6.5 Generalized Permutations and Combinations

Rule: same element can be chosen several times

Permutation with repetition:

$\boxed{\begin{array}{l}\underline{\text{Theorem 1}} \quad \#\ r\text{-permutations of an } n\text{-set with} \\ \text{repetions allowed is } n^r\end{array}}$

P: We have n possible choices for each position so by the product rule we get $n \cdot n \cdot \ldots \cdot n = n^r$ permutations

Combinations with repetition

__Example 2__ # ways to pick 4 items each of type A, B or C when #A, #B, #C all ≥ 4

Pedestrian method: List all possibilities:

4A, 4B, 4C, (3A, B), (3A, C), (3B, A), (3B, C), (3C, A), (3C, B), (2A, 2B), (2A, 2C), (2B, 2C), (2A, B, C) (2B, A, C), (2C, A, B) 15 ways

There are 3 ways to pick 4 of same kind

6 ways to pick 3 and 1

3 ways to pick 2, 2 (choose 2 of {A, B, C})

3 ways to pick 3, 1, 1

We need a general method!

Why not just $\frac{\#\,r\text{-perm}}{r!}$? $\left(\frac{3^4}{4!} = \frac{27}{8}\right)$

New example: pick 2 from (A, B, C, D)

(*) # 2-perm = 16 (*)

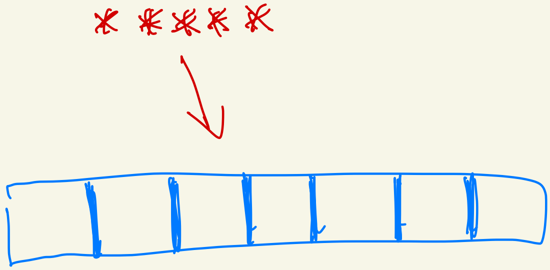
2-comb = 10 $\underbrace{2A, 2B, 2C, 2D}_{\text{counted once}}$, $\underbrace{AB, AC, AD, BC, BD, CD}_{\text{counted twice}}$

Also not 2 from {A, A, B, B, C, C, D, D}:

AA is counted once

AB is counted 4 times!

Example 3 # ways to pick 5 bank notes from 7 possible values



Same as # ways to write 5 * and 6 |

= # ways to pick 5 out of a set of 11 positions

$$= \binom{11}{5} = \binom{11}{6} = \binom{5+(7-1)}{5}$$

(each choice of 5 positions for the * correspond to a unique choice of bank notes)

Theorem # r-combinations from n-set with repetition allowed is $\binom{(n-1)+r}{r} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$

Proof We can represent each r-combination by a string of n-1 '|' and r '*', where the n-1 | are used to mark which of the n 'boxes' elements are taken from

Eg: * * * | | * | * * | | | *

$\Leftrightarrow$ 3 from 1, 1 from 3, 2 from 4, 1 from 7

There are $\binom{n-1+r}{r}$ ways to pick the

r *'s

Example 5 # solutions to $x_1 + x_2 + x_3 = 11$

= # 11-combinations from 3-set with rep

$$= \binom{11+3-1}{11} = \binom{13}{11} = \binom{13}{2} = 78$$

Can also handle lower bounds

ex $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ above!

6 items already placed so 5 remain

$\Rightarrow$ # 5-combos of a 3 set $= \binom{7}{5} = 21$

$x_i' = x_i - a_i$ when we had $x_i \geq a_i$

$\Rightarrow \sum x_i' = \text{target} - \sum a_i$

Ex 6 Find value of k after execution of

$k \leftarrow 0$
For $i_1 \leftarrow 1$ to n
  For $i_2 \leftarrow 1$ to i_1
    For $i_3 \leftarrow 1$ to i_2
      ⋮
        For $i_m \leftarrow 1$ to i_{m-1}
          $k \leftarrow k+1$

Solution: For each choice of $1 \le i_m \le i_{m-1} \le \cdots \le i_2 \le i_1 \le n$ we add one to k. Hence value of k is # ways to pick m integers all in $\{1, 2, \ldots, n\}$:

Given m such numbers, let $i_m, i_{m-1}, \ldots, i_2, i_1$ be the sorted order of them. For that choice we add one to k

Conversely every legal choice of $1 \le i_m \le i_{m-1} \le \cdots \le i_2 \le i_1 \le n$ corresponds to a unique subset of $\{1, 2 \ldots n\}$ with repetition allowed

So answer is $$\binom{m+n-1}{m}$$

Permutations with indistinguishable objects

Ex 7' # strings from SUCCESSOR

place 3 'S' in $\binom{9}{3}$ ways

2 'C' in $\binom{6}{2}$ ways

1 'E' in $\binom{4}{1}$ ways

1 'O' in $\binom{3}{1}$ ways

1 'R' in $\binom{2}{1}$ ways

1 'U' in $\binom{1}{1}$ ways

answer $\binom{9}{3}\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}$

Theorem # distinct permutations of n objects of k types with n_i of type k $\left(\sum_{i=1}^{k} n_i = n\right)$

is
$$\frac{n!}{n_1!\,n_2!\,\cdots n_k!}$$

Proof a) in example the answer is

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdot\ \cdots\ \binom{n_4}{n_4}$$

$$= \frac{n!}{\cancel{(n-n_1)!}\cdot n_1!}\cdot\frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\,n_2!}\cdot\frac{\cancel{(n-n_1-n_2)!}}{\cancel{(n-n_1-n_2-n_3)!}\,n_3!}\cdots\frac{\cancel{n_4!}}{n_4!}$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_4!}$$

Distributing objects in boxes

4 types of problems

objects	boxes
$\neq$	$\neq$
$\neq$	$=$
$=$	$\neq$
$=$	$=$

$= \Leftrightarrow$ not distinguishable

Problems are easier when boxes are distinguishable.

$\neq$ $\not\equiv$

ex 8 # ways to distribute 5 cards to each of 4 players from 52 cards

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \frac{52!}{(5!)^4 \cdot 32!}$$

NB same answer as # perms of 52 items with 5 of types 1, 2, 3, 4 and 32 type 5

List cards 1, 2, ... 51, 52

cards for player 1 $\Leftrightarrow$ positions assigned to type 1 — 5 out of 52

2 $\Leftrightarrow$ - - - - — 2: 5 out of 47

⋮ — 3: 5 out of 42

4 $\Leftrightarrow$ — 4: 5 out of 37

5 $\Leftrightarrow$ — 5: 32 - - - 32

Why 1-1 correspondance?

If two tuples (5,5,5,5,32) have same image then the cards of player i are the same in both and the remaining 32 cards are also the same

Same arguments give

Theorem # ways to distribute n $\neq$ items in k $\neq$ boxes with n_i items in box i is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

$\underline{= \quad \neq}$

ways to place n identical objects in k distinct boxes

$= \#$ n combinations of a k-set with repetitions

$$= \binom{n+k-1}{n}$$

Same argument as for bank notes using * and |

$\underline{\text{example 9}}$ # ways to place 10 identical balls in 8 distinct boxes

$= \#$ 10-combinations with repetition from an 8-set

$$= \binom{10+8-1}{10} = \binom{17}{10} = \binom{17}{7}$$

$\underline{\neq \quad =}$

$\underline{\text{example 10}}$ 4 $\neq$ employees in 3 identical offices each with capacity at least 4

1-1 correspondance between solutions and partitions of $\{A, B, C, D\}$ in 3 disjoint sets

set i $\sim$ office i

all 4 in one office	ABCD	1
3 in one 1 in another	ABC, D | ABD, C | ACD, B | BCD, A	4
2 in 2 different	AB, CD | AC, BD | AD, BC	3
2, 1, 1	AB, C, D | AC, B, D | AD, B, C BC, A, D | BD, A, C | CD, A, B	6
		14

Different way of counting!

all 3 offices used: # ways to choose 2 out of 4 — 6

2 offices used: $3 \text{ out of } 4 + \frac{2 \text{ out of } 4}{2}$ — $4+3$ — 7

1 office used: 4 out of 4 — 1

14

$\exists$ no simple formula but we shall see formula based on Stirling numbers of 2nd kind later!

= =

example 11 # ways to distribute 6 copies of same book in 4 boxes

Same as # sol, to

$$x_1 + x_2 + x_3 + x_4 = 6 \text{ and}$$

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0$$

6 | 5,1 | 4,2 | 4,1,1 | 3,3 | 3,2,1 |
3,1,1,1 | 2,2,2 | 2,2,1,1 9 ways

No simple formula !