Rosen 8.1 Recurrence relations

A recurrence relation $\{a_n\}$ is an equation that expresses a_n in terms of $a_0, a_1, \ldots, a_{n-1}$ for all $n \geq n_0$, where $n_0 \geq 0$

$\{x_n\}$ is a <u>solution</u> to $\{a_n\}$ if each term x_i satisfies the same equation as a_i

Example: $a_n = 3n$ is a solution to

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for all } n = 2, 3, \ldots$$

as $2 \cdot 3(n-1) - 3(n-2)$

$$= 6n - 6 - 3n + 6 = 3n$$

<u>Initial conditions:</u> Then specify the first terms when the recurrence starts (up to n_0)

above we could take $a_0 = 0$ and $a_1 = 3$ and see that

then $6 = 3 \cdot 2 = a_2 = 2a_1 - a_0 = 2 \cdot 3 - 0 = 6$

and $9 = 3 \cdot 3 = a_3 = 2a_2 - a_1 = 2 \cdot 6 - 3 = 9$

Initial conditions uniquely determine the sequence (induction on n)

Example 1 Fibonacci numbers

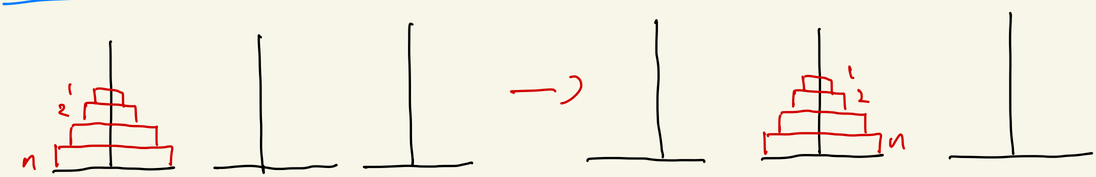
$$f_1 = f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}$$

$f_3 = f_2 + f_1 = 1 + 1 = 2$, $f_4 = f_3 + f_2 = 2 + 1 = 3$

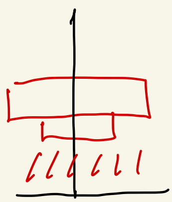
$f_5 = 5$, $f_6 = 8$ ...

Example 2 Tower of Hanoi Puzzle



Initial

rule never



H_n = # moves when there are n discs

$H_1 = 1$

$H_n \leq 2H_n + 1$

1. $n-1$ top discs → peg no 3
2. move disc n → peg 2
3. move $n-1$ discs peg 3 → peg 2

$H_n \geq 2H_{n-1}+1$:

- Cannot move disc n before all discs $1, 2, \ldots, n-1$ have been moved to another peg and this must be peg 3 as we want disc n on peg 2. Thus we need at least H_{n-1} moves for this part
- 1 move to get disc n to peg 2
- H_{n-1} moves to get discs $1, 2, \ldots, n-1$ from peg 3 to peg 2. Corresponds to problem with n replaced by $n-1$

Conclusion: $H_n = 2H_{n-1}+1$ (and $H_1 = 1$)

$H_n = 2^n - 1$: $2^n - 1 = H_n = 2H_{n-1} + 1$

$$= 2 \cdot (2^{n-1} - 1) + 1$$

$$= 2^n - 1$$

We will see a method to solve such recurrence equations in Section 8.2

Example 3 # of bitstrings of length n with no —00—

Let N be the # of such strings

By sum rule, $N = N_1 + N_0$ when N_i is # of such strings that end with i $i=0,1$

Then we see that

N_1 = # length n-1 with no ~00~

N_0 = # length n-2 with no ~00~
(since bit n-1 must be a 1 ~~10)

Hence $a_n = N = N_1 + N_0 = a_{n-1} + a_{n-2}$ when $n \geq 3$

$a_n = a_{n-1} + a_{n-2}$

initial conditions $a_1 = 2$ ('0', '1') so $a_1 = f_3$

$a_2 = 3$ ('01', '10', '11') so $a_2 = f_4$

$a_3 = a_2 + a_1 = f_4 + f_3 = f_5$

$a_4 = a_3 + a_2 = f_5 + f_4 = f_6$

So $a_n = f_{n+2}$ for $n \geq 1$

Rosen 8.2 Solving linear recurrence equations

Def 1 A linear homogeneous recurrence relation of degree k with constant coefficients is of the form

$$(*) \quad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad \text{where } c_i \in \mathbb{R} \text{ and } c_k \neq 0$$

Note the solution to $(*)$ is uniquely determined by the values of $a_0, a_1, \dots a_{k-1}$ (induction)

$(*)$ is homogeneous because a_n only depends on previous terms a_i and not on some function $f(n)$

$(*)$ has degree k as a_n is expressed as a linear combination of terms whose index is at least k.

How to solve $(*)$?

make a guess $a_n = r^n$ for some $r \neq 0$ and insert

we must have

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

divide by r^{n-k}

So

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$$

$$\Leftrightarrow$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

$(\square)$ $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$

is the characteristic equation for the sequence $\{a_n\}$

satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ $(*)$

Solutions to $(\square)$ are called characteristic roots.

<u>Lemma</u> Suppose $\{s_n\}$ and $\{t_n\}$ are solutions to $(*)$ and c is a constant then

1) $\{c \cdot s_n\}$ is a solution to $(*)$

2) $\forall c, d$ constants $\{c s_n + d t_n\}$ is a solution to $(*)$

<u>proof:</u> $s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_k s_{n-k}$

$$\Downarrow$$

$$c \cdot s_n = c(c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_k s_{n-k})$$

$$= c_1 \cdot \underbrace{c s_{n-1}} + c_2 \underbrace{c \cdot s_{n-2}} + \cdots + c_k \underbrace{c \cdot s_{n-k}}$$

so $\{c s_n\}$ solves $*$

if $s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_k s_{n-k}$

and $t_n = c_1 t_{n-1} + c_2 t_{n-2} + \cdots + c_k t_{n-k}$

then $s_n + t_n = c_1(s_{n-1} + t_{n-1}) + c_2(s_{n-2} + t_{n-2}) + \cdots + c_k(s_{n-k} + t_{n-k})$

Now 2) follows from this and 1) $\square$.

The case when $k=2$

$(*)$ $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ $\sim$ $r^2 - c_1 r - c_2 = 0$ (characteristic eq.)

$$\boxed{\begin{array}{l}\text{Theorem 1 suppose } r^2 - c_1 r - c_2 = 0 \text{ has distinct roots } r_1, r_2 \\ \text{Then } \{a_n\} \text{ solves } * \text{ if and only if} \\ a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \text{for } n = 1, 2 \ldots \text{ and } \alpha_1, \alpha_2 \in \mathbb{R}\end{array}}$$

Proof: let $r_1 \neq r_2$ be roots in $r^2 - c_1 r - c_2 = 0$

then $r_i^2 - c_1 r_i - c_2 = 0$ $\Rightarrow$ $r_i^2 = c_1 r_i + c_2$ for $i = 1, 2$ (Δ)

Now if we set $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$

we get

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1 \left(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1} \right) + c_2 \left(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2} \right) \\ &= \left[c_1 \alpha_1 r_1^{n-1} + c_2 \alpha_1 r_1^{n-2} \right] + \left[c_1 \alpha_2 r_2^{n-1} + c_2 \alpha_2 r_2^{n-2} \right] \\ &= \alpha_1 r_1^{n-2} \left[c_1 r_1 + c_2 \right] + \alpha_2 r_2^{n-2} \left[c_1 r_1 + c_2 \right] \quad \text{by } (\Delta) \\ &= \alpha_1 r_1^{n-2} \cdot r_1^2 + \alpha_2 r_2^{n-2} \cdot r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \end{aligned}$$

Hence $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution to $(*)$

Conversely, suppose $\{a_n\}$ solves (*) with

$a_0 = C_0$ and $a_1 = C_1$

This uniquely determines $\{a_n\}$ so we need to find α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ with $a_0 = C_0$, $a_1 = C_1$

we get 2 equations with 2 unknowns α_1, α_2:

$$C_0 = a_0 = \alpha_1 \cdot r_1^0 + \alpha_2 \cdot r_2^0 = \alpha_1 + \alpha_2$$

$$C_1 = a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

$$\begin{aligned} C_0 &= \alpha_1 + \alpha_2 \\ C_1 &= \alpha_1 r_1 + \alpha_2 r_2 \end{aligned} \Rightarrow$$

$C_1 - C_0 r_2 = \alpha_1 r_1 - \alpha_1 r_2$

so $\boxed{\alpha_1 = \dfrac{C_1 - C_0 r_2}{r_1 - r_2}}$

$\Rightarrow$

$C_1 - C_0 r_1 = \alpha_2 r_2 - \alpha_2 r_1$

so $\boxed{\alpha_2 = \dfrac{C_1 - C_0 r_1}{r_2 - r_1}}$

Hence there is a unique choice of α_1, α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ solves (*) with initial conditions $a_0 = C_0$, $a_1 = C_1$

Example 3

$a_n = a_{n-1} + 2a_{n-2}$ $\quad a_0 = \overset{c_0}{2},\ a_1 = \overset{c_1}{7}$

Characteristic equation: $r^2 - r - 2 = 0$

roots $r_1 = 2,\ r_2 = -1$

so $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

Determining α_1, α_2:

$$\alpha_1 = \frac{c_1 - c_0 r_2}{r_1 - r_2} = \frac{7 - 2\cdot(-1)}{2-(-1)} = \frac{9}{3} = 3$$

$$\alpha_2 = \frac{c_1 - c_0 \cdot r_1}{r_2 - r_1} = \frac{7 - 2\cdot 2}{-1-2} = -1$$

$$\boxed{a_n = 3\cdot 2^n - (-1)^n}$$

Theorem 2 Let $c_1, c_2 \in \mathbb{R}$ with $c_2 \neq 0$ and suppose $r^2 - c_1 r - c_2 = 0$ has a unique (double) root r. Then $\{a_n\}$ solves $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$

proof: exercise!

Example 5 $\quad a_n = 6a_{n-1} - 9a_{n-2} \quad \overset{c_0}{a_0 = 1} \quad \overset{c_1}{a_1 = 6}$

Characteristic eq: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$

So $\quad a_n = \alpha_1 3^n + \alpha_2 n \cdot 3^n$

Find α_1, α_2: $\quad c_0 = 1 = a_0 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot 0 \cdot 3^n = \alpha_1$

$$c_1 = 6 = a_1 = \alpha_1 \cdot 3^1 + \alpha_2 \cdot 1 \cdot 3^1$$

$$= 3\alpha_1 + 3\alpha_2$$

$$= 3 + 3\alpha_2 \text{ as } \alpha_1 = 1$$

so $\alpha_2 = (6-3)/3 = 1$

$$\boxed{a_n = 3^n + n \cdot 3^n = (n+1)3^n}$$

Theorem 3 $\quad c_i \in \mathbb{R} \quad i = 1, 2 .. k, \; c_k \neq 0$

if $\quad r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ has distinct roots $r_1, r_2, \dots r_k$ then $\{a_n\}$ solves

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$$

proof exercise

8.2.3 Linear inhomogeneous recurrence relations with constant coefficients

Example of such a recurrence relation $a_n = 3a_{n-1} + 2n$

generally $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$ ($*$)

inhomogeneous because of the term $F(n)$

Associated homogeneous equation to $*$ is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad (*)$$

Key fact: Every solution to $*$) can be obtained from any (fixed) solution

Theorem 5 Let $\{a_n^p\}$ be any solution to $*$ (called a particular solution) Then every solution to $*$ is of the form $\{a_n^h + a_n^p\}$ when $\{a_n^h\}$ is a solution to ($*$)

Proof suppose $\{b_n\}$ also solves $*$ (besides $\{a_n^p\}$). Then

$$b_n - a_n^p = c_1(b_{n-1} - a_{n-1}^p) + c_2(b_{n-2} - a_{n-2}^p) + \dots + c_k(b_{n-k} - a_{n-k}^p)$$

$F(n)$ term dies

So $\{\underbrace{b_n - a_n^p}_{a_n^h}\}$ solves ($*$) and hence $\{b_n\} = \{a_n^h + a_n^p\}$

□.

So key step when solving ⊛ is to find one solution and after that solve (*)

Ex 10 $\quad a_n = 3a_{n-1} + 2n \qquad a_1 = 3$

homogeneous part $\quad a_n = 3a_{n-1}$

Char. eq $\qquad r - 3 = 0 \quad \Rightarrow$ root $= r$

$a_n^h = \alpha 3^n$ homogeneous solution

Finding $\{a_n^p\}$:

as $F(n) = 2n$ is a linear function we guess a linear function $p_n = cn + d$ as a solution to ⊛

Then we must have $\quad cn + d = 3(c(n-1) + d) + 2n$

$= 3cn - 3c + 3d + 2n$

$= (3c+2)n + 3(d-c)$

must hold for all n so $\quad c = 3c + 2 \quad$ and $\quad d = 3(d-c)$

$\Downarrow \; c = -1 \quad$ and $\quad d = 3(d+1)$

$\Downarrow \; d = -\frac{3}{2}$

so $\quad a_n^p = -n - \frac{3}{2}$ is a solution to ⊛

By Thm 5, all solutions to ⊛ are of the form

$a_n^h + a_n^p$ and we know $a_n^h = \alpha 3^n$

so $\quad a_n = \alpha \cdot 3^n - n - \frac{3}{2}$ is the general solution

we have $a_1 = 3$ so $\quad 3 = \alpha \cdot 3^1 - 1 - \frac{3}{2} = 3\alpha - \frac{5}{2}$

$\Downarrow \; 3 = 3\alpha - \frac{5}{2} \Rightarrow \alpha = \frac{11}{6}$

$$\boxed{a_n = \frac{11}{6} \cdot 3^n - n - \frac{3}{2}}$$

Theorem 6 Suppose $\{a_n\}$ solves

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n) \quad (*)$$

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$

- If s is NOT a root in the characteristic eq

$$(\square) \quad r^k - c_1 r^{k-1} - c_2 r^{k-2} - \quad - c_k = 0$$

then there is a solution to $(*)$ of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \dots + p_0) \cdot s^n$$

- If s IS a root in $(\square)$ and has multiplicity m

then there is a solution to $(*)$ of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_0) \cdot s^n$$

Proof NOT PENSUM!