DM817 Notes: on applications of Hoffmann's circulation theorem

Jørgen Bang-Jensen, IMADA

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Theorem 0.1 (Hoffmann's circulation theorem) *Let $\mathcal{N} = (V, A, \ell, u)$ be a network. There exists a feasible circulation in $\mathcal{N}$ if and only if the following holds*

$$\ell(S, \bar{S}) \leq u(\bar{S}, S) \qquad \textit{for all } \emptyset \neq S \subset V$$

1 Generalized matchings in graphs

A **generalized matching** M^G in a graph is a collection of edges $e_1, e_2 \ldots, e_k$, $k \geq 0$ and odd cycles $C_1, \ldots, C_r$, $r \geq 0$, such that $e_1, e_2 \ldots, e_k$ form a matching M and $C_1, \ldots, C_r$ are disjoint and none of them contain a vertex from M.

Theorem 1.1 *Let $G = (V, E)$ be an undirected graph. Then G has a generalized matching if and only if G does not contain a set of vertices X which is independent (no edge inside $G[X]$) and has $|N(X)| < |X|$.*

Proof: If G has a generalized matching M^G, then there cannot exist an independent set X with $|N(X)| < |X|$ since each vertex in X will have a private neighbour outside S in M^G (there are no edges inside X). We will show how to use Hoffmann's circulation theorem to prove the other direction in the claim.

For a given undirected graph $G = (V, E)$ we denote by $\overset{\leftrightarrow}{G}$ the digraph we obtain from G be replacing each edge by a directed 2-cycle.

Observation 1: G has a generalized matching if and only if $\overset{\leftrightarrow}{G}$ has a cycle factor.

To see this, first consider a generalized matching M^G consisting of edges $e_1, e_2 \ldots, e_k$, $k \geq 0$ and odd cycles $C_1, \ldots, C_r$, $r \geq 0$. In $\overset{\leftrightarrow}{G}$ these correspond to a cycle factor consisting of a 2-cycle uvu for each edge $e_i = uv$ and an an odd directed cycle C'_i obtained by fixing an orientation of C_i as a directed cycle for each cycle C_i. For the other direction suppose that $\mathcal{F} = W_1, W_2, \ldots, W_p, W_{p+1}, \ldots, W_{p+q}$ is a cycle factor in $\overset{\leftrightarrow}{G}$, where $W_1, \ldots, W_p$ are even cycles and $W_{p+1}, \ldots, W_{p+q}$ are odd cycles. Then we obtain a generalized matching M^G by taking every second arc of the even cycles in $W_1, \ldots, W_p$, deleting the orientation of those arcs, and, for each cycle W_{p+i}, taking the odd cycle C_{p+i} in G which corresponds to W_{p+i} (delete the orientation of the arcs of W_{p+i}).

Now let D be obtained from $\overset{\leftrightarrow}{G}$ by performing the vertex-splitting technique, that is, we replace each vertex v by two copies v', v'', add the arc $v'v''$ and for each arc vw of $\overset{\leftrightarrow}{G}$ we add the arc $v''w'$. Note that by this process every 2-cycle wvw of $\overset{\leftrightarrow}{G}$ becomes a directed 4-cycle $w'w''v'v''w'$.

Observation 2: D has a cycle factor if and only if $\overset{\leftrightarrow}{G}$ has a cycle factor.

This is easy to see: if $w_1w_2 \ldots w_kw_1$ is a cycle in $\overset{\leftrightarrow}{G}$, then $w'_1w''_1w'_2w''_2 \ldots w'_kw''_kw'_1$ is a cycle in D and conversely.

Let $\mathcal{N}$ be the network that we obtain from D by adding the following lower bounds and capacities.

- Arcs of the type $v'v''$ (split arcs) have $\ell_{v'v''} = u_{v'v''} = 1$
- Arcs of the type $v''w'$ have $\ell_{v''w'} = 0$ and $u_{v''w'} = \infty$.

Observation 3: D has a cycle factor if and only if $\mathcal{N}$ has a feasible circulation

If D has a cycle factor $C_1, C_2, \ldots, C_k$, then we obtain a feasible circulation in $\mathcal{N}$ by sending one unit of flow along each of the cycles. Conversely, if x is a feasible circulation in $\mathcal{N}$, then x decomposes into cycle flows of value one along cycles $W_1, \ldots, W_r$. These are all disjoint because $u_{v'v''} = 1$ which ensures that at most one unit of flow can pass through the arcs (and $\ell_{v'v''} = 1$, then ensures that exactly one unit of flow will pass through that arc). Hence $W_1, \ldots, W_r$ is a cycle factor of D.

Now we are ready to finish the proof of the theorem. Suppose that there is no generalized matching in G. By the observations above, this means that there is no feasible circulation in $\mathcal{N}$. By Theorem 0.1 this means that there is a cut $(S, \bar{S})$ such that $\ell(S, \bar{S}) > u(\bar{S}, S)$. The only arcs that have a non-zero lower bound come from the arcs of the form $v'v''$. Let $X' \subseteq S$ be the set of tails of arcs with lower bound 1 from S to $\bar{S}$, let $X'' \subseteq \bar{S}$ be the heads of those arcs and let X be the corresponding set of vertices in $\overleftrightarrow{G}$ (so $X' = \{v' | v \in X\}$ and $X'' = \{v'' | v \in X\}$). Since $u_{v''w'} = \infty$ for every arc in $\mathcal{N}$ which is not a split arc and $\ell(S, \bar{S}) > u(\bar{S}, S)$, we conclude that there is no arc from X'' to X' in $\mathcal{N}$. Thus X is an independent set in $\overleftrightarrow{G}$ (and hence in G). As $u(\bar{S}, S) < \ell(S, \bar{S})$ the only arcs that can cross from $\bar{S}$ to S are those of the form $w'w''$ where $w' \in \bar{S}$ and $w'' \in S$. As we described above, an edge of G corresponds to a 4-cycle in $\mathcal{N}$, so if $w \in V - X$ is adjacent in G to some vertex $v \in V$, then the 4-cycle $v''w'w''v'v''$ is in $\mathcal{N}$ and since the arcs $v''w$ and $w''v'$ have infinite capacity we get that $w' \in \bar{S}$ and $w'' \in S$, implying that the arc $w'w''$ goes from $\bar{S}$ to S. But since $u(\bar{S}, S) < \ell(S, \bar{S})$ this means that there can be at most $|X| - 1$ such vertices w in G. This shows that $|N(X)| < |X|$. $\diamond$

2 Cycle subgraphs covering prescribed vertex sets

Recall that for a digraph D we denote by $\alpha(D)$ the size of a maximum independent set in D. For a subset Z of the vertices of a digraph D we denote by $D[Z]$ the subdigraph induced by the vertices in Z, that is, we keep only the vertices of Z and those arcs that have both end vertices in Z.

Theorem 2.1 *Let $D' = (V, A)$ be a k-strong digraph and let $Z \subset V$ satisfy that $\alpha(Z) \leq k$. Then D' has a cycle subdigraph which covers Z.*

Proof: Let D be obtained from D' by the vertex splitting technique as we obtained D from $\overleftrightarrow{G}$ in the proof of Theorem 1.1. Construct the network $\mathcal{N}$ by adding the following lower bounds and capacities to the arcs of D.

- Arcs of the kind $v'v''$ where $v \in Z$ have $\ell_{v'v''} = u_{v'v''} = 1$.
- Arcs of the kind $v'v''$ where $v \notin Z$ have $\ell_{v'v''} = 0$ and $= u_{v'v''} = 1$.
- Arcs of the kind $v''w'$ have $\ell_{v''w'} = 0$ and $u_{v''w'} = \infty$

As in the proof of Theorem 1.1 it is easy to see that D' has a cycle subdigraph which covers Z if and only if $\mathcal{N}$ has a feasible circulation (this time a feasible circulation still decomposes into disjoint cycles of D but these no longer need to cover all vertices, just those corresponding to Z vertices). Hence it suffices to prove that there must exist a feasible circulation on $\mathcal{N}$ when D' is k-strong.

Suppose there is no such circulation. Then by Theorem 0.1 there is a cut $(S, \bar{S})$ satisfying that $\ell(S, \bar{S}) > u(\bar{S}, S)$. Let X', X'' be the sets that we defined in the proof of Theorem 1.1 (those that are end vertices of arcs with lower bound 1 from S to $\bar{S}$. As in that proof we can conclude that $\mathcal{N}$ has no arc from a vertex in X'' to one in X' so X is an independent set in D'. We can also conclude that $|X| > 1$ since D' is k-strong and $k \geq 1$ (we just need that D is strongly connected) which implies that there is at least one arc from $\bar{S}$ to S. We are going to show that there are in fact at least $|X|$ such arcs and thus that $u(\bar{S}, S) \geq \ell(S, \bar{S})$, contradicting the assumption above.

Fix two vertices $v_1, v_2 \in X$. As D' is k-strong and there is no arc from V_1 to v_2 in D' (X is independent) it follows from Menger's theorem that D' has k internally disjoint paths $P_1, \ldots, P_k$ from v_1 to v_2. In $\mathcal{N}$ these correspond to k internally disjoint paths $Q_1, \ldots, Q_k$ from v_1'' to v_2' (by vertex splitting along each path). Each of these paths start in $\bar{S}$ and end in S so they each contribute at least one to $u(\bar{S}, S)$. But now we get the contradiction $\ell(S, \bar{S}) = |X| \leq \alpha(D[Z]) \leq k \leq u(\bar{S}, S)$, completing the proof. $\diamond$

By inspecting the proof above we can easily see that the following holds.

Corollary 2.2 *A digraph $D = (V, A)$ has a cycle factor if and only if there is no subset $W \subseteq V$ such that W is independent and we can kill all paths from W to itself by deleting less than $|W|$ vertices.*

Proof: This is because paths from the independent set X, that we identified in the proof above from the assumption that there exist a set S with $\ell(S, \bar{S}) > u(\bar{S}, S)$, to itself will correspond to paths from X'' to X' in $\mathcal{N}$ and each will contribute at least one to $u(\bar{S}, S)$. Hence if the assumption of the corollary holds, then we need to delete at least $|X|$ vertices to kill all paths from X to itself. So $u(\bar{S}, S)$ will be at least $|X|$, implying that $\ell(S, \bar{S}) > u(\bar{S}, S)$ cannot hold (if it was smaller we could delete the vertices of D corresponding to the arcs crossing from $\bar{S}$ to S). $\diamond$