

where ι as usual denotes the inclusion $\ker(f) \subseteq N^n$. The claim now follows from Proposition 3.6.1. □

Remark 3.6.3. *Let, as before,* ι *denote the inclusion* $\ker(f) \subseteq (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})$ *and consider the complex*

$$0 \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^e} \ker(f) \overset{1\otimes\iota}{\longrightarrow} \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^e} (\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^n \overset{1\otimes f}{\longrightarrow} (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^e} (\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^m \longrightarrow 0$$

from the beginning of this section. Recall that $f^{vN} : (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n \to (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^m$ *is the map induced by* $1 \otimes f$ *by applying the natural isomorphism* α_n *(respectively* α_m*) on the source-space (respectively target-space) of* $1 \otimes f$.
We have

$$\begin{aligned}\alpha_n(\mathrm{rg}\,(1 \otimes \iota)) &= \alpha_n\{\sum_i T_i \otimes x_i \mid T_i \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, x_i \in \ker(f)\} \\ &= \{\sum_i T_i x_i \mid T_i \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, x_i \in \ker(f)\}.\end{aligned}$$

Thus, $\alpha_n(\mathrm{rg}\,(1\otimes\iota))$ *is the sub-module in* $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^n$ *generated by elements of the form* Tx*, where* $x \in \ker(f)$ *and* $T \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$*. We denote this submodule by* $[\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(f)]$.
In this notation, the last equation in Proposition 3.6.1 now has the form:

$$\beta(f) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\frac{\ker(f^{vN})}{[\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(f)]}\Big).$$

This description of $\beta(f)$ *will turn out practical later.*

The following two results are not of particular importance at the moment, but they will be needed in the following section. We present them here, to avoid to many digressions later.

Lemma 3.6.4. [CS03] *For any finite subset* $E \subseteq L^2(\mathscr{M})$ *and any* $\varepsilon > 0$*, there exists a projection* $p \in \mathscr{M}$ *such that for every* $\xi \in E$ *we have*

$$p\xi \in \eta_\tau(\mathscr{M}) \text{ and } \|p\xi - \xi\|_2 \leq \varepsilon.$$

We omit the proof, since it requires some techniques from the theory of unbounded operators, not available within the context of this text. See e.g. [CS03] Lemma 2.15.

Proposition 3.6.5. [CS03] *Let* W *denote the algebraic tensor product* $L^2(\mathscr{M}) \otimes L^2(\mathscr{M}^{\mathrm{op}})$ *and let "over-lining" denote closure in the norm from* $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}}) = L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$*. Then*

$$\overline{\ker(f^{(2)}) \cap W^n} = \overline{\ker(f)},$$

and hence

$$\beta(f) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(f^{(2)})) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\overline{\ker(f^{(2)}) \cap W^n}).$$

Proof. By definition of $\beta(f)$, the second equality follows from the first.
To prove the first identity, we note that $\ker(f) = \ker(f^{(2)}) \cap (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n$ and since

$$(\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n \subseteq (L^2(\mathscr{M}) \otimes L^2(\mathscr{M}^{\mathrm{op}}))^n =: W^n,$$

the inclusion "$\supseteq$" is clear.
For the opposite inclusion, we consider an arbitrary vector $\xi \in \ker(f^{(2)}) \cap W^n$ and let $\varepsilon > 0$ be given. We then need to find a vector $\xi' \in \ker(f^{(2)}) \cap (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n$ such that $\|\xi - \xi'\|_2 < \varepsilon$.
Since $\xi \in W^n$, it has the form $(\xi_1, \ldots, \xi_n)$ with $\xi_1, \ldots, \xi_n \in W$. Each ξ_i therefore has the form $\sum_j \xi_{ij} \otimes \eta_{ij}$ with $\xi_{ij} \in L^2(\mathscr{M})$ and $\eta_{ij} \in L^2(\mathscr{M}^{\mathrm{op}})$. Note, that for any pair of projections $p, q \in \mathscr{M}$

we have

$$\begin{aligned}\|p \otimes q^{\mathrm{op}}\xi - \xi\|_2 &= \sqrt{\sum_{i=1}^{n} \|p \otimes q^{\mathrm{op}}\xi_i - \xi_i\|_2^2} \\ &\le \sum_{i=1}^{n} \|p \otimes q^{\mathrm{op}}\xi_i - \xi_i\|_2 \\ &\le \sum_{i=1}^{n}\sum_{j} \|p\xi_{i_j} \otimes q^{\mathrm{op}}\eta_{i_j} - \xi_{i_j} \otimes \eta_{i_j}\|_2 \\ &= \sum_{i=1}^{n}\sum_{j} \|p\xi_{i_j} \otimes q^{\mathrm{op}}\eta_{i_j} - \xi_{i_j} \otimes q^{\mathrm{op}}\eta_{i_j} + \xi_{i_j} \otimes q^{\mathrm{op}}\eta_{i_j} - \xi_{i_j} \otimes \eta_{i_j}\|_2 \\ &\le \sum_{i=1}^{n}\sum_{j} \|(p\xi_{i_j} - \xi_{i_j}) \otimes q^{\mathrm{op}}\eta_{i_j}\|_2 + \|\xi_{i_j} \otimes (q^{\mathrm{op}}\eta_{i_j} - \eta_{i_j})\|_2 \\ &= \sum_{i=1}^{n}\sum_{j} \|p\xi_{i_j} - \xi_{i_j}\|_2 \|q^{\mathrm{op}}\eta_{i_j}\|_2 + \|\xi_{i_j}\|_2 \|q^{\mathrm{op}}\eta_{i_j} - \eta_{i_j}\|_2\end{aligned}$$

Therefore, by applying Lemma 3.6.4 on $\{\xi_{i_j}|i,j\}$ and $\{\eta_{i_j}|i,j\}$ respectively, we may find two projections $p, q \in \mathscr{M}$ such that $p \otimes q^{\mathrm{op}}\xi \in (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n$ and $\|p \otimes q^{\mathrm{op}}\xi - \xi\|_2 < \varepsilon$.
Since $f^{(2)}$ is $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-linear, we get

$$f^{(2)}(p \otimes q^{\mathrm{op}}\xi) = p \otimes q^{\mathrm{op}} f^{(2)}(\xi) = 0,$$

and hence $p \otimes q^{\mathrm{op}}\xi \in (\mathscr{M} \otimes \mathscr{M}^{\mathrm{op}})^n \cap \ker(f^{(2)}) = \ker(f)$. Thus, $\xi' := p \otimes q^{\mathrm{op}}\xi$ has the desired properties.

□

3.7 The first Betti number — Part II

We now return to the problem of computing the first Betti number. Consider again a finite subset $F = \{x_1, \ldots, x_n\} \subseteq \mathscr{M}$. Recall, that we defined $C_1(F) := (\mathscr{M}^{\mathrm{e}}) \otimes \mathrm{span}_{\mathbb{C}}(F)$ and a map

$$C_1(F) \ni T \otimes x \overset{d_F}{\longmapsto} T(x \otimes 1 - 1 \otimes x^{\mathrm{op}}) \in \mathscr{M}^{\mathrm{e}},$$

which is the restriction of b_1 to $C_1(F) \subseteq C_1(\mathscr{M}, \mathscr{M}^{\mathrm{e}})$. Applying the functor $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^{\mathrm{e}}} -$ we get

$$1 \otimes d_F : (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M}^{\mathrm{e}}) \otimes \mathrm{span}_{\mathbb{C}}(F) \longrightarrow (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M}^{\mathrm{e}}),$$

and we denote by $d_F^{vN} : \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes \mathrm{span}_{\mathbb{C}}(F) \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ the map induced by $1 \otimes d_F$ under the natural identification of $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \otimes_{\mathscr{M}^{\mathrm{e}}} (\mathscr{M}^{\mathrm{e}})$ with $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$. (see e.g. section 3.6)
Let l denote the linear dimension of $\mathrm{span}_{\mathbb{C}}(F)$ and choose a linear basis $x_{i_1}, \ldots, x_{i_l}$ consisting of elements in F. Note, that $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes \mathrm{span}_{\mathbb{C}}(F)$ is isomorphic to $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^l$ via the map

$$T \otimes (\sum_{j=1}^{l} \mu_j x_{i_j}) \overset{\alpha}{\longrightarrow} (\mu_1 T, \ldots, \mu_l T),$$

and note also that

$$d_F^{vN} \circ \alpha^{-1}(S_1, \ldots, S_l) = \sum_{j=1}^{l} S_j(x_{i_j} \otimes 1 - 1 \otimes x_{i_j}^{\mathrm{op}}). \qquad (\dagger)$$

In the following, we will often suppress the isomorphism α and simply identify $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}\otimes\mathrm{span}_{\mathbb{C}}(F)$ with $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^l$. In exactly the same manner, we get an isomorphism

$$C_1(F) := \mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}\otimes\mathrm{span}_{\mathbb{C}}(F) \xrightarrow{\sim} (\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^l,$$

which will also be suppressed in the following. In particular, we will consider d_F as a map from $(\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^l \longrightarrow \mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$. With these identifications, we have

$$\beta(F) = \beta(d_F),$$

by Proposition 3.6.1. (see e.g. the discussion preceding Lemma 3.5.1).
This allows us to apply the results from Section 3.6. In particular, using the notation from Remark 3.6.3, we have

$$\beta(F) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\frac{\ker(d_F^{vN})}{[\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}\cdot\ker(d_F)]}\Big).$$

Throughout this section, we shall make extensive use of the isomorphism Ψ from $L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}})$ to $\mathscr{HS}(L^2(\mathscr{M}))$ introduced in Section 1.2.2.We first study its interaction with the map d_F.

Lemma 3.7.1. *Denote by Ψ_l the isomorphism*

$$(\Psi,\dots,\Psi) : (L^2(\mathscr{M})\bar{\otimes}L^2(\mathscr{M}^{\mathrm{op}}))^l \longrightarrow (\mathscr{HS}(L^2(\mathscr{M})))^l.$$

Then, for $(T_1,\dots,T_l)\in\Psi_l((\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}})^l)$, we have

$$\Psi\circ d_F\circ\Psi_l^{-1}(T_1,\dots,T_l) = \sum_{j=1}^{l}[Jx_{i_j}^*J, T_j],$$

where $J : L^2(\mathscr{M})\to L^2(\mathscr{M})$ is the conjugation from Lemma 1.2.1.

Proof. By assumption, each T_j can be written as $\Psi(S_j)$ for some $S_j\in\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$. Write S_j as $\sum_{k\in I_j} m_{j_k}\otimes n_{j_k}^{\mathrm{op}}$, for some finite index-set I_j and some $m_{j_k}, n_{j_k}\in\mathscr{M}$. The result now follows from a direct computation:

$$\begin{aligned}
\Psi\circ d_F\circ\Psi_l^{-1}(T_1,\dots,T_l) &= \Psi(\sum_{j=1}^{l} S_j\otimes(x_{i_j}\otimes 1 - 1\otimes x_{i_j}^{\mathrm{op}})) && \text{(By (†))}\\
&= \Psi(\sum_{j=1}^{l}\sum_{k\in I_j} m_{j_k}\otimes n_{j_k}^{\mathrm{op}}(x_{i_j}\otimes 1 - 1\otimes x_{i_j}^{\mathrm{op}}))\\
&= \sum_{j=1}^{l}\sum_{k\in I_j} Jx_{i_j}^*J\Psi(m_{j_k}\otimes n_{j_k}^{\mathrm{op}}) - \Psi(m_{k_j}\otimes n_{k_j}^{\mathrm{op}})Jx_{i_j}^*J && \text{(Prop. 1.2.15)}\\
&= \sum_{j=1}^{l} Jx_{i_j}^*J(\sum_{k\in I_j}\Psi(m_{j_k}\otimes n_{j_k}^{\mathrm{op}})) - (\sum_{k\in I_j}\Psi(m_{j_k}\otimes n_{j_k}^{\mathrm{op}}))Jx_{i_j}^*J\\
&= \sum_{j=1}^{l}[Jx_{i_j}^*J, T_j].
\end{aligned}$$

□

Definition 3.7.2. *We denote by $D_F : \mathscr{HS}(L^2(\mathscr{M}))^l \to \mathscr{HS}(L^2(\mathscr{M}))$ the map*

$$(T_1,\dots,T_l)\longmapsto\sum_{j=1}^{l}[Jx_{i_j}^*J, T_j].$$

The map $d_F^{vN} : (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^l \to (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$ extends d_F when $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ is considered as subspace of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ in the natural way. Since d_F^{vN} is $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-linear, it (and hence d_F) extends to a bounded $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-equivariant opeartor $d_F^{(2)} : L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^l \to L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$. By the proof of Theorem 1.2.14, the algebraic tensor product $\mathscr{M}\otimes\mathscr{M}^{\mathrm{op}}$ is dense in $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})$ and combining this with the result in Lemma 3.7.1 we get commutativity of the following diagram

$$\begin{array}{ccc} L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})^l & \xrightarrow{\Psi_l} & \mathscr{HS}(L^2(\mathscr{M}))^l \\ {\scriptstyle d_F^{(2)}}\downarrow & & \downarrow{\scriptstyle D_F} \\ L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) & \xrightarrow[\Psi]{} & \mathscr{HS}(L^2(\mathscr{M})) \end{array}$$

We now aim to prove a formula, relating the dimension of $\ker(d_F^{vN})$ to $\beta_0^{(2)}(\mathscr{M},\tau)$. (Proposition 3.7.6) For this we need the follwing two lemmas.

Lemma 3.7.3. *For fixed $S \in \mathscr{HS}(L^2(\mathscr{M}))$ the operator*

$$(\mathscr{B}(L^2(\mathscr{M})))_1 \ni T \longmapsto [T,S] \in \mathscr{HS}(L^2(\mathscr{M})),$$

is continuous from $(\mathscr{B}(L^2(\mathscr{M})))_1$ with the strong-$*$-topology, to $\mathscr{HS}(L^2(\mathscr{M}))$ with the topology induced by the Hilbert-Schmidt norm.

Proof. By linearity, it suffices to prove the continuity at zero. So, let (T_α) be a net in $(\mathscr{B}(L^2(\mathscr{M})))_1$ converging to zero in the strong-$*$-topology. We need to prove that $[T_\alpha, S]$ converges to zero in Hilbert-Schmidt norm.
Let $\varepsilon > 0$ be given. Since $\mathscr{FR}(L^2(\mathscr{M}))$ is dense in $\mathscr{HS}(L^2(\mathscr{M}))$ there exists a vector $\sum_{i=1}^n x_i \otimes y_i \in L^2(\mathscr{M}) \otimes L^2(\mathscr{M}^{\mathrm{op}})$ such that $\|S - \sum_{i=1}^n \Psi_{x_i\otimes y_i}\|_{\mathscr{HS}} < \frac{\varepsilon}{4}$. Let X denote the finite rank operator $\sum_{i=1}^n \Psi_{x_i\otimes y_i}$. We then get

$$\begin{aligned} \|[T_\alpha, S]\|_{\mathscr{HS}} &= \|[T_\alpha, S - X + X]\|_{\mathscr{HS}} \\ &= \|[T_\alpha, S - X] + [T_\alpha, X]\|_{\mathscr{HS}} \\ &\le \|T_\alpha(S-X)\|_{\mathscr{HS}} + \|(S-X)T_\alpha\|_{\mathscr{HS}} + \|[T_\alpha, X]\|_{\mathscr{HS}} \\ &\le \|T_\alpha\|_\infty \|S-X\|_{\mathscr{HS}} + \|S-X\|_{\mathscr{HS}}\|T_\alpha\|_\infty + \|[T_\alpha, X]\|_{\mathscr{HS}} \\ &\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \|[T_\alpha, X]\|_{\mathscr{HS}}. \end{aligned}$$

Note, that for any $i \in \{1, \ldots, n\}$ and $\xi \in L^2(\mathscr{M})$ we have

$$\begin{aligned} T_\alpha \Psi_{x_i\otimes y_i}(\xi) &= T_\alpha(\langle \xi \,|\, Jy_i\rangle x_i) = \langle \xi \,|\, Jy_i\rangle T_\alpha x_i = \Psi_{(T_\alpha x_i)\otimes y_i}(\xi) \\ \Psi_{x_i\otimes y_i} T_\alpha(\xi) &= \langle T_\alpha \xi \,|\, Jy_i\rangle x_i = \langle \xi \,|\, T^* Jy_i\rangle x_i = \Psi_{x_i \otimes (JT_\alpha^* Jy_i)}(\xi). \end{aligned}$$

From this it follows that

$$\begin{aligned}
\|[T_\alpha, X]\|_{\mathscr{HS}} &\leq \sum_{i=1}^{n} \|[T_\alpha, \Psi_{x_i \otimes y_i}]\|_{\mathscr{HS}} \\
&\leq \sum_{i=1}^{n} \|T_\alpha \Psi_{x_i \otimes y_i}\|_{\mathscr{HS}} + \|\Psi_{x_i \otimes y_i} T_\alpha\|_{\mathscr{HS}} \\
&= \sum_{i=1}^{n} \|\Psi_{(T_\alpha x_i) \otimes y_i}\|_{\mathscr{HS}} + \|\Psi_{x_i \otimes (J T_\alpha^* J y_i)}\|_{\mathscr{HS}} \\
&= \sum_{i=1}^{n} \|(T_\alpha x_i) \otimes y_i\|_2 + \|x_i \otimes (J T_\alpha^* J y_i)\|_2 \qquad (\Psi \text{ isometric}) \\
&= \sum_{i=1}^{n} \|T_\alpha x_i\|_2 \|y_i\|_2 + \|x_i\|_2 \|J T_\alpha^* J y_i\|_2 \\
&\leq \sup_i \{\|x_i\|_2, \|y_i\|_2\} \Big(\sum_{i=1}^{n} \|T_\alpha x_i\|_2 + \|T_\alpha^* (J y_i)\|_2 \Big) \\
&\leq \sup_i \{\|x_i\|_2, \|y_i\|_2\} \Big(\sum_{i=1}^{n} \|T_\alpha x_i\|_2 + \|T_\alpha^* x_i\|_2 + \|T_\alpha (J y_i)\|_2 + \|T_\alpha^* (J y_i)\|_2 \Big)
\end{aligned}$$

Since (T_α) converges to zero in the strong-$*$-topology, the sum in the last expression converges to zero. Hence, there exists an α_0 such that

$$\sup_i \{\|x_i\|_2, \|y_i\|_2\} \Big(\sum_{i=1}^{n} \|T_\alpha x_i\|_2 + \|T_\alpha^* x_i\|_2 + \|T_\alpha (J y_i)\|_2 + \|T_\alpha^* (J y_i)\|_2 \Big) < \frac{\varepsilon}{2},$$

when $\alpha \geq \alpha_0$. This completes the proof. □

Lemma 3.7.4. *Let $\mathscr{N}$ be any von Neumann algebra and let $\mathscr{J}$ be a strongly closed left ideal in $\mathscr{N}$. Then $\mathscr{J}$ has the form $\mathscr{N}p$ for some projection $p \in \mathscr{N}$. In particular $\mathscr{J}$ is projective, when considered a left module over $\mathscr{N}$.*

Proof. If $\mathscr{J} = \{0\}$ there is nothing to prove. So assume that $\mathscr{J}$ is non-trivial and choose a non-zero vector $x \in \mathscr{J}$. Then x^*x is also a non-zero element in $\mathscr{J}$ and since $\mathscr{J}$ is an ideal every polynomial expression, without constant term, in x^*x is in $\mathscr{J}$. Because $\mathscr{J}$ is strongly (especially uniformly) closed we get $f(x^*x) \in \mathscr{J}$ for every $f \in C(\sigma(x^*x))$ with $f(0) = 0$.
Since x^*x is non-zero, we can find an $\varepsilon > 0$ such that $e := \chi_A(x^*x) \neq 0$ where $A = [\varepsilon, \|x\|^2] \cap \sigma(x^*x)$. The characteristic function χ_A can be approximated pointwise by a (uniformly) bounded sequence of continuous functions $(f_n)_{n \in \mathbb{N}}$ with $f_n(0) = 0$ and hence $f_n(x^*x) \in \mathscr{J}$ converges strongly to e. By hypothesis, $\mathscr{J}$ is strongly closed and hence $e \in \mathscr{J}$.
This shows that $\mathscr{J}$ contains non-trivial projections and we now define p to be the union of all projections in $\mathscr{J}$ and aim to show that p has the desired property.
We first show that p belongs to $\mathscr{J}$. Since $\mathscr{J}$ is strongly closed, it suffices to prove that each finite union of projections in $\mathscr{J}$ also belongs to $\mathscr{J}$ and an easy induction-argument reduces the problem to the case of two projections.

So, let $p, q \in \mathscr{J}$ be projections, and recall that $p \vee q = R(p + q)$, where $R(p + q)$ denotes the range-projection of the (positive) operator $x := p + q$. (see e.g. [KR1] Proposition 2.5.14)
We need to prove that $R(x) \in \mathscr{J}$. Since $R(x) = R(\frac{1}{\|x\|}x)$ we may assume that $\|x\| \leq 1$ such that $x \leq 1$.
We now prove that $(x^{\frac{1}{n}})_{n \in \mathbb{N}}$ converges strongly to $R(x)$. This is sufficient since $\mathscr{J}$ is stable

under continuous functional calculus from $\{f \in C(\sigma(x)) | f(0) = 0\}$ by what is already proven and strongly closed by assumption. By considering the corresponding functions on $[0,1]$, it is easy to see that $(x^{\frac{1}{n}})_{n\in\mathbb{N}}$ is an increasing sequence of positive operators bounded from above by 1 and it therefore converges strongly to its least upper bound q.
Since $(x^{\frac{1}{n}})_{n\in\mathbb{N}}$ is a bounded sequence of pairwise commuting elements, each element in the sequence commutes with q and from this it follows that the squared sequence $(x^{\frac{2}{n}})_{n\in\mathbb{N}}$ converges strongly to q^2. But since $(x^{\frac{2}{n}})_{n\in\mathbb{N}}$ contains $(x^{\frac{1}{n}})_{n\in\mathbb{N}}$ as a subsequence (the even terms), we conclude that $q^2 = q$.
Each $x^{\frac{1}{n}}$ is self-adjoint and since the sequence $(x^{\frac{1}{n}})_{n\in\mathbb{N}}$ converges weakly to q, it follows that q is self-adjoint — and hence a projection.
We now need to see that q is the range-projection of x. Since x is self-adjoint, we see that

$$\ker(x)^{\perp} = \overline{\mathrm{rg}\,(x^*)} = \overline{\mathrm{rg}\,(x)},$$

and hence it suffices to prove that $\ker(q) = \ker(x)$.
Assume first that $x\xi = 0$ and consider a fixed $n \in \mathbb{N}$. Since the function $f : t \mapsto t^{\frac{1}{n}}$ has $f(0) = 0$, it can be approximated uniformly by a sequence of polynomials without constant terms; i.e. of the form $t \mapsto p_k(t)t$ for some polynomial sequence $(p_k)_{k\in\mathbb{N}}$. So, if $x\xi = 0$ we have $p_k(x)x\xi = 0$ for all $k \in \mathbb{N}$ and hence $x^{\frac{1}{n}}\xi = 0$. Since this holds for every $n \in \mathbb{N}$ and q is the strong operator limit of $(x^{\frac{1}{n}})_{n\in\mathbb{N}}$, we conclude that $q\xi = 0$.
Conversely, assume that $q\xi = 0$. Then, since q dominates each $x^{\frac{1}{n}}$, we get

$$0 = \langle q\xi | \xi\rangle \geq \langle x^{\frac{1}{2}}\xi | \xi\rangle = \langle x^{\frac{1}{4}}\xi | x^{\frac{1}{4}}\xi\rangle = \|x^{\frac{1}{4}}\xi\|_2,$$

and hence

$$x\xi = x^{\frac{1}{4}}x^{\frac{1}{4}}x^{\frac{1}{4}}x^{\frac{1}{4}}\xi = 0$$

Hence q is the range-projection of x and we conclude from this that $\mathscr{J}$ is stable under union of projections. Thus, $p \in \mathscr{J}$.

We now wish to see that $\mathscr{J} = \mathscr{N}p$. Since $p \in \mathscr{J}$, the inclusion "$\supseteq$" is clear. For the opposite inclusion, we assume, towards a contradiction, that there exists some element $x \in \mathscr{J} \setminus \mathscr{N}p$. Then $x - xp \in \mathscr{J}$ is nonzero and we have

$$z := (x - xp)^*(x - xp) = (1 - p)x^*x(1 - p) \in \mathscr{J} \setminus \{0\}.$$

Choose an $\varepsilon > 0$ such that $\chi_A(z) \neq 0$ where $A = [\varepsilon, \|z\|] \cap \sigma(z)$. Then, by what was proven above, $\chi_A(z) \in \mathscr{J}$ and we now prove that $\chi_A(z) \leq 1 - p$. Given this fact the proof is complete, since $\chi_A(z)$ then is a non-zero projection in $\mathscr{J}$ orthogonal to p, contradicting the construction of p.
Since 0 is isolated from A, we may choose a uniformly bounded sequence $(f_n)_{n\in\mathbb{N}} \subseteq C(\sigma(z))$ converging pointwise to χ_A and with the property that $f_n(0) = 0$ for all $n \in \mathbb{N}$.
Then each f_n can be uniformly approximated by polynomials and since $f_n(0) = 0$ we may choose the polynomials of the form $t \mapsto p_k(t)t$. (i.e. without constant term)
Because of this (and the construction of z) we see that $p_k(z)z(1-p) = p_k(z)z$ for each $k \in \mathbb{N}$ and hence the same holds for the uniform limit $f_n(z)$. Since right-multiplication with a fixed operator is strong-operator continuous, this implies that the strong limit $\chi_A(z)$ of the sequence $f_n(z)$ also fulfills the relation

$$\chi_A(z)(1 - p) = \chi_A(z),$$

and hence $\chi_A(z) \leq 1 - p$. Since $\chi_A(z) \neq 0$, this contradicts the choice of p.

□

Having these two lemmas at our disposal, we now return to the problem of computing the first Betti number. So, let $\mathscr{M}$ be a finite von Neumann algebra, endowed with a fixed, faithful, normal, tracial state τ. In the following, the dimension $\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\cdot)$ will always be computed with respect to the tensor-trace $\tau \otimes \tau^{\mathrm{op}}$ on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$.

Remark 3.7.5. *Consider two finite subsets $F, G \subseteq \mathscr{M}$ and assume $F \subseteq G$. Recall, from Section 3.5, that we defined $\varphi^{G,F}$ to be the inclusion $C_1(F) \subseteq C_1(G)$. If we apply the functor $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes_{\mathscr{M}^{\rm e}} -$, we get an induced map*

$$1 \otimes \varphi^{G,F} : \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes_{\mathscr{M}^{\rm e}} C_1(F) \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes_{\mathscr{M}^{\rm e}} C_1(G).$$

Applying the natural isomorphism $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes_{\mathscr{M}^{\rm e}} \mathscr{M}^{\rm e} \xrightarrow{\sim} \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$ on both source- and target space of $1 \otimes \varphi^{G,F}$, identifies $1 \otimes \varphi^{G,F}$ with the inclusion

$$\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes {\rm span}_{\mathbb{C}}(F) \subseteq \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes {\rm span}_{\mathbb{C}}(G).$$

To avoid getting the notation to cumbersome, we will denote also the latter inclusion by $\varphi^{G,F}$ in stead of $\varphi^{G,F^{vN}}$, which would be more consistent with the notation introduced so far.

In the rest of this section, we shall primarily consider self-adjoint[3] finite subsets of $\mathscr{M}$. Note, that if $F \subseteq \mathscr{M}$ is self-adjoint then the algebra generated by F is automatically a (possibly non-unital) $*$-subalgebra of $\mathscr{M}$.

Proposition 3.7.6. [CS03] *Let $F = \{x_1, \ldots, x_n\}$ be a finite, self-adjoint subset of $\mathscr{M}$ and let A denote the algebra generated by F. Assume that A contains the unit of $\mathscr{M}$ and is strongly dense in $\mathscr{M}$. If G is another finite self-adjoint subset containing F, then*

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(\varphi^{G,F}(\ker(d_F^{vN}))) = l - 1 + \beta_0^{(2)}(\mathscr{M}, \tau),$$

where l denote the linear dimension of ${\rm span}_{\mathbb{C}}(F)$.

Proof. The general situation follows from the special case $F = G$, since $\varphi^{G,F}$ is injective and hence an isomorphism of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}$-modules from $\ker(d_F^{vN})$ to $\varphi^{G,F}(\ker(d_F^{vN}))$.
So, we may assume $F = G$. We then need to show

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(\ker(d_F^{vN})) = l - 1 + \beta_0^{(2)}(\mathscr{M}, \tau). \qquad (\dagger)$$

Recall, that the zero'th L^2-homology of $\mathscr{M}$ can be computed as the zero'th homology of the Hochschild complex $(C_*(\mathscr{M}, \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}), b_*)$. That is, $H_0^{(2)}(\mathscr{M}, \tau) = \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}/{\rm rg}\,(b_1)$.
Using additivity of the dimension function on the associated short-exact sequence

$$0 \longrightarrow {\rm rg}\,(b_1) \longrightarrow \mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \longrightarrow H_0^{(2)}(\mathscr{M}, \tau) \longrightarrow 0,$$

we get

$$\begin{aligned}\beta_0^{(2)}(\mathscr{M}, \tau) :&= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(H_0^{(2)}(\mathscr{M}, \tau)) \\ &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}({\rm rg}\,(b_1)) \\ &= 1 - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}({\rm rg}\,(b_1)).\end{aligned}$$

If we can show that

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}({\rm rg}\,(d_F^{vN})) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}({\rm rg}\,(b_1)), \qquad (\ddagger)$$

the desired formula follows, since (‡) implies that

$$\begin{aligned}\beta_0^{(2)}(\mathscr{M}, \tau) &= 1 - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}({\rm rg}\,(b_1)) \\ &= 1 - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}({\rm rg}\,(d_F^{vN})) \\ &= 1 - \Big(\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op} \otimes {\rm span}(F)) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(\ker(d_F^{vN}))\Big) \\ &= 1 - (l - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\rm op}}(\ker(d_F^{vN}))).\end{aligned}$$

[3]I.e. for all $x_i \in F$ we have $x_i^* \in F$

We therefore aim to prove (‡). Since b_1 is $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-linear, its range is a left ideal in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$. We first prove that

$$\overline{\operatorname{rg}(b_1)}^{s} \subseteq \overline{\operatorname{rg}(b_1)}^{\text{alg}},$$

where the closure on the right-hand side is in the sense of Definition 1.1.8 (relative to $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$) and the closure on the left-hand side is in the strong operator topology on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \subseteq \mathscr{B}(L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}))$. Let $f : \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ be an $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-linear map vanishing on $\operatorname{rg}(b_1)$. Then f is given by right-multiplication with $f(1\otimes 1)$ and since multiplication from the right with a fixed operator is strong-operator continuous, f vanishes on $\overline{\operatorname{rg}(b_1)}^{s}$. Since this holds for all such f, the inclusion follows. We therefore get

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\operatorname{rg}(b_1)) \leq \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\overline{\operatorname{rg}(b_1)}^{s}) \leq \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\overline{\operatorname{rg}(b_1)}^{\text{alg}}),$$

and by continuity of the dimension function (Theorem 1.4.7), we the conclude that

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\operatorname{rg}(b_1)) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\overline{\operatorname{rg}(b_1)}^{s}).$$

Since $\overline{\operatorname{rg}(b_1)}^{s}$ is a strongly closed left ideal in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$, Lemma 3.7.4 implies that $\overline{\operatorname{rg}(b_1)}^{s}$ has the form $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}p$ for a suitable projection $p \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$.
A similar argument shows that

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\operatorname{rg}(d_F^{vN})) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}}(\overline{\operatorname{rg}(d_F^{vN})}^{s}),$$

and that $\overline{\operatorname{rg}(d_F^{vN})}^{s}$ is of the form $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}q$, for a suitable projection $q \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$.
By the results of Chapter 1 (Theorem 1.3.17), it suffices to see that $\nu(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}p, \langle\cdot|\cdot\rangle_{\text{st}})$ and $\nu(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}q, \langle\cdot|\cdot\rangle_{\text{st}})$ are isomorphic as finitely generated Hilbert $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$-modules.
By construction, $\operatorname{rg}(b_1)$ is dense in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}p$ in the strong operator topology, and hence also in the topology induced from the norm on $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})$. Thus,

$$\overline{\operatorname{rg}(b_1)}^{L^2} = \overline{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}p}^{L^2} = \nu(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}p, \langle\cdot|\cdot\rangle_{\text{st}}),$$

and similarly

$$\overline{\operatorname{rg}(b_F^{vN})}^{L^2} = \overline{\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}q}^{L^2} = \nu(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}q, \langle\cdot|\cdot\rangle_{\text{st}}).$$

We therefore have to show that

$$\overline{\operatorname{rg}(d_F^{vN})}^{L^2} = \overline{\operatorname{rg}(b_1)}^{L^2}.$$

Since $b_1 : \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}} \otimes \mathscr{M} \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ is given by

$$T \otimes a \longmapsto T(a\otimes 1 - 1\otimes a^{\text{op}}),$$

$\operatorname{rg}(b_1)$ is the left ideal in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ generated by $\{a\otimes 1 - 1\otimes a^{\text{op}} \mid a \in \mathscr{M}\}$.
We now wish to transport the problem from $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})$ into $\mathscr{HS}(L^2(\mathscr{M}))$ via the isometry Ψ from Proposition 1.2.12. For any $T \in \mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ and $a \in \mathscr{M}$, we have

$$\begin{aligned}\Psi(T(a\otimes 1 - 1\otimes a^{\text{op}})) &= \Psi(T(a\otimes 1) - T(1\otimes a^{\text{op}}) \\ &= Ja^*J\Psi(T) - \Psi(T)Ja^*J \qquad \text{(by Proposition 1.2.15)} \\ &= [Ja^*J, \Psi(T)].\end{aligned}$$

Since $a \mapsto Ja^*J$ maps $\mathscr{M}$ onto $\mathscr{M}'$ (Corollary 1.2.7) and since $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$ is dense in $L^2(\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}})$, we get

$$\Psi(\overline{\operatorname{rg}(b_1)}^{L^2}) = \overline{\operatorname{span}\{[S, m'] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), m' \in \mathscr{M}'\}}^{\mathscr{HS}}.$$

Similarly, since $d_F^{vN} = b_1|_{C_1(F)}$ we get that $\operatorname{rg}(d_F^{vN})$ is the left ideal in $\mathscr{M}\bar{\otimes}\mathscr{M}^{\text{op}}$, generated by the set

$$\{x_i \otimes 1 - 1\otimes x_i^{\text{op}} \mid x_i \in F\},$$

and applying Ψ we get

$$\Psi(\overline{\mathrm{rg}\,(d_F^{vN})}^{L^2}) = \overline{\mathrm{span}\{[S, Jx_i^* J] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), x_i \in F\}}^{\mathscr{HS}}.$$

Since $\Psi(\overline{\mathrm{rg}\,(d_F^{vN})}^{L^2}) \subseteq \Psi(\overline{\mathrm{rg}\,(b_1)}^{L^2})$, we just need to prove that

$$\mathrm{span}\{[S, Jx_i^* J] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), x_i \in F\}$$

is Hilbert-Schmidt-dense in

$$\mathrm{span}\{[S, m'] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), m' \in \mathscr{M}'\}.$$

Recall that F is assumed to generate $\mathscr{M}$ as a von Neumann algebra, so if A denotes the algebra generated by F then $\mathscr{M}$ is the strong-$*$-closure of A. Note, that by the Kaplansky density theorem (see e.g. [Brat] Theorem 2.4.16), every $m \in \mathscr{M}$ is the strong-$*$-limit of a bounded net in A.
For fixed $S \in \mathscr{HS}(L^2(\mathscr{M}))$ the map

$$(\mathscr{B}(L^2(\mathscr{M})))_1 \ni T \longmapsto [S, T] \in \mathscr{HS}(L^2(\mathscr{M})),$$

is (strong-$*$, Hilbert-Schmidt)-continuous by Lemma 3.7.3 and hence

$$\overline{\mathrm{span}\{[S, Ja^* J] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), a \in A\}}^{\mathscr{HS}} = \overline{\mathrm{span}\{[S, m'] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), m' \in \mathscr{M}'\}}^{\mathscr{HS}}.$$

It is thus sufficient to prove that

$$\mathrm{span}\{[S, Ja^* J] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), a \in A\} = \mathrm{span}\{[S, Jx_i^* J] \mid S \in \mathscr{HS}(L^2(\mathscr{M})), x_i \in F\}.$$

To see this, we must prove that for all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ and $S \in \mathscr{HS}(L^2(\mathscr{M}))$ we have

$$[S, (Jx_{i_1}^* \ldots x_{i_k}^* J)] \in \mathrm{span}\{[T, Jx_i^* J] \mid T \in \mathscr{HS}(L^2(\mathscr{M})), x_i \in F\}.$$

For any $S, y_1, \ldots, y_k \in \mathscr{B}(L^2(\mathscr{M}))$ we have:

$$\begin{aligned}
[S, y_1 \cdots y_k] &= [y_2 \cdots y_k S, y_1] \\
&\quad + [y_3 \cdots y_k S y_1, y_2] \\
&\quad + [y_4 \cdots y_k S y_1 y_2, y_3] \\
&\qquad \vdots \\
&\quad + [S y_1 \cdots y_{k-1}, y_k].
\end{aligned}$$

Put $y_j := Jx_{i_j}^* J$ for $j \in \{1, \ldots, k\}$ and note that

$$y_1 \cdots y_k = (Jx_{i_1}^* J)(Jx_{i_2}^* J) \cdots (Jx_{i_k}^* J) = J(x_{i_1}^* \cdots x_{i_k}^*)J.$$

Furthermore, for $S \in \mathscr{HS}(L^2(\mathscr{M}))$ we have

$$y_j \ldots y_k S y_1 \ldots y_{j-2} \in \mathscr{HS}(L^2(\mathscr{M})),$$

since $\mathscr{HS}(L^2(\mathscr{M}))$ is a two-sided ideal. Thus

$$[S, (Jx_{i_1}^* \cdots x_{i_k}^* J)] \in \mathrm{span}\{[T, Jx_i^* J] \mid T \in \mathscr{HS}(L^2(\mathscr{M})), x_i \in F\},$$

as desired. □

In the light of Proposition 3.7.6, we introduce the following notation.

Definition 3.7.7. *Let $\mathscr{M}$ be a finite von Neumann algebra with normal, faithful tracial state τ and let F be a finite, self-adjoint subset of $\mathscr{M}$, spanning a linear space of linear dimension l. For any finite self-adjoint subset G containing F we define*

- $\Delta(F) := l - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_F)])$.
- $\Delta(G : F) := l - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}} \Big(\varphi^{G,F}(\ker(d_F^{vN})) \cap ([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_G)])\Big)$

We place ourselves under the hypotheses of the above definition and assume moreover that the algebra generated by F contains the unit of $\mathscr{M}$ and is strongly dense in $\mathscr{M}$. We then have

$$\begin{aligned}\beta(F) &= \beta(d_F)\\ &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(d_F^{vN})) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_F)]) && \text{(Remark 3.6.3)}\\ &= l - 1 + \beta_0^{(2)}(\mathscr{M},\tau) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_F)]) && \text{(Proposition 3.7.6)}\\ &= \Delta(F) - 1 + \beta_0^{(2)}(\mathscr{M},\tau).\end{aligned}$$

Similarly we get

$$\begin{aligned}\beta(G : F) &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\varphi_*^{G,F}(H(F)))\\ &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big(\frac{\varphi^{G,F}(\ker(d_F^{vN}))}{\varphi^{G,F}(\ker(d_F^{vN})) \cap [\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_G)]}\Big) && \text{(Remark 3.5.4)}\\ &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\varphi^{G,F}(\ker(d_F^{vN}))) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\varphi^{G,F}(\ker(d_F^{vN})) \cap [\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_G)])\\ &= l - 1 + \beta_0^{(2)}(\mathscr{M},\tau) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\varphi^{G,F}(\ker(d_F^{vN})) \cap [\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_G)])\\ &= \Delta(G : F) - 1 + \beta_0^{(2)}(\mathscr{M},\tau)\end{aligned}$$

Also note, that

$$\Delta(F) = l - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\overline{\ker(d_F)}^{L^2}) = l - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\overline{\ker(d_F^{(2)} \cap (L^2(\mathscr{M}) \otimes L^2(\mathscr{M}))^l}^{L^2}),$$

by the results of Proposition 3.6.1 and Proposition 3.6.5. This last formula of course holds whether or not F generates a strongly dense algebra.

Definition 3.7.8. *We shall call a von Neumann algebra $\mathscr{M}$* finitely generated*, if there exists a finite self-adjoint subset F, such that the algebra generated by F is strongly dense in $\mathscr{M}$ and contains the unit of $\mathscr{M}$. We will refer to such a subset F, as a* finite generating *subset of $\mathscr{M}$.*

In the case when $\mathscr{M}$ is finitely generated we also make the following definition.

Definition 3.7.9. *Assume $\mathscr{M}$ to be finitely generated and let $\mathscr{G}$ denote the family of finite generating subsets of $\mathscr{M}$. We then define*

$$\Delta(\mathscr{M},\tau) := \sup_{F\in\mathscr{G}} \inf_{\substack{G\in\mathscr{G}\\ F\subseteq G}} \Delta(G : F).$$

By repeating the argument from the proof of Proposition 3.5.2 we get

$$\beta_1^{(2)}(\mathscr{M},\tau) = \sup_{F\in\mathscr{G}} \inf_{\substack{G\in\mathscr{G}\\ F\subseteq G}} \beta(G : F),$$

in the case when $\mathscr{M}$ is finitely generated. Using this formula, we see that

$$\begin{aligned}\beta_1^{(2)}(\mathscr{M},\tau) &= \sup_{F\in\mathscr{G}} \inf_{\substack{G\in\mathscr{G}\\ F\subseteq G}} \Big(\Delta(G : F) - 1 + \beta_0^{(2)}(\mathscr{M},\tau)\Big)\\ &= \Delta(\mathscr{M},\tau) - 1 + \beta_0^{(2)}(\mathscr{M},\tau).\end{aligned}$$

So, $\Delta(\mathscr{M},\tau)$ measures the difference between $\beta_1^{(2)}(\mathscr{M},\tau)$ and $\beta_0^{(2)}(\mathscr{M},\tau)$. In particular, $\Delta(\mathscr{M},\tau) = 1$ implies $\beta_1^{(2)}(\mathscr{M},\tau) = \beta_0^{(2)}(\mathscr{M},\tau)$.

Proposition 3.7.10. [CS03] *Let F be a finite, self-adjoint subset of $\mathscr{M}$ and assume that A (the algebra generated by F) contains the unit of $\mathscr{M}$.*
Then $\Delta(F)$ only depends on the algebra A and the restriction of the trace τ to A.

Proof. Let $\mathscr{N}$ be the strong closure of A inside $\mathscr{M}$ and endow $\mathscr{N}$ with the trace-state arising from the restriction of τ. We need to prove that we get the same $\Delta(F)$-value, whether we compute it relative to $\mathscr{M}$ or relative to $\mathscr{N}$. To avoid confusion, we will decorate the relevant notation with a sub-script indicating which von Neumann algebra it is constructed with respect to. For instance, $\Delta_{\mathscr{M}}(F)$ will denote the Δ-quantity of F computed relative to $\mathscr{M}$ and $\Delta_{\mathscr{N}}(F)$ will denote the Δ-quantity of F computed relative to $\mathscr{N}$. Similar notation will be used for $\beta(F)$ and d_F^{vN}.
Applying Proposition 3.7.6 to the pair $(\mathscr{N},\tau)$, gives

$$\dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}}(\ker(d^{vN}_{F,\mathscr{N}})) = l - 1 + \beta_0^{(2)}(\mathscr{N},\tau),$$

where l is the linear dimension of $\mathrm{span}_{\mathbb{C}}(F)$. We now consider the exact sequence

$$0 \longrightarrow \ker(d^{vN}_{F,\mathscr{N}}) \stackrel{\iota}{\longrightarrow} \mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}} \otimes \mathrm{span}_{\mathbb{C}}(F) \stackrel{d^{vN}_{F,\mathscr{N}}}{\longrightarrow} \mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}.$$

By Theorem 1.5.1, the induced sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \ker(d^{vN}_{F,\mathscr{N}}) \\ \downarrow {\scriptstyle 1\otimes\iota} \\ \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}} \otimes \mathrm{span}(F) \\ \downarrow {\scriptstyle 1\otimes d^{vN}_{F,\mathscr{N}}} \\ \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \end{array}$$

is still exact and

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(1\otimes d^{vN}_{F,\mathscr{N}})) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}\Big((\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}})\otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \ker(d^{vN}_{F,\mathscr{N}})\Big) = \dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}}(\ker(d^{vN}_{F,\mathscr{N}}))$$

The natural isomorphism $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}} \simeq \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ identifies $1\otimes d^{vN}_{F,\mathscr{N}}$ with $d^{vN}_{F,\mathscr{M}}$ and hence

$$\dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(d^{vN}_{F,\mathscr{M}})) = \dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}}(\ker(d^{vN}_{F,\mathscr{N}})).$$

From this we get

$$\begin{aligned} \beta_{\mathscr{M}}(F) &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\ker(d^{vN}_{F,\mathscr{M}})) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_{F,\mathscr{M}})]) && \text{(Remark 3.6.3)} \\ &= \dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}}(\ker(d^{vN}_{F,\mathscr{N}})) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_{F,\mathscr{M}})]) \\ &= l - 1 + \beta_0^{(2)}(\mathscr{N},\tau) - \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}([\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \cdot \ker(d_{F,\mathscr{M}})]) && \text{(Proposition 3.7.6)} \\ &= \Delta_{\mathscr{M}}(F) - 1 + \beta_0^{(2)}(\mathscr{N},\tau), \end{aligned}$$

and hence

$$\Delta_{\mathscr{M}}(F) = \beta_{\mathscr{M}}(F) + 1 - \beta_0^{(2)}(\mathscr{N},\tau). \tag{$\dagger$}$$

From Lemma 3.5.1 and Proposition 3.6.2 we have

$$\beta_{\mathscr{M}}(F) = \beta(d_{F,\mathscr{M}}) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}} \mathrm{Tor}_1^{\mathscr{M}^{\mathrm{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A),$$

and we now claim that

$$\mathrm{Tor}_1^{\mathscr{M}^{\mathrm{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A) \simeq \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathrm{Tor}_1^{\mathscr{N}^{\mathrm{e}}}(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A).$$

To see this, we choose a resolution (F_n, f_n) of $\mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A$ by free $\mathscr{N}^{\mathrm{e}}$-modules. Then the complex

$$(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}} \otimes_{\mathscr{N}^{\mathrm{e}}} F_* \ ,\ 1 \otimes f_*)$$

computes $\mathrm{Tor}_*^{\mathscr{N}^{\mathrm{e}}}(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A)$. If we apply the functor $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} -$ to this complex, we arrive at

$$(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}} \otimes_{\mathscr{N}^{\mathrm{e}}} F_* \ ,\ 1 \otimes 1 \otimes f_*), \tag{$*$}$$

and since $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} -$ is exact, (Theorem 1.5.1) the first homology of the complex $(*)$ is (see e.g. [CE] Ch. IV, Thm. 7.2)

$$\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathrm{Tor}_1^{\mathscr{N}^{\mathrm{e}}}(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A).$$

We now wish to identify this module with $\mathrm{Tor}_1^{\mathscr{M}^{\mathrm{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A)$.
If we apply the exact functor $\mathscr{M}^{\mathrm{e}} \otimes_{\mathscr{N}^{\mathrm{e}}} -$ (see e.g. Remark 1.5.2) to the free resolution (F_n, f_n), the resulting complex is a resolution of

$$\mathscr{M}^{\mathrm{e}} \otimes_{\mathscr{N}^{\mathrm{e}}} \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A \simeq \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A,$$

by free $\mathscr{M}^{\mathrm{e}}$-modules. So, if we apply the functor $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} -$ to this resolution, the first homology of the resulting complex

$$(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{M}^{\mathrm{e}}} \mathscr{M}^{\mathrm{e}} \otimes_{\mathscr{N}^{\mathrm{e}}} F_n \ ,\ 1 \otimes 1 \otimes f_n), \tag{$**$}$$

is

$$\mathrm{Tor}_1^{\mathscr{M}^{\mathrm{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A).$$

One easily checks, that the both $(*)$ and $(**)$ are isomorphic to the complex

$$(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}^{\mathrm{e}}} F_* \ ,\ 1 \otimes f_*),$$

and we conclude that

$$\mathrm{Tor}_1^{\mathscr{M}^{\mathrm{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A) \simeq \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathrm{Tor}_1^{\mathscr{N}^{\mathrm{e}}}(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A).$$

Using Theorem 1.5.1, we now get

$$\begin{aligned}
\beta_{\mathscr{M}}(F) &= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\mathrm{Tor}_1^{\mathscr{M}^{\mathrm{e}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathscr{M}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A)) \\
&= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \otimes_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathrm{Tor}_1^{\mathscr{N}^{\mathrm{e}}}(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A)) \\
&= \dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}} \mathrm{Tor}_1^{\mathscr{N}^{\mathrm{e}}}(\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}, \mathscr{N}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A) \\
&= \beta_{\mathscr{N}}(F).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta_{\mathscr{M}}(F) &= \beta_{\mathscr{M}}(F) + 1 - \beta_0^{(2)}(\mathscr{N}, \tau) && \text{(by (†))} \\
&= \beta_{\mathscr{N}}(F) + 1 - \beta_0^{(2)}(\mathscr{N}, \tau) \\
&= \Delta_{\mathscr{N}}(F),
\end{aligned}$$

as desired. □

Note, that we also proved $\beta_{\mathscr{M}}(F) = \beta_{\mathscr{N}}(F)$ in the last part of the proof of Proposition 3.7.10, so that also $\beta(F)$ is independent of the choice of von Neumann algebra, relative to which it is computed.

Recall, that $\mathscr{M}$ denotes a finite von Neumann algebra, endowed with a normal, faithful, tracial state τ.

Theorem 3.7.11. [CS03] *Let F be a finite, self-adjoint subset of $\mathscr{M}$, generating an algebra A which contains the unit of $\mathscr{M}$. Assume furthermore, that F contains a (normal) element x with diffuse spectrum, such that x or x^* commutes with every other element in F. Then $\Delta(F) = 1$.*

Proof. Since x is normal, Fugledes Theorem (see e.g. [MV] Theorem 17.23) implies that if x commutes with F then so does x^*; and vice versa. So, in either case, both x and x^* commutes with F.
Let $\mathscr{N}$ be the strong closure of A in $\mathscr{M}$. By Proposition 3.7.10, we may compute $\Delta(F)$ and $\beta(F)$ relative to $(\mathscr{N}, \tau)$. Since F obviously generates $\mathscr{N}$, the remarks preceding Proposition 3.7.10 implies that

$$\Delta(F) := \beta(F) + 1 - \beta_0^{(2)}(\mathscr{N}, \tau).$$

Since $\sigma(x)$ is diffuse, Theorem 3.3.5 implies that $\beta_0^{(2)}(\mathscr{N}, \tau) = 0$ and hence we have $\Delta(F) \geq 1$.
The opposite inequality requires a bit more work:
Let l denote the linear dimension of $\mathrm{span}_{\mathbb{C}}(F)$ and choose a basis $\{x_1, \dots, x_l\}$ containing x, consisting of elements from F. Assume without loss of generality that $x = x_1$.
We now consider $\mathscr{N}$ in its GNS-representation on $L^2(\mathscr{N}, \tau) =: L^2(\mathscr{N})$ and let J denote the conjugation operator on $L^2(\mathscr{N})$.
Let $y_2, \dots, y_l \in \mathscr{FR}(L^2(\mathscr{N}, \tau))$ be arbitrary operators and define $s_i := [y_i, Jx^*J]$ for $i \in \{2, \dots, l\}$ and $s := -\sum_{i=2}^{l} [y_i, Jx_i^*J]$. Put $\tilde{x}_i := Jx_i^*J$ for all $i \in \{1, \dots, l\}$. Since $F^* = F$ and x^* commutes with F we have

$$[\tilde{x}_i, \tilde{x}] := Jx_i^*JJx^*J - Jx^*JJx_i^*J = Jx_i^*x^*J - Jx^*x_i^*J = Jx^*x_i^*J - Jx^*x_i^*J = 0,$$

for any $i \in \{2, \dots, l\}$ and using the Jacobi identity we now get

$$0 = [[y_i, \tilde{x}], \tilde{x}_i] + [[\tilde{x}_i, y_i], \tilde{x}] + [[\tilde{x}, \tilde{x}_i], y_i] = [[y_i, \tilde{x}], \tilde{x}_i] + [[\tilde{x}_i, y_i], \tilde{x}].$$

Since this holds for any $i \in \{2, \dots, n\}$, we have

$$\begin{aligned} [s, \tilde{x}] + \sum_{i=2}^{n} [s_i, \tilde{x}_i] &= \sum_{i=2}^{l} [-[y_i, \tilde{x}_i], \tilde{x}] + [s_i, \tilde{x}_i] \\ &= \sum_{i=2}^{l} [-[y_i, \tilde{x}_i], \tilde{x}] + [[y_i, \tilde{x}], \tilde{x}_i] \\ &= \sum_{i=2}^{l} [[\tilde{x}_i, y_i], \tilde{x}] + [[y_i, \tilde{x}], \tilde{x}_i] \\ &= 0. \end{aligned}$$

This shows, that $(s, s_2, \dots, s_l) \in \ker(D_F) \cap \mathscr{FR}(L^2(\mathscr{N}))^l$. [4]
Define $\Phi : \mathscr{HS}(L^2(\mathscr{N}))^{l-1} \longrightarrow \mathscr{HS}(L^2(\mathscr{N}))^l$ by

$$\Phi : (y_2, \dots, y_l) \longmapsto (-\sum_{i=2}^{l} [y_i, \tilde{x}_i], [y_2, \tilde{x}], \dots, [y_l, \tilde{x}]),$$

[4] See e.g. Definition 3.7.2 and the remarks following it

and note that Φ is Hilbert-Schmidt continuous. This follows from the fact, that for every $a, b \in \mathscr{B}(L^2(\mathscr{N}))$ and $y \in \mathscr{HS}(L^2(\mathscr{N}))$ we have $\|ayb\|_{\mathscr{HS}} \le \|a\|_\infty \|y\|_{\mathscr{HS}} \|b\|_\infty$, such that each of the commutators appearing in the definition of Φ is Hilbert-Schmidt continuous.
By what was just proven, we get

$$\Phi(\mathscr{HS}(L^2(\mathscr{N}))^{l-1}) = \Phi(\overline{\mathscr{FR}(L^2(\mathscr{N}))^{l-1}}^{\mathscr{HS}}) \subseteq \overline{\ker(D_F) \cap \mathscr{FR}(L^2(\mathscr{N}))^l}^{\mathscr{HS}}. \qquad (\dagger)$$

We now consider $\mathscr{HS}(L^2(\mathscr{N}))$ as a left Hilbert $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$-module, with respect to the action

$$T \cdot y := \Psi(T\Psi^{-1}(y)), \quad \text{for } T \in \mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}, y \in \mathscr{HS}(L^2(\mathscr{N})),$$

and $\mathscr{HS}(L^2(\mathscr{N}))^k$ ($k \in \{l-1, l\}$) as a left $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$-Hilbert module with respect to the associated diagonal action. Note, that if $T = a \otimes b^{\mathrm{op}} \in \mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}$ then $T \cdot y = ayb$ by Proposition 1.2.15.
We now prove that Φ is $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$-equivariant. For $a, b \in \mathscr{N}$ we have

$$[ay_i b, \tilde{x}_i] = ay_i b\tilde{x}_i - \tilde{x}_i a y_i b = a y_i \tilde{x}_i b - a\tilde{x}_i y_i b = a[y_i, \tilde{x}_i]b,$$

and from this it follows that Φ is equivariant with respect to the action of $\mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}$. Since Φ is continuous and $\mathscr{N} \otimes \mathscr{N}^{\mathrm{op}}$ is dense in $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$, it follows that Φ is $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$-equivariant.
Next we show that Φ is injective.
Let $\mathscr{N}_0$ denote the von Neumann algebra generated by $\tilde{x} := Jx^*J$ in $\mathscr{B}(L^2(\mathscr{N}))$. Since x is normal we have

$$\ker(\lambda 1 - x) = \ker(\bar{\lambda} 1 - x^*) \quad \text{for any } \lambda \in \mathbb{C},$$

and since x is assumed to have diffuse spectrum, we see that also x^* has diffuse spectrum. Using this, it is not hard to show that also $\tilde{x} = Jx^*J$ has diffuse spectrum and by Remark 3.3.7 this implies that $\mathscr{N}_0'$ intersects trivially with the compact operators on $L^2(\mathscr{N})$.
Since Hilbert-Schmidt operators in particular are compact, we conclude from this that Φ is injective.

We are now in position to prove the equality $\Delta(F) \le 1$. Note first, that both $\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}}$ and $\overline{\ker(D_F) \cap \mathscr{FR}(L^2(\mathscr{N}))^l}^{\mathscr{HS}}$ are finitely generated Hilbert $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$-modules. By what was proven above, Φ is a weak isomorphism from $\mathscr{HS}(L^2(\mathscr{N}))^{l-1}$ to $\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}}$ and since ν^{-1} preserves weak exactness (Lemma 1.4.6) we get a weak isomorphism

$$\nu^{-1}(\Phi) : \nu^{-1}(\mathscr{HS}(L^2(\mathscr{N}))^{l-1}) \longrightarrow \nu^{-1}(\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}}).$$

By continuity of the dimension function (Theorem 1.4.7 part *3.*) we therefore have

$$\begin{aligned} \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\mathscr{HS}(L^2(\mathscr{N}))^{l-1}) &:= \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\nu^{-1}(\mathscr{HS}(L^2(\mathscr{N})^{l-1}))) \\ &= \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\nu^{-1}(\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}})) \\ &=: \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}}). \end{aligned}$$

Since $\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}}$ is a finitely generated Hilbert $\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}$-submodule in $\overline{\ker(D_F) \cap \mathscr{FR}(L^2(\mathscr{N}))^l}^{\mathscr{HS}}$ (see the inclusion ($\dagger$)) we now have

$$\begin{aligned} \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\overline{\ker(D_F) \cap \mathscr{FR}(L^2(\mathscr{N}))^l}^{\mathscr{HS}}) &\ge \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\overline{\mathrm{rg}\,(\Phi)}^{\mathscr{HS}}) \\ &= \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(\mathscr{HS}(L^2(\mathscr{N})^{l-1}) \\ &= \dim_{\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}}}(L^2(\mathscr{N} \bar{\otimes} \mathscr{N}^{\mathrm{op}})^{l-1}) \\ &= l - 1. \end{aligned} \qquad (\ddagger)$$

Since Ψ maps $L^2(\mathscr{N}) \otimes L^2(\mathscr{N}^{\mathrm{op}})$ onto $\mathscr{FR}(L^2(\mathscr{N}))$, the restriction of $\Psi_l := (\Psi, \dots, \Psi)$ gives rise to an isomorphism: (see the discussion following Definition 3.7.2)

$$\overline{\ker(d_F^{(2)}) \cap (L^2(\mathscr{N}) \otimes L^2(\mathscr{N}^{\mathrm{op}}))^l}^{L^2} \xrightarrow{\sim} \overline{\ker(D_F) \cap \mathscr{FR}(L^2(\mathscr{N}))^l}^{\mathscr{HS}},$$

and by Proposition 3.6.5 we have

$$\overline{\ker(d_F^{(2)}) \cap (L^2(\mathscr{N}) \otimes L^2(\mathscr{N}^{\mathrm{op}}))^l}^{L^2} = \overline{\ker(d_F)}^{L^2}.$$

Combining this with the inequality (‡) above yields

$$\dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}}(\overline{\ker(d_F)}^{L^2}) \geq l-1.$$

Using Proposition 3.6.1 we now have

$$\Delta(F) = l - \dim_{\mathscr{N}\bar{\otimes}\mathscr{N}^{\mathrm{op}}}(\overline{\ker(d_F)}^{L^2}) \leq l-(l-1) = 1,$$

and since the opposite inequality is already proven we have $\Delta(F) = 1$. □

As a consequence we get the following.

Corollary 3.7.12. [CS03] *Let $\mathscr{M}$ be a finitely generated von Neumann algebra, endowed with a faithful, normal, tracial state τ. Assume furthermore that $\mathscr{M}$ contains a central element with diffuse spectrum. Then $\Delta(\mathscr{M},\tau) = 1$ and $\beta_1^{(2)}(\mathscr{M},\tau) = 0$.*

Proof. Let F be a finite generating subset of $\mathscr{M}$ (in the sense of Definition 3.7.8) and consider the set $G := F \cup \{x, x^*\}$, where x is the central element with diffuse spectrum. Then G is also self-adjoint and generating and fulfills the requirements of Theorem 3.7.11. Thus, $\Delta(G) = 1$. From this we get

$$\begin{aligned} \Delta(G:F) &= \beta(G:F) + 1 - \beta_0^{(2)}(\mathscr{M},\tau) \\ &\leq \beta(G) + 1 - \beta_0^{(2)}(\mathscr{M},\tau) && \text{(Remark 3.5.4)} \\ &= \Delta(G) \\ &= 1. \end{aligned}$$

If $\mathscr{G}$ denotes the family of finite generating subsets of $\mathscr{M}$, we therefore have

$$\inf_{\substack{G\in\mathscr{G} \\ F\subseteq G}} \Delta(G:F) \leq 1.$$

Since $F \in \mathscr{G}$ was arbitrary, we get

$$\Delta(\mathscr{M},\tau) := \sup_{F\in\mathscr{G}} \inf_{\substack{G\in\mathscr{G} \\ F\subseteq G}} \Delta(G:F) \leq 1.$$

Recall that $1 - \Delta(\mathscr{M},\tau) = \beta_0^{(2)}(\mathscr{M},\tau) - \beta_1^{(2)}(\mathscr{M},\tau)$ by the observations preceding Proposition 3.7.10. Since $\mathscr{M}$ contains an element with diffuse spectrum, $\beta_0^{(2)}(\mathscr{M},\tau) = 0$ and hence

$$0 \leq \beta_1^{(2)}(\mathscr{M},\tau) = \Delta(\mathscr{M},\tau) - 1.$$

Since we proved $\Delta(\mathscr{M},\tau) \leq 1$, we conclude from this that $\Delta(\mathscr{M},\tau) = 1$ and $\beta_1^{(2)}(\mathscr{M},\tau) = 0$. □

Remark 3.7.13. *Let m denote the Lebesgue measure on $[0,1]$ and consider the multiplication algebra $\mathscr{M} := \{M_f | f \in L^\infty([0,1],m)\}$ in $\mathscr{B}(L^2([0,1],m))$.*
Endow $\mathscr{M}$ with at faithful, normal, tracial state τ and let id *denote the identity-function $t \mapsto t$ on $[0,1]$. Note that $\{M_{\mathrm{id}}, 1\}$ is a finite generating subset of $\mathscr{M}$ and that*

$$\sigma(M_{\mathrm{id}}) = \operatorname{essrg}(\mathrm{id}) = \operatorname{rg}(\mathrm{id}) = [0,1].$$

The spectral measure E *for* M_{id} *is given by*

$$\mathcal{B}([0,1]) \ni A \longmapsto M_{\chi_A} \in \{M_f | f \in L^\infty([0,1], m)\}.$$

(see e.g. [MV] Example 18.11 for a proof that this is the right spectral measure)
In particular, it can not have atoms since for any $g \in L^2([0,1], m)$ *and* $\lambda \in [0,1]$ *we have*

$$\|E(\{\lambda\})g\|_2^2 = \int_0^1 |\chi_{\{\lambda\}} g|^2 dm = \int_{\{\lambda\}} |g|^2 dm = 0.$$

From Corollary 3.7 we conclude that $\beta_1^{(2)}(\mathscr{M}, \tau) = 0$. *(Compare e.g. with* [CS03] *Corollary 5.4)*

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List of Notation

Sets	$\mathcal{P}(X)$	The family of subsets of a set X.
	$\mathcal{P}_e(X)$	The family of finite subsets of a set X.
	$\mathcal{B}(X)$	The Borel σ-algebra on a topological space X.
	$C(X)$	The continuous functions on a topological space X.
Rings & Modules	R	A unital $*$-ring.
	R^n	The direct sum of n copies of a ring R.
	$R^{(X)}$	The free R-module with generators the elements of the set X.
	$\mathrm{Mod}(R)$	The category of left R-modules.
	$\langle r_i \vert i \in I\rangle$	The left ideal in R generated by $(r_i)_{i\in I}$.
	$X \otimes_R Y$	The tensor product of R-modules.
	$\simeq_R$	Isomorphism of R-modules.
	$\mathbb{C}G$	The group-algebra of a discrete group G.
Hilbert Spaces & Operator Algebras	$\mathscr{H}, \mathscr{K}$	Hilbert spaces.
	$\mathscr{B}(\mathscr{H})$	All bounded operators on $\mathscr{H}$.
	$\mathscr{FR}(\mathscr{H})$	The finite rank operators on $\mathscr{H}$
	$\mathscr{HS}(\mathscr{H})$	The Hilbert Schmidt operators on $\mathscr{H}$.
	$\mathscr{H}\bar{\otimes}\mathscr{K}$	The tensor product of Hilbert spaces.
	$\mathscr{M}, \mathscr{N}$	von Neumann algebras
	$\mathscr{M}\bar{\otimes}\mathscr{N}$	The tensor products of von Neumann algebras.
	$\mathscr{M}_+$	The positive cone in $\mathscr{M}$.
	$(\mathscr{M})_1$	The unit ball of $\mathscr{M}$.
	$M_n(\mathscr{M})$	$n \times n$-matrices over $\mathscr{M}$.
	$\mathscr{N}(G)$	The group von Neumann algebra of a discrete group G.
Operators	$\sigma(T)$	Spectrum of an operator.
	$\vert T\vert$	Absolute value of T ($=\sqrt{T^*T}$).
	$\ker(T)$	The kernel of T.
	$\mathrm{rg}\,(T)$	The range of T.
	$\mathrm{coker}\,(T)$	The cokernel of T.
	$R(T)$	The range-projection of T.
	$N(T)$	The null-projection of T.
	$\Vert T\Vert_\infty$	The operator norm of T.
	Ad_U	The map $T \mapsto U^*TU$ on $\mathscr{B}(\mathscr{H})$, for a unitary $U \in \mathscr{B}(\mathscr{H})$.