Institut for Matematik og Datalogi
Syddansk Universitet, Odense

11 November 2022
JBJ

DM551-MM851 – 2. Exam assignment

Hand in by Friday December 2 at 12:15.

Rules

This is the second of two sets of problems which together with the oral exam in January constitute the exam in DM551-MM851. **This second set of problems must be solved individually.** Any collaboration with other students will be considered as exam fraud. Thus you are not allowed to show your solutions to fellow students. On the other hand, you can learn a lot from discussing the problems with each other so you may do this to some extend, such as which methods can be used or similar problems from the book or exercise classes.

Remember that this counts as part of your exam, so do a good job and try to answer all questions carefully. It is important that you **argue so that the reader can follow your calculations and explanations**.

How to hand in your report

Your report must be handed in on itslearning by Friday December 2 at 12:15

On the first page you must write your **name** and **first 6 digits of your CPR-number** .

Problems

Solve the following problems and **Remember to justify all answers:**

Problem 1 (15 points)

Consider the following recursive method for finding the maximum and minimum element from a set $S = \{x_1, x_2, \ldots x_{2^n}\}$ consisting of 2^n, $n \geq 1$, distinct real numbers, where M (m) denotes the largest (smallest) number in S:

1. If $n = 1$, then let $M := \max\{x_1, x_2\}$; let m be the other element and return M, m;
2. If $n \geq 2$, then let $S_1 := \{x_1, \ldots, x_{2^{n-1}}\}$, $S_2 := \{x_{2^{n-1}+1}, \ldots, x_{2^n}\}$;
3. Find (recursively using the same method) largest and smallest elements M_i, m_i in S_i, $i = 1, 2$;
4. Let $M := \max\{M_1, M_2\}$ and $m := \min\{m_1, m_2\}$ and return these;

Question a:

Prove that the method above is correct, that is, the numbers M, m returned will be the maximum, respectively, the minimum elements in S.

Question b:

Let a_n be the number of comparisons used by the algorithm on a set S of size 2^n, $n \geq 1$. Prove that a_n satisfies recurrence relation $a_n = 2a_{n-1} + 2$ and $a_1 = 1$.

Question c:

Prove that for all $n \geq 1$ we have $a_n = \frac{3}{2}2^n - 2$. Hint: Solve the recurrence relation for a_n that you derived above. Remember to explain carefully how you do this.

Question d:

Compare the running time of the recursive algorithm to that of the naive algorithm that first finds the maximum element and then the minimum element in the rest. You may still assume that $|S| = 2^n$ for some $n \geq 1$.

Problem 2 (10 points)

Solve the linear recurrence equation $a_n = 7a_{n-1} - 12a_{n-2}$ with initial conditions $a_1 = 1$, $a_2 = 4$.

Problem 3 (20 points)

This problem is about 2-colouring subsets of a set. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set with n elements and let $\mathcal{F}$ be a collection of subsets of S such that every subset $X \in \mathcal{F}$ has size k, where $k \geq 2$ is a fixed integer. Let $c : S \to \{1, 2\}$ be a 2-colouring of the elements in S. We say that a set $X \in \mathcal{F}$ is **monochromatic** if $c(x_i) = c(x_j)$ for all $x_i, x_j \in X$ (all elements of X have the same colour).

Now consider a random 2-colouring c of S. So for each element $x_i \in S$ the probability that $c(x_i) = j$ is $\frac{1}{2}$ for $j = 1, 2$.

Question a:

Prove that if $k = 3$ (all sets in $\mathcal{F}$ have size 3) and $p = |\mathcal{F}|$, then the expected number of sets in $\mathcal{F}$ which are not monochromatic with respect to the colouring c is $\frac{3}{4}p$.

Question b:

Again let $\mathcal{F}$ have p sets, all of which have size 3. Derive an upper bound on the expected number of times one needs to generate a random 2-colouring of S before we obtain a 2-colouring such that at most $\frac{1}{4}p$ of the sets in $\mathcal{F}$ are monochromatic. Hint: use a similar analysis as in Kleinberg-Tardos Section 13.4.

Question c:

Now assume that all sets in $\mathcal{F}$ have size k for some k. Prove that if $p = |\mathcal{F}| \leq 2^{k-1} - 1$, then there exists a 2-colouring c such that no set in $\mathcal{F}$ is monochromatic.

Problem 4 (20 points)

This problem considers robustness of a score (grade) given to students based on a multiple choice test.

Assume that a multiple choice test consists of n questions, each having 4 choices. For each question precisely one choice is correct. Students are allowed to make zero or one "check" (cross) for each question. The score for a question is 1 if the student has checked the correct choice, $-\frac{1}{3}$ if the student has checked a wrong choice and 0 if no choices are checked. The score for the test is computed as the sum of the scores for all questions. The maximal score is therefore n. We assume that the test is used only to decide pass/fail, and the threshold for passing is a 50% score, i.e. a score $\geq \frac{n}{2}$.

Define a **challenged** student to be a student that knows the answers to at most 40% of the questions.

Let us assume that a challenged student leaves no questions unanswered. Then clearly (s)he has nothing to lose by guessing the answers to the questions (s)he does not know. So assume that (s)he accordingly puts down checks at uniformly random choices (one per question).

Define a multiple choice test to be **good**, if the probability that a challenged student passes is at most 5%.

A teacher has to make a test, and naturally he wants it to be good. He suspects that if he has enough questions in the test then it will be good. This is indeed correct as we shall see below.

Consider a challenged student and assume that (s)he guesses uniformly at random one of the 4 possible answers to each of the $m = \frac{3}{5}n$ questions to which (s)he does not know the answer. Define

$$X_i = \begin{cases} 1, & \text{if the } i\text{th guess is correct} \\ 0, & \text{otherwise} \end{cases}$$

Define $X = \sum_{i=1}^{m} X_i$

Question a:

Determine $E[X]$.

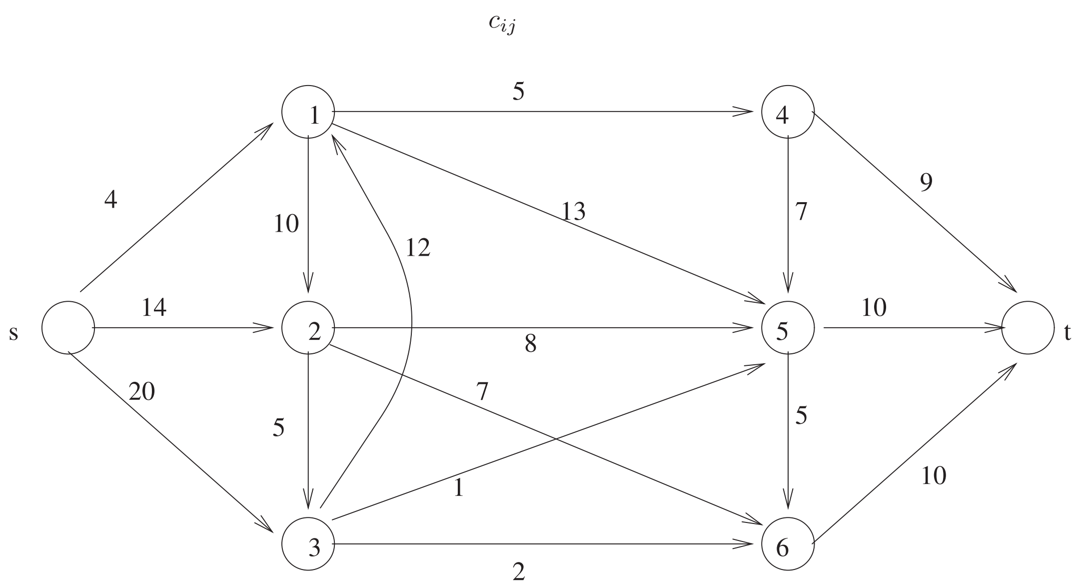
Question b:

Show that the challenged student only passes if $X \geq \frac{3}{2}E[X]$.

Question c:

Use the Chernoff bound technique to determine a size n of the test for which the challenged student only passes with probability at most .05.

Problem 5 (25 points)



Figur 1: The network $N = (V, A, c, s, t)$. The value of the capacity function c is shown along each arc.

Question a:

Give a short description of the Edmonds-Karp algorithm for finding a maximum flow and illustrate the algorithm on the network in Figure 1. Remember to say how much you augment by along each path.
In order to make the correction process easier you should do this exactly as follows: each new augmenting path should not only be a shortest path in the current residual network, it should also have the smallest name lexicographically. That is, the path $s14t$ is the first augmenting path.

You should draw the residual network each time there are no more augmenting paths of the current shortest path length. Thus after listing a set of augmenting paths of length 3 (according to the rule above) so that no more augmenting paths of length 3 can be found (in the current residual network), you give the current residual network and then go on to the next set of paths (which will have length 4, as you will see).

Question b:

Give the values on every arc of the resulting maximum flow f^*, give its value and also a minimum cut whose capacity shows that f^* is a maximum flow.

Question c:

Suppose now that we increase the capacity of the arc from vertex 5 to t to 19. Use your final residual network above to say what the value of a new maximum flow will be and give a new cut that shows that this new value is indeed the maximum.

Problem 6 (10 points)

An **independent set** in a graph $G = (V, E)$ is a set $X \subseteq V$ of vertices so that no edge of E has both end vertices in X. Consider the following method for generating an independent set of a given graph G. Let $\sigma = v_1, v_2, \dots, v_n$ be an ordering (a permutation) of the vertices of V and let $S(\sigma)$ be defined as follows $v_i \in S(\sigma)$ if and only if v_i has no edge to the set $\{v_1, \dots, v_{i-1}\}$ for $i = 1, 2, \dots, n$.

Question a:

Show that $S(\sigma)$ is an independent set of G.

Question b:

Prove that if $\sigma = v_1, v_2, \dots, v_n$ is a random permutation of the vertices in V, then the expected cardinality of the resulting set $S(\sigma)$ is

$$\sum_{i=1}^{n} \frac{1}{d_i + 1},$$

where d_i denotes the degree of the vertex v_i in G.

Question c:

Prove that G has an independent set of size at least $\sum_{i=1}^{n} \frac{1}{d_i+1}$.