Institut for Matematik og Datalogi
Syddansk Universitet, Odense

October 3, 2022
JBJ

DM551/MM851 – 1. Exam assignment

Hand in by Monday October 31, 2022 09:00.

Rules

This is the first of two sets of problems which together with the oral exam in January constitute the exam in DM551/MM851. This first set of problems may be solved in groups of up top three. Any collaboration between different groups will be considered as exam fraud. Thus you are not allowed to show your solutions to fellow students, not from your group and you may not discuss the solutions with other groups. On the other hand, you can learn a lot from discussing the problems with each other so you may do this to some extend, such as which methods can be used or similar problems from the book or exercise classes.

It is important that you **try to be as concise as possible but still argue so that the reader can follow your calculations and explanations.** You must use combinatorial arguments to solve the problems. For example in a counting problem it is not enough to generate all solutions and count them. **It is also not enough just to say that the solution follows from an example in the book or similar**. In such a case you should repeat the argument in your own words.

Remember that this (and the second set of assignment to follow later) counts as part of your exam, so do a good job and try to answer all questions carefully.

How to hand in your report

Your report, which should be written in Danish or English, must be handed in on itslearning by Monday October 31 at 09:00.

On the first page you must write your **name(s)** and the first 6 digits of your **CPR-number(s)**. Do not write the last 4 digits!

Exam problems

Solve the following problems. **Remember to justify all answers.**

Problem 1 (6p)

Consider an arrangement of 26 distinct houses $H_1, \ldots, H_{26}$ (so they are listed in that order). Let v_i denote the value of H_i for $i = 1, 2, \ldots, 26$. Note that several houses may have the same value. Show that no matter how we order the houses there will always be indices $1 \leq i_1 < i_2 < \ldots < i_6 \leq 26$ so that we either have $v_{i_1} \leq v_{i_2} \leq \ldots \leq v_{i_6}$ or $v_{i_1} \geq v_{i_2} \geq \ldots \geq v_{i_6}$ **You must give the argument in your solution. It is not**

enough to refer to a result from the book.

Problem 2 (14p)

At a party there are 12 men and 8 women.

(a) In how many ways can we split these 20 persons into two sets so that each set has 6 men and 4 women?

(b) In how many ways can we arrange the 8 women in two circles around two round tables if each table has 4 women and we consider two arrangements identical when each woman has the same two women next to her (so for example $w_1w_2w_3w_4w_1$ is the same as $w_1w_4w_3w_2w_1$ and we do not distinguish between the two tables)?

(c) Now consider two fixed cyclic orderings $w_{i_1}w_{i_2}w_{i_3}w_{i_4}w_{i_1}$ and $w_{i_5}w_{i_6}w_{i_7}w_{i_8}w_{i_5}$ of the 8 women, where $w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}$ are the women seated at table 1 and $w_{i_5}, w_{i_6}, w_{i_7}, w_{i_8}$ are the women seated at table 2.

We want to place the men into the circles in such a way that 6 men are seated at each table and no two women sit next to each other. In how many ways can this be done if we do not distinguish between the men but we do distinguish between the women? Hint: Compare with Exercise 6.5.48.

Problem 3 (10p)

(a) Suppose we choose a random letter x from the 17 letter string 'IMMUNOSUPPRESSIVE' and a random letter y from the string 'RECURRENCE'. What is the probability that $x = y$?

(b) How many different permutations are there of the string 'IMMUNOSUPPRESSIVE'?

Problem 4 (5p)

Prove that for all non-negative integers n we have

$$\sum_{k=0}^{n} \binom{n}{k} 7^k 9^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 8^n$$

Problem 5 (10p)

Consider an experiment where we roll a dice twice. Let X denote the value of the first roll and let Y denote the value of the second roll. Let the random variable Z be the product of the random variables X and Y, so $Z = X \cdot Y$.

(a) What are the different values that Z can take?

(b) Determine $E(Z)$.

Problem 6 (20p)

This problem is on random permutations of the integers $\{1, 2, \ldots, n\}$, that is, random orderings of the integers $\{1, 2, \ldots, n\}$, where each fixed ordering has probability $\frac{1}{n!}$. A permutation π is denoted $(\pi(1), \pi(2), \ldots, \pi(n))$ where $\pi(i)$ is the image of i under the mapping $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Below π is a random permutation of $\{1, 2, \ldots, n\}$.

1. Let $i \in \{1, 2, \ldots, n\}$ be fixed. What is the probability that $\pi(i) = 1$? Hint: how many of the $n!$ permutations satisfy this?

2. What is the expected number of positions i such that $\pi(i) = i$? Hint: use indicator random variables.

3. Let i and j be distinct integers from $\{1, 2, \ldots, n\}$. What is the probability that $\pi(i) = i$ and $\pi(j) = j$? and what is the probability that $\pi(i) = i$ and $\pi(j) \neq j$?

4. Suppose we learn that $\pi(j) = j$. What is that probability that we also have $\pi(i) = i$?

5. Suppose we learn that $\pi(j) \neq j$. What is that probability that we have $\pi(i) = i$?

6. Compare the last probabilities and give an intuitive explanation for why one is (slightly) larger than the other.

Problem 7 (15p)

Let $k \geq 2$ be an integer. A **k-partition** of a set V is any collection of disjoint sets $V_1, V_2, \ldots, V_k$ such that $V_1 \cup \ldots \cup V_k = V$. Note that some of the sets may be empty. We say that a pair of k-partitions $V_1, V_2, \ldots, V_k$ and $V_1', V_2', \ldots, V_k'$ of the same set V are **identical** if there exist distinct integers $i_1, i_2, \ldots, i_k$, all between 1 and k so that $V_j = V_{i_j}$ for $j = 1, 2, \ldots, k$. That is, we can permute the sets of the second partition so that we get the first partition. If two k-partitions are not identical, we say that they are **distinct**.

(a) How many distinct 2-partitions are there of an n-set (set of size n)?

(b) Show that there are 41 distinct 3-partitions of a 5-set.

(c) Derive a closed formula for the number of distinct k-partitions of an n-set.

Hint for (b) and (c): use Stirling numbers of the second kind. Remember to argue why you can do that.

Problem 8 (20p)

The object of the assignment is to use the probabilistic method to prove the following. See the definition of a k-partition in Problem 7 above. Note that now we are not concerned whether two k-partitions are identical or not. We just consider one (random) such partition.

Theorem 1 *Every graph $G = (V, E)$ has a k-partition $V_1, V_2, \ldots, V_k$ such that $|E^*| \geq \frac{k-1}{k}|E|$, where $E^* \subseteq E$ is the of edges from G that do not have both endpoints inside the same set V_i.*

You should prove this by following the steps (a)-(e) below and answering each question on the way.

Consider the following randomized algorithm $\mathcal{A}$: for each vertex $v \in V$ put v in V_i with probability $1/k$, e.g. by picking a random integer i from $\{1, 2, \ldots, k\}$ and putting v in V_i. We say that $\mathcal{A}$ **colours** v by colour i if v is put in V_i.

We now study the resulting k-partite graph H whose vertex partition is the output $V_1, V_2, \ldots, V_k$ from $\mathcal{A}$ and whose edge set E' consists of those edges $e \in E$ whose endpoints have different colours. Below $s \in S$ refers to the random k-partition $V_1, V_2, \ldots, V_k$ that we generated above (and S is the set of all k-partitions).

(a) Argue that the probability that an edge uv belongs to E' is precisely $\frac{k-1}{k}$.

(b) Define, for each edge $e = uv$, the indicator random variable $X_e = X_e(s)$ so that $X_e(s) = 1$ if u and v are in distinct sets in the k-partition s and $X_e(s) = 0$ if u and v are in the same V_i for some i. Prove that the expected value of X_e is $\frac{k-1}{k}$.

(c) Define the random variable $X = X(s)$ to be $X(s) = |E'|$. Show that $X(s) = \sum_{e \in E} X_e(s)$.

(d) Use (c) to prove that the expected value of X is given by $E(X) = \frac{k-1}{k}|E|$.

(e) We saw above that the expected value (size of $|E'|$) of the k-partition generated by $\mathcal{A}$ is $\frac{k-1}{k}|E|$. Use this and the general formula for the expected value of $X(s)$ to prove that there is at least one k-partition with $|E'| \geq \frac{k-1}{k}|E|$. Hint: suppose all values are less than $\frac{k-1}{k}|E|$ and derive a contradiction to (d).