Last lecture:

ex $C_2 = (X_3 + \bar{X}_5 + X_6)$ $(k=3)$

Thm Let $\mathcal{F} = C_1 * C_2 * \cdots * C_m$ be an instance of k-SAT over variables $X_1, X_2, \ldots, X_n$

If $m < 2^k$, then $\mathcal{F}$ is satisfiable

(that is, we can choose a $\{0,1\}$ assignment to $x_1, x_2 \ldots x_n$ such that this satisfies all C_j at the same time).

Recall the proof:

Consider a random assignment φ

$\{X_1, X_2 \ldots, X_n\} \xrightarrow{\varphi} \{0,1\}^n$

For a given clause C_j the probability that it is NOT satisfied (all literals are 0) is $\left(\frac{1}{2}\right)^k$

Let $X_j = \begin{cases} 1 & \text{if } C_j \text{ NOT satisfied} \\ 0 & \text{else} \end{cases}$ $\quad p = \left(\frac{1}{2}\right)^k = 2^{-k}$

Then $X = X_1 + \cdots + X_m$ counts # of non-satisfied clauses.

$$E(X) = \sum_{i=1}^{m} E(X_i) = \sum_{i=1}^{m} 2^{-k} = m \cdot 2^{-k}$$

So since $m < 2^k$ we get $E(X) < 1$

By Markov's ineq.

$$P(X \geq 1) \leq \frac{E(X)}{1} = E(X) < 1$$

So $P(X=0) > 0$

This is best possible.

Take $n = k$ and $m = 2^k$

clauses $C_1, C_2 \dots C_{2^k}$ that represent all possible distinct clauses from $x_1, x_2 \dots x_k$

eg $C_1 = (x_1 + x_2 + x_3 + \dots + x_k)$

$C_2 = (x_1 + x_2 + \dots + x_{k-1} + \bar{x}_k)$

$\vdots$

$C_{2^k} = (\bar{x}_1 + \bar{x}_2 + \dots + \bar{x}_k)$

This cannot be satisfied since no matter which truth assignment we make precisely one clause will be non satisfied

e.g if $k = 4$ and $x_1 = 0, x_2 = 1, x_3 = 0$ $x_4 = 0$

then $C_j = (x_1 + \bar{x}_2 + x_3 + x_4)$ not satisfied

If $\mathcal{F} = C_1 * C_2 * \cdots * C_m$

and $\sum_{j=1}^{m} 2^{-|C_j|} < 1 \quad (*)$ $p(C_j$ not satisfied by random truth ass.$)$

then $\mathcal{F}$ is satisfiable.

P: as before let

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ not satisfied} \\ 0 & \text{else} \end{cases} \quad p = 2^{-|C_j|}$$

$X = X_1 + \cdots + X_m$ is # of not satisfied clauses

$$E(X) = \sum_{i=1}^{m} E(X_i) = \sum_{i=1}^{m} 2^{-|C_i|}$$

So $E(X) < 1$ when $(*)$ holds

same conclusion as before:

$\exists$ an assignment s.t. all C_j's are ok.

<u>Thm</u> For all $\varepsilon > 0$ there is pol alg to decide satisfiability of formulas $\mathcal{F} = C_1 * C_2 * \dots * C_m$ when $|C_i| \geq \varepsilon n$, when $n = \#$variables.

P: if $m < 2^{\varepsilon n}$ then

$$E(X) = \sum_{i=1}^{m} 2^{-|C_i|} \leq \sum_{i=1}^{m} 2^{-\varepsilon n} = \frac{m}{2^{\varepsilon n}} < 1$$

So $\mathcal{F}$ is satisfiable (but we do not <u>find</u> a solution)

So suppose $m \geq 2^{\varepsilon n}$

then input size $|\mathcal{F}| \geq \varepsilon n \cdot \underbrace{2^{\varepsilon n}}_{\stackrel{\wedge|}{m}}$

$$|\mathcal{F}| \geq 2^{\varepsilon n} \quad \text{as each } |C_i| \geq 1$$

$$\Downarrow$$

$$|\mathcal{F}|^{\frac{1}{\varepsilon}} \geq 2^n$$

But then we can try all 2^n possible truth assignments in time

$$2^n |\mathcal{F}| \leq |\mathcal{F}|^{\frac{1}{\varepsilon}} |\mathcal{F}| = |\mathcal{F}|^{(1+\frac{1}{\varepsilon})}$$

which is polynomial as ε is a constant

Rosen 8.5 Inclusion-Exclusion

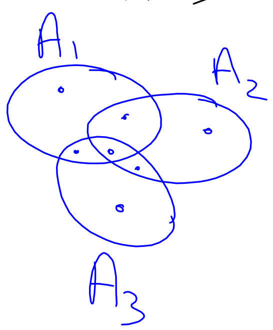
We saw early in the course

$n=2$ $\quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$n=3$ $\quad |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$

$\qquad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$

$\qquad + |A_1 \cap A_2 \cap A_3|$



Thm $\quad |A_1 \cup A_2 \cup \cdots \cup A_n|$

$$= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le j_1 < j_2 < j_3 \le n} |A_{j_1} \cap A_{j_2} \cap A_{j_3}|$$

$$- \cdots\cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

P: We need to show that each element $a \in A_1 \cup \dots \cup A_n$ is counted precisely once on right hand side.

Fix $a \in A_1 \cup \dots \cup A_n$ and let r be the number of sets A_i s.t $a \in A_i$

Then a is counted

$\binom{r}{1}$ times in $\sum |A_i|$

$\binom{r}{2}$ ------ $\sum |A_{i_1} \cap A_{i_2}|$

$\vdots$

$\binom{r}{r}$ ------ $\sum |A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_r}|$

times r is counted on rhs:

$$\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \dots + (-1)^{r+1}\binom{r}{r}$$

recall binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

so with $x=1$ and $y=-1$ we get

$$0 = (-1+1)^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k$$

$$0 = \binom{r}{0} - \underline{\binom{r}{1} + \binom{r}{2} - \dots + (-1)^r \binom{r}{r}}$$

$$0 = 1 - \left[\binom{r}{1} - \binom{r}{2} + \dots + (-1)^{r+1}\binom{r}{r}\right]$$

So $\binom{r}{1} - \binom{r}{2} + \dots + (-1)^{r+1}\binom{r}{r} = 1$

showing that a is counted precisely once on rhs. qed

Applications of incl - excl.

N = total # of elements

Counting # elements without specified properties $P_1, P_2, \ldots, P_n$

Define $N(P'_{i_1} P'_{i_2} \cdots P'_{i_k})$ = # elements with none of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$

and $N(P_{i_1} P_{i_2} \cdots P_{i_k})$ = # elements with each of the properties $P_{i_1}, \ldots, P_{i_k}$

Let A_i be the set of elements with property P_i

Then $N(P_{i_1} P_{i_2} \cdots P_{i_k}) = |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$

and by incl-excl formula:

$$N(P'_1 P'_2 \cdots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$= N - \left(\sum |A_i| - \sum |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \right)$$

$$= N - \left(\sum N(P_i) - \sum N(P_i P_j) + \cdots + (-1)^{n+1} N(P_1 \cdots P_n) \right)$$

Ex 1 p 542 Find # sols to

$$x_1+x_2+x_3=11, \quad x_i \geq 0 \text{ integer}$$

and $x_1 \leq 3, x_2 \leq 4, x_3 \leq 6$

$N = ?$ 11 '*''s in 3 boxes

11 '*' out of 13 symbols from '*', '|'

*** | **** | ****

$$N = \binom{11+(3-1)}{11} = \binom{13}{11}$$

Define properties P_1, P_2 and P_3

$P_1: x_1 \geq 4 \quad P_2: x_2 \geq 5 \quad P_3: x_3 \geq 7$

$$N(P_1) = \#\text{ sols to } x_1'+x_2'+x_3' = 7 = 11-4$$
$$= \binom{9}{7}$$

$$N(P_2) = \binom{8}{6} \quad N(P_3) = \binom{6}{4}$$

$N(P_1P_2) = \#$ sols with $x_1 \geq 4, x_2 \geq 5$ $\binom{4}{2}$

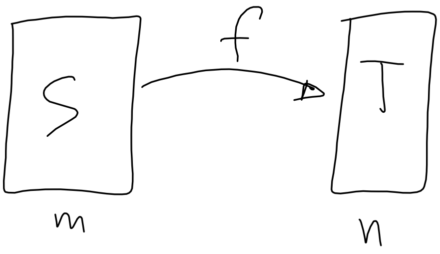
$N(P_1P_3) = \#$ sols with $x_1 \geq 4, x_3 \geq 7 = 1 \left(= \binom{2}{0}\right)$

$N(P_2P_3) = \#$ sols with $x_2 \geq 5, x_3 \geq 7 = 0$

$$N(P_1P_2P_3) = 0$$

insert these in formula..

P 543 — # of onto functions



$T = \{t_1, t_2, \ldots, t_n\}$

total # mappings is

$$N = n^m$$

Define P_i: t_i was not 'hit' by f

$$(\forall s \in S : f(s) \neq t_i)$$

$$N(P_i) = (n-1)^m$$

and $N(P_{i_1} P_{i_2} \cdots P_{i_k}) = (n-k)^m$ and $N(P_1 \cdots P_n) = 0$

$$\text{\# onto functions} = N - \sum N(P_i) + \sum N(P_i P_j) - \cdots - (-1)^n N(P_1 \cdots P_n)$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}$$

onto functions $\Longleftrightarrow$ Stirling numbers

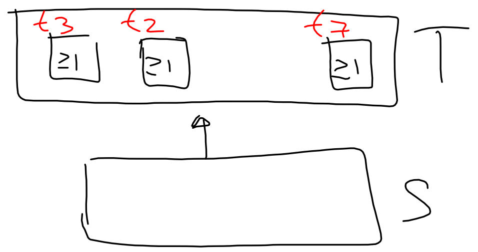
$\uparrow$ $S(m,n)$

distributing distinct objects in identical boxes. (s.t. no empty box)

Consider the following two step construction of an onto function

1. Distribute the m (distinct) elements from S in n identical boxes (with label hidden) s.t no box is empty
2. reveal labels on boxes (now we can distinguish them)

1.



$n!$ distinct labellings

Conclusion

$$\#\text{ of onto fcts} = n!\, S(m,n)$$

so $S(m,n) = \frac{1}{n!}\left(n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots\right)$