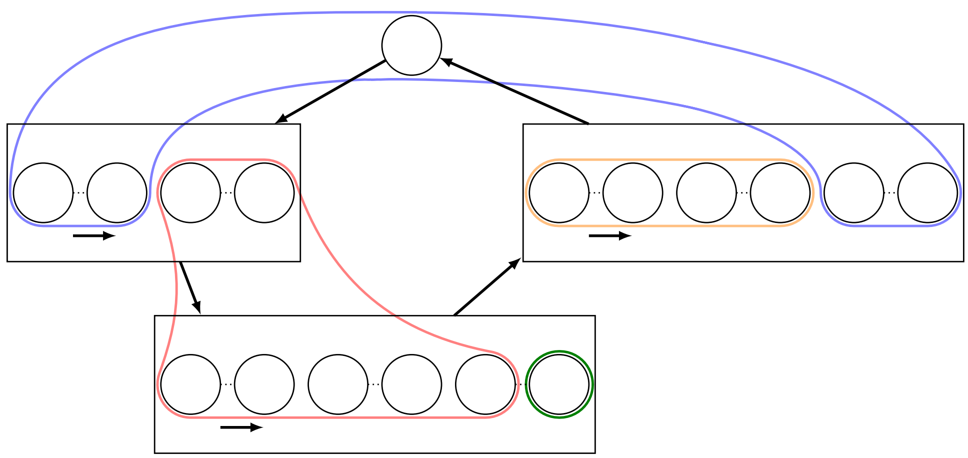
Algorithmic and structural problems in digraphs



PhD dissertation by
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English Abstract

This dissertation presents new results on structures of digraphs obtained during this authors PhD studies. More concretely three graph problems are considered and new results are obtained for each of these. The first problem is the linkage problem, where we are looking for disjoint paths between predefined vertices. We consider generalizations of tournaments and find both polynomial algorithms and sufficient conditions in these digraphs.

The second problem is the problem of partitioning a digraph into disjoint subdigraphs, such that each of the partitions induces a digraph with certain properties. We consider variations of minimum degree properties and find some, but not all, polynomial algorithms for tournaments and semicomplete digraphs.

The third problem, is the problem of finding sufficient conditions for a semicomplete digraph to contain a spanning k-strong tournament. We improve the best known bound for $k = 3$, by showing that every 6-strong semicomplete digraphs contains a 3-strong spanning tournament. This does not confirm a conjecture by Bang-Jensen and Jordán saying that 5-strong is sufficient, and we discuss how one might obtain this bound.

The material found in this dissertation is both a presentation of results from two published papers and of new unpublished results.

Danish Abstract

Denne afhandling præsenterer nye resultater omhadlende strukturelle egenskaber i digrafer fundet i løbet af denne forfatters PhD studier. Mere konkret er der betragtet tre grafproblemer og nye resultater for hver af disse er opnået. Det første af disse problemer, er det såkaldte linkage problem, søger man efter disjunkte veje mellem prædefinerede knuder. Vi har betragtet generaliseringer af turneringer og for disse både fundet polynomielle algoritmer og tilstrækkelige betingelser i disse grafer.

Det andet problem vi har betragtet er problemet med at opdele grafens knuder, således at hvert knudesæt inducerer en digraf med prædefinerede egenskaber. For disse problemer har vi set på forskellige variationer af minimum-valens egenskaber og fundet nogle, men ikke alle, polynomielle algorithmer for problemet på semikomplette digrafer og turneringer.

I det tredje problem forsøges der at opstille nødvendige betingelser for den stærke sammenhængsgrad af semikomplette digrafer, således at vi er sikret at de indeholder en k-stærk udspændende turnering. Vi forbedre det bedst kendte grænse for $k = 3$, ved at vise at en 6-stærk semikomplet digraf indeholder en 3-stærk udspændende turnering. Dette bekræfter ikke en formodning af Bang-Jensen og Jordán som siger at 5-stærk er nok, og vi diskuterer hvordan man kan opnå denne grænse.

Indholdet i denne afhandling er både en præsentation af resultater fra to publiserede artikler og af nye upubliserede resultater.

Preface

In February 2014 I started my PhD in discrete mathematics. Through the last 5 years I have studied a variety of problems on digraphs, and especially considered structural problems on tournaments or generalizations of tournaments. This has resulted in two published papers and a handful of unpublished results.

Apart from research I have contributed with a variety of tasks at the department such as teaching, exam correction and project supervision. I have followed courses that not only broaden my knowledge in mathematics and computer science, but also given me more applicable tools for future use. I have participated in the conferences ICGT 2014, GT2015 and ARCO November 2016, the last with a contributed talk, while a mistimed illness prohibited me to participate and give a talk at BGW16. I also gave a presentation at DTU during my environmental change. This dissertation now marks the end of my life as a PhD student.

In this dissertation I will present much of my research, both published and unpublished. Three graph problems are considered and the dissertation is divided in three parts accordingly. The structure of each part is the same: First I give a 'state-of-the-art' introduction to the problem, then my contribution to the field is presented and third a discussion and summary of open problems is given. While the presentation of published results will be done by extracting certain elements, new results will be followed by full proofs and illustrations. The dissertation ends with a short recap of open problems that, had time permitted it, would have been subject to future work. On a technical note, all published results is marked by citation, while unpublished work is marked on the form [NEW: name of author(s)]. Sometimes results have been proven but not explicitly stated, and for these results I write ['NEW': name of author(s)]. Onward I will refer to the author(s) as we, and only when needed to clarify that it is my opinion/claim write 'this author'.

Now a short introduction to the main contributions found in this dissertation. Part I considers the k-linkage problem and Chapter 3 contains the main results of the paper 'Disjoint paths in decomposable digraphs'. In this paper we proved that for a wide class of digraphs, the k-linkage problems is polynomial. The most prominent digraphs covered by this are the locally semicomplete, extended semicomplete, round decomposable and quasi-transitive digraphs. We also confirm two conjectures by proving sufficient strong connectivity on round decomposable digraphs respectively locally semicomplete digraphs to be k- respectively 2-linked. Now it is known that there exists a function $f(k)$ such that every $f(k)$-strong locally semicomplete digraph is k-linked and Bang-Jensen conjectured that $f(k)$ is linear. In Chapter 4 we prove (for the first time) that this is true. The bound of $f(k)$ is, not unexpectedly, strongly related to the equivalent bound for semicomplete digraphs. Hence improving this bound for semicomplete digraphs will directly imply an improvement on the bound for locally semicomplete digraphs.

In Part II, we consider partition problems, where we wish to decide or find a partition of the vertices of a digraph, such that each part induces a digraph with certain properties. In the paper 'Degree constrained 2-partitions of semicomplete digraphs', we prove a number of such results on semicomplete digraphs. We describe all these in Chapter 6. To mention some; We prove that partitioning a semicomplete digraph such that each partition induces digraphs with minimum out-degree k can be done in polynomial time, while it is NP complete for general digraphs. On the other hand we could not prove or disprove similar polynomial algorithm for semicomplete digraphs in the case where one part induces a digraph with minimum out-degree k_1 and the other a digraph with minimum in-degree k_2, unless $k_1 = 1$. In Chapter 7 we prove a new result, by deriving an algorithm to find a partition where each part induces a tournament with minimum semi-degree at least k, given that the minimum semi-degree is large in the original digraph. This algorithm improves the best known algorithm derived from an existence proof of the partition by Lichiardopol.

Finally in Part III we consider the problem of finding a k-strong spanning tournament in a $g(k)$-strong semicomplete digraph. While the best known result for this problem was obtained by Guo, showing that a $(3k-2)$-strong semicomplete digraph contains a k-strong spanning tournament, Bang-Jensen and Jordán conjectured that the correct bound is $2k-1$. In Chapter 10 we consider the problem for $k = 3$. We prove three subresults which all improves Guo's '7' for $k = 3$, but none of them reaches the conjecture of Bang-Jensen and Jordán. One of the three results is that every 6-strong semicomplete digraph contains a 3-strong spanning tournament. In Chapter 8, we discuss whether the correct bound might be $2k+1$ and suggest how the conjecture can be proven both for $k = 3$ and for general k.

I hope you will enjoy reading this dissertation and might even be inspired to solve some of the open problems found here.

Publication list

- Disjoint paths in decomposable digraphs. Joint work Jørgen Bang-Jensen and Alessandro Maddaloni. Published in Journal of Graph Theory 2016, vol 85 no 2, 545-567
- Degree constrained 2-partitions of semicomplete digraphs. Joint work with Jørgen Bang-Jensen. Theoretical Computer Science 2018, In Press. Available online.

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Contents

Chapter 1

Terminology, preliminaries and graph classes

In this chapter we will introduce most of the terminology and preliminary results used in this dissertation. As this is standard notation and results, a fast skim through should be sufficient for readers familiar with directed graph theory.

This chapter also contains an introduction to graph classes. Again most are well-known and can also be found in [10]. An exception to this is the refined structure of the class ***evil locally semicomplete digraph***. This was introduces in [7] and should be read to understand the proofs in Chapter 2.

1.1 Terminology and preliminaries

We will often have to pick an index belonging to a set of integers. For this reason we will use the simplified notation $[k]$=$\{1, 2, \ldots, k\}$, while $[l, k]$ is the set $\{l, l + 1 \ldots, k\}$ for $1 < l < k$. Sets and set-operations are standard. If X, Y are two sets containing the same type of elements, then $X - Y$ denotes the set of elements that are in X and not in Y. The number of elements in a set X is denoted $|X|$ and we will either call this number the ***cardinality***, the ***size*** or the ***order*** of X. If we have a set consisting of just one element we may skip the brackets to simplify notation, i.e if $X = \{x\}$ we write $X = x$.

A ***digraph*** D is defined by the two sets: the ***vertices*** $V(D)$ and the ***arcs*** $A(D)$. When clear in the context. (D) is omitted, and we just write V, respectively A. The small letter n is reserved for the size of V. All digraphs considered are ***simple*** (no two arcs between the same pair of vertices and in the same direction) and without ***loops*** (arc where both ends are identical).

Let $a = uv \in A$ be an arc of D, then u is called the ***tail*** of a, v the ***head*** of a and u, v are the ***ends*** of a. We say that u is ***adjacent to*** v and v is ***adjacent from*** u. We may also say that a is ***incident with*** u, if u is either the head or tail of a and when specifying direction we say ***incident to*** or ***incident from***.

Given a digraph D, a ***subdigraph*** of D is a digraph obtained by taking a subset of the vertices of D and a subset of the arcs among those arcs with both ends in this chosen vertex set. If D' is the subdigraph with vertexset $X \subseteq V$ and it contains every arc with both ends in X, then D' is called an ***induced subgraph*** of D and is denoted $D\langle X\rangle$. For $Y \subseteq V$, we simplify notation by writing $D - Y$ instead of $D\langle V - Y\rangle$. Similarly, for $E \subseteq A$, we may write $D - E$ to denote the digraph obtained by deleting the arcs E from the digraph D.

Let $v \in V(D)$, then $d^+(v)$ ($d^-(v)$) denotes the number of arcs that has tail (head) in v and is called the ***out-degree*** (***in-degree***) of v. Furthermore, $N^+(v)$ ($N^-(v)$) is the set of ***out-neighbours*** (***in-neighbours***) of v. More general let $X \subset V(D)$, then $N^+[X]$ is the union of the out-neighbours of the elements of X and $N^+(X)$ $= N^+[X] - X$. The smallest out-degree (in-degree) of a vertex in D is called the ***minimum out-degree*** (***minimum in-degree***) of D and is denoted $\delta^+(D)$ ($\delta^+(D)$) and the ***minimum semi-degree*** of D is $\delta^0(D) = min\{\delta^+(D), \delta^-(D)\}$. Similar the largest out-, in- and semi-degree is denoted $\Delta^+(D)$, $\Delta^-(D)$ and $\Delta^0(D)$. If we need to specify which digraph these refer to, we add a subscript D. We say that a digraph is ***regular*** or k-***regular*** if $\delta^0 = \Delta^0 = k$.

A sequence of numbers is said to be a ***degree sequence*** of a (di)graph D if its vertices has (out/in) degree accordantly to this sequence. For the class of digraphs called tournament (se later) this sequence is also called the ***score sequence***.

Let X and Y be disjoint subsets of $V(D)$. Then we say X ***dominates*** Y, and write $X \rightarrow Y$, if every vertex in X is adjacent to every vertex in Y. Further, if no vertices of Y is adjacent to any vertex of X and want to clarify this we write $X \Rightarrow Y$ and say that X ***completely dominates*** Y[1].

A (directed) ***path*** in D is a subdigraph consisting of a sequence of distinct vertices $P = v_1 v_2, \dots v_{|V(P)|}$ and arcs $v_i v_{i+1}$ for all $i \in [|V(P)| - 1]$. The ***length*** of the path is the number of arcs in P. The ***internal*** vertices of P is $v_2, v_3, \dots, v_{|V(P)|-1}$, while the first (last) vertex of the path is the ***tail*** (***head***) and is denoted $t(P)$ ($h(P)$). We may also say that the path ***starts*** in v_1 and ***ends*** in $v_{|V(P)|}$. For each $i \in [2, |V(P)|]$, v_i is the ***predecessor*** of v_{i+1} and for $j \in [|A(P)|]$ v_j is the ***successor*** of v_{j-1}. For

[1] For tournaments where 2-cycles are not allowed we will say X dominates Y though the correct formulation is X completely dominates Y

two distinct vertices $u, v \in V(D)$ a (u, v)***-path*** is a path with tail u and head v and a ***minimal*** (u, v)-paths in D is a path in D that among all (u, v)-path uses the fewest number of vertices. If $v_i, v_j \in V(P)$ and $i < j$ then we denote by $P[v_i, v_j]$ the ***subpath*** of P with tail v_i and head v_j.

Consider two paths P, Q. Then P is ***internally disjoint*** from Q if no internal vertices of P belongs to Q. Furthermore P and Q are ***internally disjoint*** if both P is internally disjoint from Q and Q is internally disjoint from P. If P and Q do not share any vertices, then we say that P and Q are ***vertex disjoint***.

A ***linkage***, L, in a digraph D is a collection of vertex disjoint paths, though we may sometimes allow the ends of these paths to be shared. Further a k***-linkage*** is a linkage consisting of k paths. Once again $t(L)$ is the tail of the linkage and is the collection of the tails of the paths in the linkage. Similar $h(L)$ is the head of the linkage and is the collection of the heads of the paths in the linkage.

The collection of all heads and tails in a linkage is also called the ***terminals*** of L.

A ***cycle*** in a digraph is a path P where $h(P) = t(P)$ and a ***proper cycle*** is a cycle consisting of more than one vertex. To specify the length of a cycle, a cycle of k vertices is called a k***-cycle***.

A digraph D is ***acyclic*** if it does not contain any cycles. For such digraphs there always exist an ***acyclic ordering*** (though not necessary unique) of the vertices $v_1, v_2, \dots v_n$, such that if v_iv_j is an arc of D for some $i, j \in [n]$ then $i < j$. A ***feedback vertex set*** of D is a set $X \subset V$ such that $D - X$ is an acyclic digraph. A digraph D is said to be ***transitive*** if there for every pair of arcs uv and vw also is the arc uw. A digraph that do not contain any cycles of length 2 is called an ***oriented graph***

A ***connected*** digraph is a digraph where the underlying undirected graph is connected, i.e a graph where there is an undirected path between every pair of vertices. A ***strong*** digraph is a digraph where there for every pair of distinct vertices $u, v \in V(D)$ exist a (u, v)-path and a (v, u)-path in D. A ***strong component*** of a digraph D is a maximal induced subdigraph of D which is strong. A strong component is ***trivial*** if it has size 1. Let D be a non-strong digraph with strong components $D_1, D_2, \dots, D_p$ for $p \leq 2$. Then the ***strong component digraph*** $SC(D)$ is the digraph obtained by replacing D_i with a single vertex v_i for each $i \in [p]$ and deleting every parallel arcs. Clearly $SC(D)$ is an acyclic digraph and its acyclic ordering gives a natural acyclic ordering of the strong components of D.

If D is strong, then a set S of vertices is said to be a ***separator*** of D, if $D - S$ are not strong. D is ***k-strongly connected*** or ***k-strong*** if $|V| \geq k + 1$ and D has no separator of less than k vertices. The largest integer k such that D is k-strong is called the ***vertex strong connectivity***. If S is a separator, but $S' \subset S$ is not a separator for any subset S', then S is a ***minimal separator***. Further, for a pair of distinct vertices $u, v \in V$ an (u, v)-***separator*** is a set $S' \subseteq V - \{u, v\}$ such that there is no (u, v)-path in $D - S'$. For such a pair we call the minimum size of such an (u, v)-separator for the ***local connectivity*** of (u, v) and its size is denoted $\kappa(u, v)$. The following result by Thomassen and Bang-Jensen gives a usefull relation between disjoint paths and the local connectivity in digraphs.

Proposition 1.1. *[4, 48] Let D be a digraph and x, y, u, v be distinct vertices of D. Assume that $\kappa(u, v) \geq q + 2$ and $P_1, P_2, \ldots, P_p$ are internally disjoint (x, y)-paths in D such that $D\langle V(P_1) \cup \cdots \cup (V(P_p)\rangle$ has no (x, y)-path of length less than or equal to 3 and the predecessor of y on P_j dominates that successor of x on P_i for all $i, j \in [p]$. Then D has q internally disjoint (u, v)-paths, the union of which intersects at most $2q$ of the paths $P_1, \ldots, P_p$.*

When we want to remove arcs instead of vertices, the equivalence to a separator is called a ***cut***. This is a collection of arcs C in a strong digraph D such that $D - C$ is not strong. Now we can state the classical result by Menger.

Theorem 1.2. *[39] Let D be a digraph and u, v be distinct vertices of D. Then the following holds:*

1. *The maximum number of arc-disjoint (u, v)-paths is equal to the size of a minimum (u, v)-cut of D.*
2. *If uv is not an arc of D, then the maximal number of vertex-disjoint (u, v)-paths in D is equal to the minimum size of a (u, v)-separator of D.*

A subset $X \subseteq V(D)$ is said to be an ***independent set*** in D if there are no arcs between any pair of vertices in X. The cardinality of the maximal independent set in D is called the ***independence number*** of D and is denoted $\alpha(D)$. The opposite of an independent set is a ***clique***(or a complete graph) and is a set where every pair of vertices are adjacent. The classical result by Turan gives a bound on the number of edges (undirected arcs) in a graph not containing large cliques.

Theorem 1.3. *[51] Let D be a graph without any cliques of size $r+1$, then the number of edges in G is at most*

$$\left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

A 2-***partition*** of a digraph D is a partition of $V(D)$ into disjoint sets V_1 and V_2 such that $V_1 \cup V_2 = V(D)$. Similarly, an r-***partition*** of D is a partition of $V(D)$

into exactly r disjoint sets $V_1, V_2, \dots, V_r$ such that $V = \cup_{i\in[r]} V_i$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be graph properties, then (V_1, V_2) is a $(\mathcal{P}_1, \mathcal{P}_2)$-partition, if V_1 induces a digraph with property $\mathcal{P}_1$ and V_2 induces a digraph with property $\mathcal{P}_2$.

1.2 Graph classes

We can now define the different graph classes that will be used in the dissertation.

A ***tournament*** is a digraph, where there between any pair of vertices are exactly one arc, while a ***semicomplete digraph*** is a tournament where we allow an arc in both direction. A ***locally in-semicomplete digraph*** is a digraph where $N^-(v)$ induced a semicomplete digraph for every vertex $v \in V(D)$. Similar a ***locally out-semicomplete digraph*** is a digraph where $N^+(u)$ induces a semicomplete digraph for every vertex $u \in V(D)$. Now a digraph that is both locally in-semicomplete and locally out-semicomplete is called a ***locally semicomplete digraph***. In Section 1.2.1, we will describe locally semicomplete digraphs in more detail. We may replace 'semicomplete' with 'tournament' in the above and define ***local in-tournament***, ***local out-tournament*** and ***local tournament*** similarly.

As mentioned in the last section, if D is a non-strong digraph, then $SC(D)$ has an acyclic ordering. Now if D is a (local) semicomplete digraphs, then this ordering is ***unique***. We will use this ordering to label the strong components of D, $D_1, D_2, \dots, D_p$, $p \geq 2$. D_1 is called the ***initial component*** or ***source*** and D_p the ***terminal component*** or ***sink*** of D. When we know that D is a tournament, we replace $'D'$ with $'T'$ to empathize this, i.e the strong components is denoted $T_1, T_2, \dots, T_p$. An acyclic tournament, or a tournament where all strong components are trivial, is also called a ***transitive tournament***.

We may construct a new digraph on the 'backbone' on another digraphs. Let D be a digraph with vertices $v_1, v_2, \dots v_n$ and let $G_1, G_2, \dots G_n$ be n pairwise vertex disjoint digraphs. Then we define the ***composition*** $C = D[G_1, \dots G_n]$ to be a digraph where each vertex v_i of D is replaced by the digraph G_i, and where an arc v_iv_j of D implies that $G_i \Rightarrow G_j$. A special composition graph, is the one where vertices are replaced by independent sets, this is called the ***extension*** of D.

Let Φ be a class of digraphs. Then we say that a digraph D is Φ***-decomposable***, if either $D \in \Phi$ or $D = H[S_1, \dots S_h]$ for some $H \in \Phi$ with $h = |V(H)| \geq 2$ and some choice of digraphs $S_1, S_2, \dots S_h$. We call $D = H[S_1, \dots S_h]$ the Φ***-decomposition*** of D. Further, a digraph D is ***totally*** Φ***-decomposable*** if either $D \in \Phi$ or there exists a Φ-decomposition $D = H[S_1, \dots S_h]$ such that each S_i is totally Φ-decomposable.

A ***round*** digraph is a digraph where there exist an enumeration $v_1, v_2, \dots v_n$ of the vertices of D such that for each vertex v_i, $N^+(v_i) = \{v_{i+1}, v_{i+2}, \dots v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, v_{i-d^-(v_i)+1}, \dots v_{i-1}\}$ and a ***round decomposable*** digraph $D = R[S_1, \dots S_r]$ is a digraph where R is a round local tournament, $r \geq 2$ and S_i are strong semicomplete digraphs. We call $R[D_1, D_2 \dots, D_r]$ a ***round decomposition*** of D.

A ***quasi-transitive digraph*** is a digraph where for every pair of arcs uv and uw there will either be an arc uw or an arc wu. Notice quasi-transitive digraph is not generally transitive. Bang-Jensen and Huang found a complete classification of the quasi-transitive digraphs.

Theorem 1.4. *[13] Let D be a quasi-transitive digraph.*

1. *If D is not strong, then there exist a transitive acyclic digraph T on t vertices and strong quasi-transitive digraphs $H_1, ..., H_t$ such that $D = T[H_1, ..., H_t]$*
2. *If D is strong, then there exist a strong semicomplete digraph S on s vertices and quasi-transitive digraphs $Q_1, ..., Q_s$ such that each Q_i is either a single vertex or is non-strong and $D = S[Q_1, ..., Q_s]$.*

Moreover one can find the above decomposition in polynomial time.

We will end this section with two classes of graphs:

$$\Phi_1 := \{ \text{ Semicomplete digraphs } \} \bigcup \{ \text{ Acyclic digraphs } \}$$

$$\Phi_2 := \{ \text{ Semicomplete digraphs } \} \bigcup \{ \text{ Round digraphs } \}$$

Notice that Theorem 1.4 shows that quasi-transitive digraphs are totally Φ_1-decomposable. Also extended semicomplete digraphs are totally Φ_1-decomposable. Furthermore, there exist an algorithm that decides and finds a Φ_2-decomposable. Now combining that we can decide in polynomial time whether a digraph is round decomposable with polynomial algorithms for totally ϕ_i-decomposition obtained by Bang-Jensen and Gitin [10], we can find Φ_i-decompositions of totally ϕ_i-decomposable digraphs.

Theorem 1.5. *[10] For each $i \in [2]$, there exists a polynomial algorithm that finds s total Φ_i -decomposition of every totally Φ_i-decomposable digraphs.*

1.2.1 Locally semicomplete digraphs

In this section we will describe locally semicomplete digraph and introduce a number of results for this class. We will especially give a detailed description of a subclass that in many cases has proven to be the most difficult and tedious subclass to handle. For this reason this subclass is also called ***the evil class***. Most of the results presented here are also collected in [10], while the refinement of the evil class were introduced in [7]. This refinement plays a major role in the proof of the main theorem in [7] (Se Chapter 2). As this refinement is self-contained, it seemed natural to introduce this here together with a new Lemma. One could hope that these might help to prove other results for the evil class.

As mentioned above, a ***locally semicomplete digraph*** is a digraph where both the in-neighbourhood and out-neighbourhood of each vertex induces a semicomplete digraph. If the neighbourhood induces tournaments for every vertex, then the digraph is called a ***local tournament***.

We will not mention all results on locally semicomplete digraphs. There are many results on semicomplete digraphs that also applies to locally semicomplete digraphs. For example one could mention the existence of Hamiltonian path and Hamiltonian cycle in connected respectively strong locally semicomplete digraph [3].

As mentioned in the last section non-strong locally semicomplete digraphs has a unique acyclic ordering, and we denote the strong components $D_1, D_2, \ldots, D_p$ following this ordering. Furthermore, if we consider a strong locally semicomplete digraph with separator S, then $D_1, D_2, \ldots, D_p$ is the strong components of $D - S$ and $D_{p+1}, \ldots, D_q$ is the strong components of S.

We will now present some of the structural results on locally semicomplete digraphs. Notice that these are presented given a natural flow in this dissertation, and not by the order in which they are proven. First locally semicomplete digraphs might be round decomposable, and we can decide whether this is the case in polynomial time.

Theorem 1.6. *[9] It can be decided (and found) in polynomial time whether a locally semicomplete digraph is round decomposable, and if so the round decomposition* $D = R[D_1, D_2, ..., D_r]$, $r \geq 2$ *is unique.*

Considering a non strong locally semicomplete digraph D, then we will collect the strong components of D into a unique ordering of ***semicomplete components*** in the following way.

Theorem 1.7. *[25] Let* D *be a non-strong locally semicomplete digraph. Then* D *can* *be decomposed into $r \geq 2$ disjoint subdigraphs $D'_1, D'_2, ..., D'_r$ as follows:*

$$D'_1 = D_p, \quad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\{\, j \mid N^+(D_j) \cap V(D'_i) \neq \emptyset\},$$

$$\textit{and} \quad D'_{i+1} = D\langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \cdots \cup V(D_{\lambda_i - 1})\rangle.$$

The subdigraphs $D'_1, D'_2, ..., D'_r$ satisfy the properties below:

(a) D'_i consists of some strong components that are consecutive in the acyclic ordering of the strong components of D and is semicomplete for $i = 1, 2, ..., r$;

(b) D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, ..., r-1$;

(c) if $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|j-i| \geq 2$. $\square$

The unique sequence $D'_1, D'_2, ..., D'_r$ defined in Theorem 1.7 will be referred to as the ***semicomplete decomposition*** of D.

Now considering strong locally semicomplete digraphs, it can be shown that minimal separators either them-self are semicomplete, or leaves the remaining of the locally semicomplete digraphs as a semicomplete digraph.

Lemma 1.8. *[9] Let S be a minimal separator of a strong locally semicomplete digraph D. Then either $D\langle S\rangle$ is semicomplete, or $D\langle V - S\rangle$ is semicomplete.*

We will call a locally semicomplete digraph which is not semicomplete and not round decomposable for an ***evil locally semicomplete digraph***. With this definition we can now present a full classification of locally semicomplete digraphs.

Theorem 1.9. *[9] If D is a locally semicomplete digraph. Then it has the structure of exactly one of the following and deciding which can be done in polynomial time.*

(a) $D = R[D_1, D_2, ..., D_r]$ is round decomposable with a unique round decomposition, where R is a round local tournament on $r \geq 2$ vertices.

(b) D is evil.

(c) D is a semicomplete digraph which is not round decomposable.

In the remaining of this section we will give detailed description on the structure of the evil class. Let us call a separator S of a locally semicomplete digraph D ***good*** if S is a minimal separator and $D - S$ is not semicomplete.

Theorem 1.10. *[9] Let D be an evil locally semicomplete digraph. Then D is strong and satisfies the following properties.*

(a) *There is a good separator S such that the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3 (and $D\langle S\rangle$ is semicomplete by Lemma 1.8);*

(b) *Furthermore, for each such S, there are integers α, β, μ, ν with $\lambda_2 \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that*

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \quad \text{and} \quad N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset,$$

$$\text{or} \quad N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \quad \text{and} \quad N^+(D_\mu) \cap V(D_\beta) \neq \emptyset,$$

where $D_1, D_2, ..., D_p$ and $D_{p+1}, ..., D_{p+q}$ are the strong decomposition of $D - S$ and $D\langle S\rangle$, respectively, and D_{λ_2} is the initial component of D'_2 (See Figure 1.1).

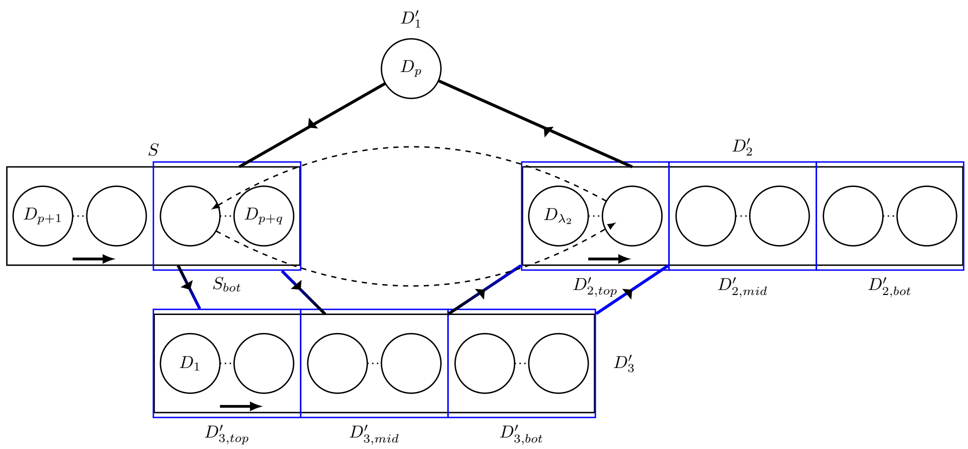


Figure 1.1

Corollary 1.11. *[9] If D is an evil locally semicomplete digraph, then it has independence number at most 2.*

The following lemma gives important information about the arcs in an evil locally semicomplete digraph.

Lemma 1.12. *[9] Let D be an evil locally semicomplete digraph and let S be a good separator of D. Then the following holds:*

(i) $D_p \Rightarrow S \Rightarrow D_1$.

(ii) *If sv is an arc from S to D'_2 with $s \in V(D_i)$ and $v \in V(D_j)$, then*

$$D_i \cup D_{i+1} \cup \ldots \cup D_{p+q} \Rightarrow D_1 \cup \ldots \cup D_{\lambda_2 - 1} \Rightarrow D_{\lambda_2} \cup \ldots \cup D_j.$$

(iii) $D_{p+q} \Rightarrow D'_3$ *and* $D_f \Rightarrow D_{f+1}$ *for* $f \in [p+q]$, *where* $p+q+1 = 1$.

(iv) If there is any arc from D_i to D_j with $i \in [\lambda_2 - 1]$ and $j \in [\lambda_2, p-1]$, then $D_a \Rightarrow D_b$ for all $a \in [i, \lambda_2 - 1]$ and $b \in [\lambda_2, j]$.

(v) If there is any arc from D_k to D_ℓ with $k \in [p+1, p+q]$ and $\ell \in [\lambda_2 - 1]$, then $D_a \Rightarrow D_b$ for all $a \in [k, p+q]$ and $b \in [\ell]$.

We are now ready to refine the structure of the evil locally semicomplete digraph D as done in [7]. In this paper D is assumed 5-strong, but in the following we will assume k-strong for any integer $k \geq 1$.

Refinement of k-strong evil locally semicomplete digraph:

The following indexes are well-defined by Theorem 1.10 (b):

- $\mu \in [q]$ is the smallest index such that there is an arc from $D_{p+\mu}$ to D'_2
- $\gamma \in [p-1]_{\lambda_2}$ is the largest index such that there is an arc from S to D_γ.

The blocks of D'_2.

- $\boldsymbol{D'_{2,top}}$ is the union of the strong components of D'_2 that are dominated by all vertices of D'_3. By Lemma 1.12 (ii), $D_{\lambda_2}, D_{\lambda_2+1} \ldots, D_\gamma$, are all in $D'_{2,top}$.
- $\boldsymbol{D'_{2,mid}}$ is the (possibly empty) union of those strong components of $D'_2 - D'_{2,top}$ that are dominated by some vertex of D'_3. By Lemma 1.12 (iv) we have $D_{\lambda_2 - 1} \Rightarrow D'_{2,top} \cup D'_{2,mid}$.
- $\boldsymbol{D'_{2,bot}}$ is the (possibly empty) union of those strong components of D'_2 that have no neighbours in D'_3.

The blocks of D'_3.

- $\boldsymbol{D'_{3,top}}$ is the set of strong components of D'_3 that are dominated by all vertices of S.
- $\boldsymbol{D'_{3,bot}}$ is the set of strong components of D'_3 that dominate all vertices of $D'_{2,top} \cup D'_{2,mid}$. By Lemma 1.12 (iv), $D_{\lambda_2 - 1}$ is contained in $D'_{3,bot}$.
- $\boldsymbol{D'_{3,mid}}$ is the (possibly empty)[2] set of strong components of $D'_3 - D'_{3,top} - D'_{3,bot}$.

The blocks of S. Only one part of S plays a special role, namely $\boldsymbol{S_{bot}}$ which is the union of the strong components $D_{p+\mu}, D_{p+\mu+1}, \ldots, D_{p+q}$. By Lemma 1.12 (ii) every vertex of S_{bot} dominates all of D'_3.

With these definitions and by Theorem 1.10 respectively Lemma 1.12 we can conclude the following:

[2] Note that we may have $D'_{3,top} \cap D'_{3,bot} \neq \emptyset$ (in which case $D'_{3,mid} = \emptyset$)

E1) There is at least one arc sv from S_{bot} to $D'_{2,top}$ and at least one arc from $D'_{2,top}$ to S_{bot}.

E2) The set $S^* = D'_{2,top} \cup D'_{2,mid}$ is also a good separator. (Follows as there is no arc from D'_3 to $D'_{2,bot} \cup D'_1 \cup S$).

Theorem 1.13. *[7] Let D be a k-strong evil locally semicomplete digraph with semicomplete decomposition D'_1, D'_2, D'_3 and S minimal good separator. Furthermore, let $S^* = D'_{2,top} \cup D'_{2,mid}$ and $\overleftarrow{D}$ be the locally semicomplete digraph where all arcs of D are reversed. Then we have the following semicomplete decomposition:*

1. For $\overleftarrow{D}$ with minimal separator S, the semicomplete decomposition of $\overleftarrow{D} - S$ is:

1.1 $\overleftarrow{D'_1} = D_1$, that is, the first strong component of $D'_{3,top}$.

1.2 $\overleftarrow{D'_2} = \overleftarrow{D}'_{2,top} \cup \overleftarrow{D'_3} - \overleftarrow{D_1}$.

1.3 $\overleftarrow{D'_3} = D'_{2,mid} \cup D'_{2,bot} \cup D'_1$.

See Figure 1.2b.

2. For D with minimal separator S^*, the semicomplete decomposition of $D - S^*$ is:

2.1 $D'^*_1 = D_{\lambda_2 - 1}$ is the last strong component of $D'_{3,bot}$.

2.2 $D'^*_2 = \widetilde{S} \cup D'_3 - D'^*_1$.

2.3 $D'^*_3 = D'_{2,bot} \cup D'_1 \cup S - \widetilde{S}$.

where $\widetilde{S} \subseteq S$ is the set of vertices of S that dominate all vertices of $D'_{3,bot}$. Notice that, by Lemma 1.12 and the definition of S_{bot}, we have $S_{bot} \subseteq \widetilde{S}$. See Figure 1.3a.

3. For $\overleftarrow{D}$ with minimal separator S^*, the semicomplete decomposition of $\overleftarrow{D} - S^*$ is: *The semicomplete decomposition $\overleftarrow{D}^*_1, \overleftarrow{D}^*_2, \overleftarrow{D}^*_3$ of $\overleftarrow{D} - S^*$ has the following form. If $D'_{2,bot} = \emptyset$ ($S^* = D'_2$) then we have $\overleftarrow{D}^*_1 = D_p, \overleftarrow{D}^*_2 = S$ and $\overleftarrow{D}^*_3 = D'_3$. Otherwise, r is defined, and we have*

3.1 $\overleftarrow{D}^*_1 = D_r$.

3.2 $\overleftarrow{D}^*_2 = X \cup D_p \cup Y$.

3.3 $\overleftarrow{D}^*_3 = D'_3 \cup S - Y$.

See Figure 1.3b.

Proof. First notice that the proof is a copy of Lemma 6.2, 6.3 and 6.4 of [7].

1: By the definition of the semicomplete decomposition, $\overleftarrow{D'_1}$ must be a strong component of $\overleftarrow{D} - S$ and this dominates S in $\overleftarrow{D}$, so $\overleftarrow{D'_1} = D_1$. For 1.2. notice first that, by the definition of the semicomplete decomposition, $\overleftarrow{D'_2}$ is formed by those strong components of $\overleftarrow{D}$ that dominate $\overleftarrow{D'_1}$. Thus, as the first component of D'_3 dominates all other vertices of D'_3 in D, clearly when reorienting the arcs, all other vertices of D'_3 dominate $\overleftarrow{D'_1}$. Secondly, by Lemma 1.12 (ii) $D'_{2,top}$ is dominated by all vertices of D'_3 implying that $\overleftarrow{D'}_{2,top}$ dominates $\overleftarrow{D'_1}$ in $\overleftarrow{D}$. By the definition of $D'_{2,mid}$ and Lemma 1.12 (iv), no vertex in $D'_{2,mid}$ is dominated by a vertex in D_1 in D. 1.3. follows from Theorem 1.10 since the semicomplete decomposition of an evil locally semicomplete digraph has exactly three components.

2: This is again based on the structural information from Lemma 1.12 and is very similar to the proof of 1.

3: If $D'_{2,bot} \neq \emptyset$, then let $r \in [p-1]_{\lambda_2}$ be the smallest index such that $D_r \subseteq D'_{2,bot}$ and let $X = D_{r+1} \cup \ldots \cup D_{p-1}$. Now, since every vertex of $D'_{2,top}$ dominates every vertex of D_r we get that $D_r \Rightarrow D_y$ for every $y \in [p+q]_{p+1}$ such that there is an arc from $D'_{2,top}$ to D_y and there is at least one such y by (E1)). Let $y' \in [p+q]_{p+1}$ be the largest index such that there is an arc from D_r to $D_{y'}$ and let $Y = D_{p+1} \cup \ldots \cup D_{y'}$. If r is not defined above, then let $X = \emptyset$ (note that X is also empty if $r = p-1$) and $Y = \emptyset$. With this the result follows easy.

□

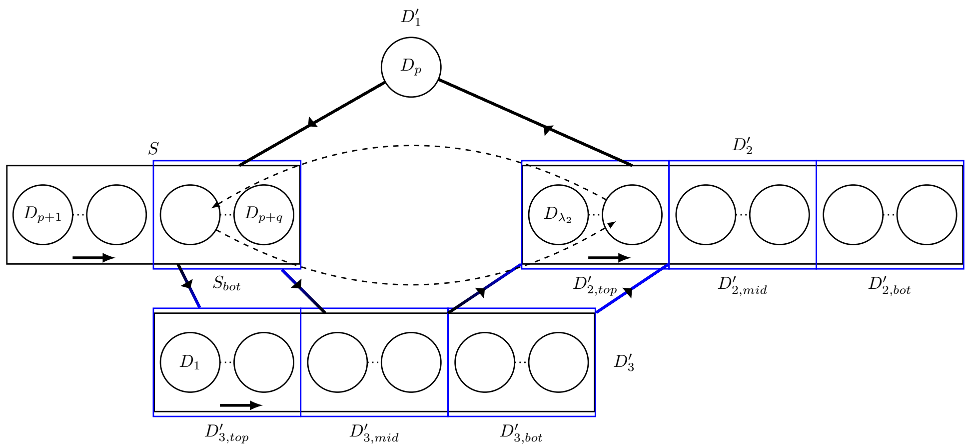
The following lemma shows that between any pair of vertices in an evil locally semicomplete digraph, there is a path of length at most 4 and if there is no path shorter than 4, then there is a path of length 4 containing non-adjacent vertices.

Lemma 1.14 ('New': Bang-Jensen and Christiansen)**.** *Let D be an evil locally semicomplete digraph, then for every $x, y \in V(D)$ either D contains a (x,y)-path of length at most 3, or $yx \in A(D)$ and D contains a (x,y)-path of length 4 containing non-adjacent vertices.*

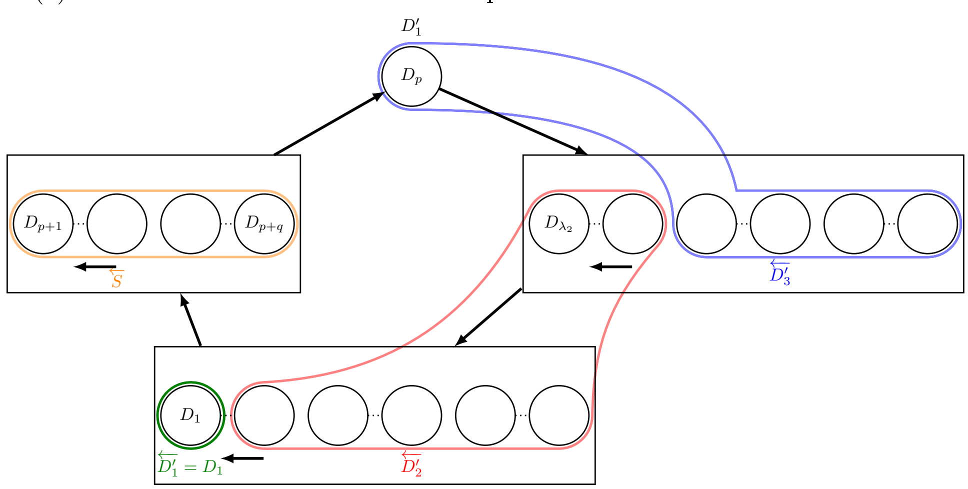
Proof. The proof repeatedly uses the dominating structure in D between the refined blocks of semicomplete components of the evil locally semicomplete digraph (See Figure 1.2a). We will consider any possible position of x, y and for each of these show that we can find a (x,y)-path fulfilling the statement of the theorem. We may assume that xy is not an arc of D. Now the following three observations are almost identical and will find the (x,y)-paths for almost all positions of x, y.

1. As $S_{bot} \Rightarrow D'_3 \Rightarrow D'_{2,top} \Rightarrow D'_1 \Rightarrow S_{bot}$ we have a collection of 4-cycles

$$\mathcal{C}_1 = \{v_{sb}v_3v_{2t}v_1v_{sb} \mid v_{sb} \in S_{bot}, v_3 \in D'_3, v_{2t} \in D'_{2,top}, v_1 \in D'_1\}.$$

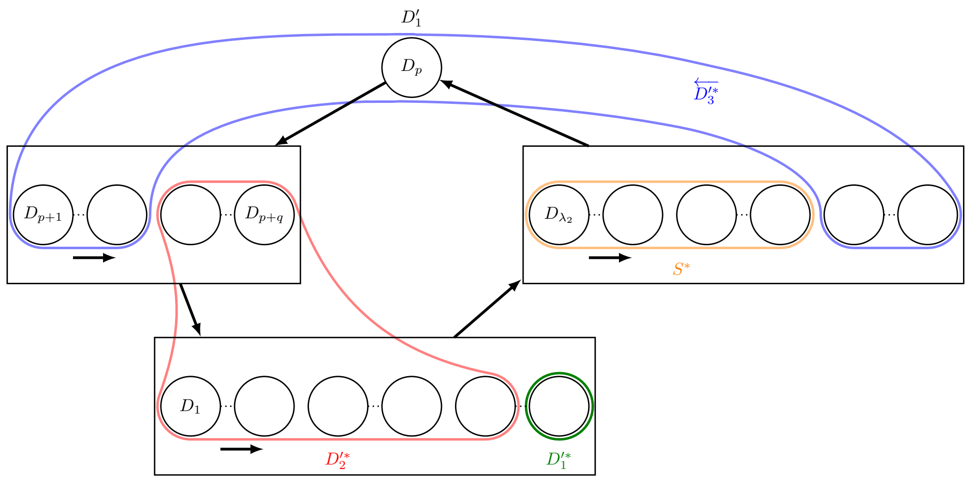


(a) D with the blocks marked and the pair of evil arcs showed as dotted arcs.

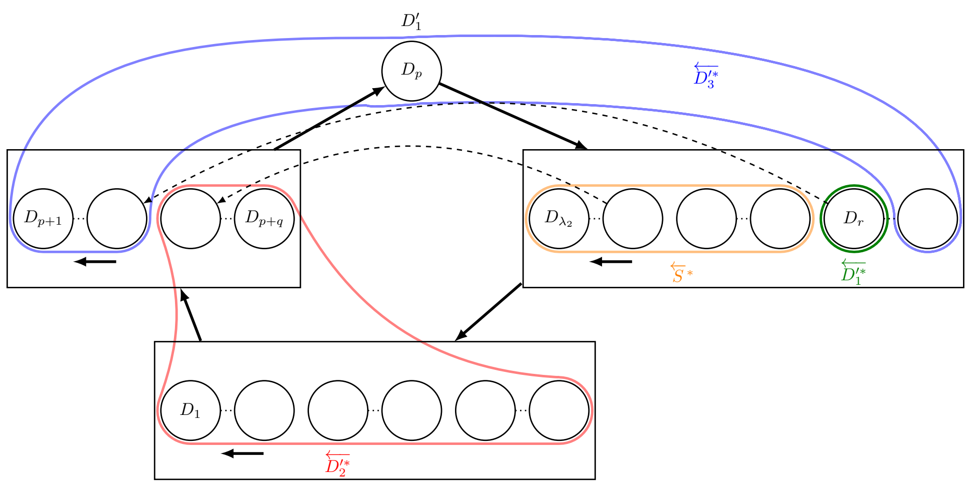


(b) The semicomplete decomposition of $\overleftarrow{D}$ with respect to the separator $\overleftarrow{S}$. The colored sets indicate the four parts.

Figure 1.2



(a) The semicomplete decomposition of D with respect to S^*. The colored sets indicate the four parts.



(b) The semicomplete decomposition of $\overleftarrow{D}$ with respect to the separator $\overleftarrow{S^*}$. The colored sets indicate the four parts. The dotted arcs are in D and indicate the arcs with head y respectively y', where y,y' are as defined before Theorem 1.13 E1).

Figure 1.3

Furthermore for every $C_1 \in \mathcal{C}_1$ and every vertex $v \in V$, there is a vertex of C_1 that dominates v and v dominates a vertex of C_1. This implies that $x, y \in (S - S_{bot}) \cup D'_{2,mid} \cup D'_{2,bot}$.

2. As $S \Rightarrow D'_{3,top} \Rightarrow D'_{2,top} \Rightarrow D'_1 \Rightarrow S$ we have the collection of 4-cycles

$$\mathcal{C}_2 = \{v_s v_{3t} v_{2t} v_1 v_s \mid v_s \in S, v_{3t} \in D'_{3,top}, v_{2t} \in D'_{2,top}, v_1 \in D'_1\}.$$

Furthermore for every $C_2 \in \mathcal{C}_2$ and every vertex $v \in V$, there is a vertex of C_2 that dominates v and v dominates a vertex of C_2. This implies that $x, y \in D'_{3,mid} \cup D'_{3,bot} \cup D'_{2,mid} \cup D'_{2,bot}$.

3. As $S_{bot} \Rightarrow D'_{3,bot} \Rightarrow D'_{2,mid} \Rightarrow D'_1 \Rightarrow S_{bot}$ we have the collection [3]

$$\mathcal{C}_3 = \{s_b v_{3b} v_{2t} v_1 s_b \mid v_{sb} \in S_{bot}, v_{3b} \in D'_{3,bot}, v_{2t} \in D'_{2,mid}, v_1 \in D'_1\}.$$

Furthermore for every $C_3 \in \mathcal{C}_3$ and every vertex $v \in V$, there is a vertex of C_3 that dominates v and v dominates a vertex of C_3. This implies that $x, y \in (S - S_{bot}) \cup D'_{3,top} \cup D'_{3,mid} \cup D'_{2,top} \cup D'_{2,bot}$.

Proof of 1. By considering the definitions of the blocks of the semicomplete components, it is easy to realize that for every $v \in V$ and every $C_1 \in \mathcal{C}_1$ there is a vC_1 and a $C_1 v$ arc in D. For example as $D'_{2,top} \Rightarrow (D'_{2,mid} \cup D'_{2,bot})$ and $(D'_{2,mid} \cup D'_{2,bot}) \Rightarrow D'_1$, this is true for all $v \in (D'_{2,mid} \cup D'_{2,bot})$. Now clearly if $x, y \in C_1$ for some $C_1 \in \mathcal{C}_1$ then we have a (x, y)-path P of length at most 4, and if is has length 4 it contains the non-adjacent vertices v_1 and v_3 and yx is an arc. Assume that x belongs to some cycle $C_1 \in \mathcal{C}_1$ and let $C_1 = xu_1u_2u_3x$. Furthermore $i \in [3]$ be the smallest index such that $u_i y \in A(D)$. Then $C_1[x, u_i]y$ is a (x, y)-path. Furthermore if $i < 3$ this path has length at most 3 and if $i = 3$ it has length 4 and contains the non-adjacent vertices v_1 and v_3. Also as u_i dominates x, y there is an arc between x and y and hence yx is an arc of D. Similar can be argued if $y \in C'_1 \in \mathcal{C}_1$. Hence we can assume that neither x nor y is on a cycle in $\mathcal{C}_1$ implying that $x, y \in D'_{3,mid} \cup D'_{3,bot} \cup D'_{2,mid} \cup D'_{2,bot}$.

2. and 3. follows by the exact same arguments. Hence the only position left for x, y is $D'_{2,bot}$. Now let $v_1 \in D'_1$, and $s_b v_{2t}$ be an arc of D with $s_b \in S_{bot}$ and $v_{2t} \in D''_{2,top}$. Then $xv_1s_bv_{2t}y$ is an (x, y)-path of length 4 and as $y \in D'_{2,bot}$ it is not adjacent with s_b. This completes the proof.

□

[3] If $D'_{2,mid} = \emptyset$ then this do not exist, but in this case we do not need these 4-cycles

Part

I

Linkages in digraphs

Chapter 2

Introduction to the Linkage problem

Consider a digraph D and a set of vertices (***terminals***) $\Pi = \{s_1, t_1, s_2, t_2, \ldots, s_k, t_k\}$ of V. Deciding whether D has k vertex disjoint (s_i, t_i)-paths is called the k-linkage problem or k-path problem. If the k paths exist for every set of $2k$ terminals of D, then D is said to be k-linked.

In the following we will only consider the k-linkage problem in digraphs where k is a fixed integer. For this problem we will both consider the complexity and whether it is possible to set up sufficient conditions for the digraph to be k-linked.

Starting with the complexity. Fortune, Hopcroft and Wyllie [22] proved that the k-linkage problem is NP complete for general digraphs already when $k = 2$. On the other hand they proved that the problem is polynomial on acyclic digraphs.

Theorem 2.1. *[22] The k-linkage problem is polynomial on acyclic digraphs.*

In the proof of this, they constructed an auxiliary digraph in polynomial time and proved that a single path in this will always induce the k-linkage in the acyclic graph given that such a k-linkage exist. Both the result and the approach proving this has later been used to obtain other polynomial results on the k-linkage problem. Johnson, Seymour and Thomas [30] found polynomial algorithms for digraphs with bounded directed tree-width and digraphs with bounded DAG-width. Later Bang-Jensen and this author [15] [1] did the same for digraphs with bounded circumference.

Now Thomassen considered strong digraphs and proved that the k-linkage problem is also NP complete for these [50], while Schrijver proved that the problem is

[1] Changed last name in 2016 to Christiansen

polynomial in planar digraphs [42]. Until 2015 only small progress had been made on semicomplete digraphs, where Bang-Jensen and Thomassen had proved that the 2-linkage is polynomial [17]. Then Chudnovsky, Scott and Seymour did not only confirm polynomial k-linkage for semicomplete digraphs, but also for a larger class they call d-path dominant digraph. For $d = 1$ this is exactly the semicomplete digraphs.

Theorem 2.2. *[19] The k-linkage problem is polynomial on d-path dominant digraphs for all fixed integer d.*

Proving this is non-trivial, but is also another example of the useful approach first used on acyclic digraph. They construct an auxiliary digraph and find a single path in this inducing a k-linkage in the d-path dominant digraph. Later Chudnovsky, Scott and Seymour were able to expand their result to also apply to digraphs consisting of disjoint semicomplete components.

Theorem 2.3. *[20] The k-linkage problem is polynomial on digraphs consisting of unions of disjoint semicomplete digraphs.*

With the affirmative answer for semicomplete digraphs, one of the next natural question to consider is on the complexity for locally semicomplete digraphs. This together with some totally Φ-decomposable digraphs, was done in [7]. Here Bang-Jensen, Maddaloni and this author proved that the k-linkage problem is polynomial on both round and round decomposable digraphs.

Theorem 2.4. *[7] Let $D = R[S_1, S_2, \ldots, S_r]$ for $r \geq 2$, then k-linkage problem is polynomial on D, also when $|S_i| = 1$ for all $i \in [2]$.*

As the subclass of locally semicomplete digraphs, called the evil class, can be covered by the union of three semicomplete components and as locally semicomplete digraphs, which is not evil, is either semicomplete or round decomposable, Theorem 2.2, 2.3 and 2.4 implies that the k-linkage problem is polynomial on locally semicomplete digraphs.

Theorem 2.5. *[7] The k-linkage problem is polynomial on locally semicomplete digraph.*

Bang-Jensen proved that the 2-linkage problem is polynomial for quasi-transitive digraphs and extended this to also apply to extended semicomplete digraphs.

Theorem 2.6. *[4] Let Φ be a class of strongly connected digraphs and denote by Φ_0, the class of all extensions of graphs in Φ. Finally let*

$$\Phi^* = \{F[D_1, D_2, \ldots, D_{|F|}] : F \in \Phi, D_i \textit{ arbitrary digraph } \}$$

Then there exist a polynomial algorithm for the 2-linkage problem in Φ^*, if and only if there exist one for all digraphs in Φ_0.

In [7] we extended this result by proving that we can decide the k-linkage problem in polynomial time for certain classes of totally Φ-decomposable digraphs. The following is an implication of this result:

Theorem 2.7. *[7] The k-linkage problem is polynomial on the quasi-transitive and extended semicomplete digraphs.*

Let us now turn to the problem of finding sufficient conditions for a digraph to be k-linked. In 1985 Heydemann and Sotteau [28] imposed conditions on the degree of the vertices and proved that $\delta^0 \geq n/2 + 1$ is sufficient for 2-linkage in a digraph. In 1990 Manoussakis conjectured[2] [38] that $\delta^0 \geq n/2 + k - 1$ is sufficient for all integers k . Kühn and Osthus confirmed this for large digraphs and proved that it is best possible.

Theorem 2.8. *[33] Let $k \geq 2$ and D be a digraph of order $n \geq 1600k^3$ such that $\delta^0(D) \geq n/2 + l - 1$. Then D is k-linked.*

Proposition 2.9. *[33] Let $k \geq 2$ be a fixed integer. For every $n \geq 2k$ there exist a digraph on n vertices and $\delta^0 \geq n/2 + k - 2$ which is not k-linked.*

Later Ferrara, Jacobsen and Pfender proved the following that not only applies to large digraphs.

Lemma 2.10. *[21] Let D be a digraph and $k \geq 1$ a fixed integer. Furthermore let $\sigma_2 = \min_{u,v \in V}\{d^+(u) + d^-(v)\}$. If $\sigma_2 \geq |V| + 3k - 4$, then D is k-linked*

The bound on the minimum semi-degree, though tight, also demands many arcs and one would hope that 'more usable' sufficient conditions could be found. The probably most obvious condition to consider is the strong connectivity of a digraph. Remember that a k-strong digraph is a digraph where there are k disjoint paths between every pair of vertices. For general digraphs Thomassen proved in 1991 that there exists a family of k-strong digraphs that are not even 2-linked, and hereby disproving this intuition [50]. Ferrara, Jacobsen and Pfender combined strong connectivity and degree conditions to obtain the following general result

Theorem 2.11. *[21] Let D be a $(9/2)k$-strong digraph and let $\sigma_2 = \min_{u,v \in V}\{d^+(u) + d^-(v)\}$. If $\sigma_2 \geq |V| + \frac{1}{2}k - 2$, then D is k-linked.*

[2]Manoussakis also finds a sufficient condition on number of arcs, but we will not repeat this here.

Now for tournaments and generalizations of tournaments a lot more can be said. Thomassen proved that there is an exponential function $f(k)$, such that every $f(k)$-strong tournament is k-linked [48]. This bound was first improved by Kühn, Lapinskas, Osthus and Patel to $10^4 k \log k$ [35] and later by Pokrovskiy to $452k$. In [11] Bang-Jensen and Havet generalized the proof of Pokrovskiy to also apply to semicomplete digraphs.

Theorem 2.12. *[40, 11] $452k$-strong semicomplete digraphs are k-linked.*

For small k Kim, Kühn and Osthus improved Pokrovskiys bound as they proved that $(k^2 + 3k)$-strong tournaments contains a k-path-factor between $2k$ predefined vertices.

Theorem 2.13. *[31] Let D be a $(k^2 + 3k)$-strong tournament. For any set $\{x_1, y_1, \ldots, x_k, y_k\}$ of distinct vertices, D contains a k-path factor $P_1 \cup P_2 \cup \cdots \cup p_k$ such that P_i is an (x_i, y_i)-path for $i \in [k]$.*

Now Pokrovskiy constructed a family of nearly $2k$-strong tournaments that are not k-linked and posed the following conjecture, stating that the correct bound to ensure k-linkage in semicomplete digraphs is close to $2k$.

Conjecture 2.14. *[40] For every fixed integer k there exist a function $d(k)$ such that every $2k$-strong tournament with $\delta^0 \geq d(k)$ is k-linked.*

In a preprinted version Girão and Snyder give evidence towards this conjecture.

Theorem 2.15. *[23] For every positive integer k there exists a function $f(k)$ such that every $4k$-strong tournament T with $\delta^+(T) \geq f(k)$ is k-linked.*

Considering generalizations of semicomplete digraphs Bang-Jensen proved that $(3k-2)$-strong round decomposable locally semicomplete digraphs are k-linked. This was improved by Bang-Jensen, Maddaloni and this author. We proved that the necessary bound on $2k-1$ strong connectivity is also sufficient.

Theorem 2.16. *[7] Let D be a round or round decomposable digraph on at least $2k$ vertices that are not semicomplete. Then D is $(2k-1)$-strong if and only if it is k-linked.*

Now for locally semicomplete digraphs Bang-Jensen prove that there exist a function $f(k)$, such that every $f(k)$-strong locally semicomplete digraph are k-linkage.

Theorem 2.17. *[4] For every fixed integer k, there exist a function such that every $f(k)$-strong locally semicomplete digraph is k-linked.*

The function $f(k)$ is not linear in k, but as both round decomposable and semicomplete digraphs are k-linked when a linear function on the strong connectivity is imposed, it seems reasonable to believe that this do also applies to locally semicomplete digraphs. Bang-Jensen also conjectures this in [11] and in Chapter 4 we will prove that this conjecture is true.

Conjecture 2.18. *[11] There exists a constant B such that every Bk-strong locally semicomplete digraph is k-linked.*

Finally Bang-Jensen and Thomassen, proved that 5-strong semicomplete digraphs and quasi-transitive digraphs are 2-linked [48, 4]. Bang-Jensen conjectured that this is also true for locally semicomplete digraph, and we confirm this in [7].

Conjecture 2.19. *[4] Every 5-strong locally semicomplete digraph is 2-linked.*

In the next chapter we will give a more detailed introduction to the results obtained in [7]. First we will introduce the notion of linkage ejector, which give us the necessary tool to prove the polynomial k-linkage for classes of totally Φ-decomposable digraphs. Then we consider the sufficiency result of the paper, and while only restating Theorem 2.16, we will go into detail with the proof for Conjecture 2.19. In this description we will extract certain self-contained results from the original proof, while also giving a sketch of the proof. A few subresults are proven to give the reader a good understanding both of the proof, and also of the usefulness of the refined structure of evil locally semicomplete digraph that were introduced in Section 1.2.1. Finally in Chapter 4 we will prove Conjecture 2.18, using one of the self-contained results of Chapter 3. We will also mention a few open problems and related results.

Chapter 3

Disjoint paths in decomposable digraphs

In this chapter we will introduce the results found in the paper 'Disjoint paths in decomposable digraphs', published in Journal of Graphs Theory 2016 [7]. This is joint work of this author, Jørgen Bang-Jensen and Alessandro Maddaloni and the full paper can be found in Part IV. In the first part of the paper we consider the complexity of the k-linkage problem and prove that it is polynomial for locally semicomplete digraphs and several classes of decomposable digraphs, amongst others extended semicomplete digraphs, round decomposable digraphs and quasi-transitive digraphs. In the second part of the paper we consider whether strong connectivity on round decomposable digraphs and locally semicomplete digraphs will ensure that the digraphs is k-linkage. Here we prove that $(2k-1)$-strong round decomposable digraphs are k-linked and 5-strong locally semicomplete digraphs are 2-linked. Both is best possible and the last answers Conjecture 2.19 from 1990.

3.1 Polynomial k-linkage

We consider different classes of decomposable digraphs, $D = S[M_1, M_2, \ldots, M_s]$. We start by defining a property, ***a linkage ejector***, of such classes of decomposable digraphs and describe a polynomial algorithm for the k-linkage problem for all classes having this property.

Definition 3.1. We say that a class Φ of digraphs is a linkage ejector if

1. It is polynomial to find a total Φ-decomposition for every totally Φ-decomposable digraph.
2. For fixed k, it is polynomial to solve the k-linkage problem on Φ.

3. The class Φ is closed with respect to blow-up [1]
4. It is polynomial to construct a digraph of Φ given the totally Φ-decomposition digraph $D = S[M_1, \ldots, M_s]$, where the construction replaces arcs inside each of the modules, M_i.

Theorem 3.2. *Let Φ be a linkage ejector. For every fixed k, there exists a polynomial algorithm to solve the k-linkage problem on totally Φ-decomposable digraphs.*

To prove this theorem we distinguish between terminal pairs, where both belongs to the same module (***internal*** pairs Π^i), and terminal pairs where the vertices are in distinct modules (***external*** pairs Π^e). We notice that every minimal path in a decomposable digraph with ends in different modules do not use any arcs inside a module. Hence in a Π-linkage in D, the external paths (path not using arcs inside any module) uses at most 2 vertices in each module. This will give a polynomial algorithm. Consider every partition of the internal terminal pairs $\Pi^i = \Pi_1 \cup \Pi_2$ and check if D has a $(\Pi^e \cup \Pi_1)$-linkage consisting only of external paths and Π_2-linkage consisting only of internal paths. In order for these to be disjoint, we preserve vertices of each module to one of the two linkages. As the external paths uses at most $|(\Pi^e \cup \Pi_1)|$ vertices in each module, there are only a bounded number of partition of each module to consider.

Having established Theorem 3.2 we can prove polynomial linkage of two totally Φ-decomposable digraphs, by showing that each of these are linkage ejectors. First

$$\Phi_1 := \{ \text{ Semicomplete digraphs } \} \bigcup \{ \text{ Acyclic digraphs } \}$$

is a linkage ejector. This is obtained by combine Theorem 1.5, Theorem 2.2 and Theorem 2.1. Secondly in order to prove that

$$\Phi_2 := \{ \text{ Semicomplete digraphs } \} \bigcup \{ \text{ Round digraphs } \}$$

is also a linkage ejector, we prove that the k-linkage problem is polynomial for round digraphs using Theorem 2.1. Combining this with Theorem 1.5 it will imply that Φ_2 is a linkage ejector.

Theorem 3.3. *[7] Let k be fixed, then the k-linkage problem is polynomial for the classes of digraphs Φ_1 and Φ_2.*

Now by Theorem 1.4, quasi-transitive digraphs are totally Φ_1-decomposable. Also the extended semicomplete digraphs are totally Φ_1-decomposable. Hence Theorem 3.3

[1] Blowing up a vertex v to K, means that v is replaced by K in the digraph.

implies polynomial k-linkage for both these classes. Now the final complexity result obtained is on locally semicomplete digraphs.

Theorem 2.5. [7] The k-linkage is polynomial for locally semicomplete digraph.

Remember that locally semicomplete digraphs is either round decomposable, evil or semicomplete (See Theorem 1.9 in Section 1.2.1). Chudnovsky, Scott and Seymour have proved two parts of this theorem. First semicomplete digraphs in Theorem 2.2 and then unions of semicomplete digraphs in Theorem 2.3. Here the evil class can be covered by three semicomplete digraphs (D'_3, D'_2 and $D\langle V(S) \cup V(D'_1)\rangle$ where D'_1, D'_2 and D'_3 are the semicomplete decomposition of $D - S$, see Section 1.2.1). Now the theorem follows as we can recognize a round decomposable locally semicomplete digraph in polynomial time by Theorem 1.6.

3.2 Sufficient conditions of k-linkage

The second part of the paper considers sufficient conditions for a digraph to be k-linked. We first prove that $(2k-1)$-strong connectivity is equivalent to k-linked when considering digraphs that are not semicomplete but round decomposable digraphs. Then proof of this is an application of earlier results in the paper and some structural observations.

Theorem 2.16. [7] Let D be a round or round decomposable digraph on at least $2k$ vertices that are not semicomplete. Then D is $(2k-1)$-strong if and only if it is k-linked.

Finally we prove that 5-strong locally semicomplete digraphs are 2-linked. This confirms Conjecture 2.19 by Bang-Jensen from 1999.

Theorem 3.4. *[7] Every 5-strong semicomplete digraph is 2-linked*

As Thomassen already proved this for semicomplete digraphs [48] and as the result follows for round decomposable digraphs by Theorem 2.16, it remains to prove this for evil locally semicomplete digraphs. The proof of this is very technical and it is here the refined structure of the evil locally semicomplete digraphs introduced in Section 1.2.1 are used.

Theorem 3.5. *[7] Let D be a 5-strong evil locally semicomplete digraph. Then D is 2-linked.*

3.2.1 Sketch of proof of Theorem 3.5

In this section we will give an outline of the proof of Theorem 3.5 together with the proof of some of sub-results in order to illustrate the techniques used. Before reading this section it is important that the reader has understood the refined structure of the evil locally semicomplete digraphs described in Section 1.2.1. The figures illustrating this refined structure of D together with the reversing of arc and choosing new good separators are presented again in Figure 3.1 and Figure 3.2. Notice that D is 5-strong and if we reverse all arcs of D we obtain a 5-strong digraph $\overleftarrow{D}$. Furthermore if we are looking for a $\{s_1, t_1, s_2, t_2\}$-linkage in D, then this corresponds to a $\{t_1, s_1, t_2, s_2\}$-linkage in $\overleftarrow{D}$. Hence by interchanging the names of the terminals when reversing all arcs, we obtain a new equivalent linkage problem.

We start by restate two lemmas from Section 1.2.1 together with a new one. While the first lemma is presented as a separate lemma in the original paper, the remaining are integrated parts in the original proof. This author has chosen to extract these here for two reasons. It makes the sketch of the proof more clean and secondly as these preliminary do not dependent on the specific problem (5-strong imply 2-linked) they can be used to solve other problems.

Lemma 1.8. [9] Let S be a minimal separator of a strong locally semicomplete digraph D. Then either $D\langle S\rangle$ is semicomplete, or $D\langle V - S\rangle$ is semicomplete.

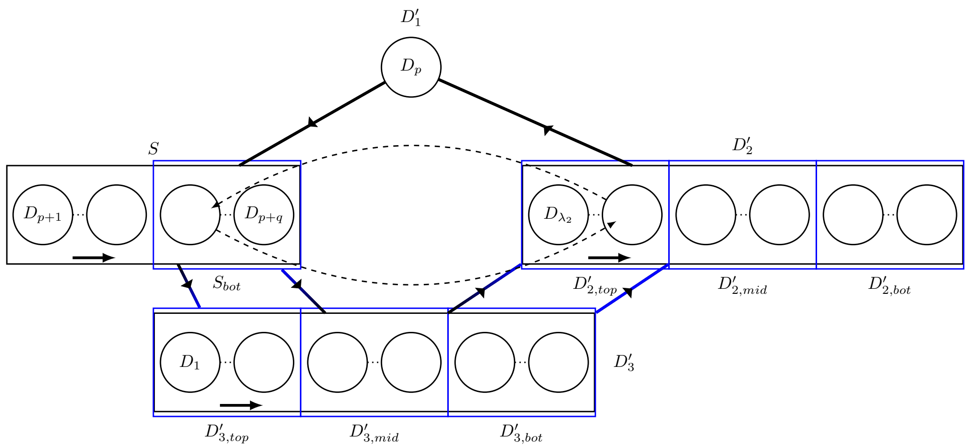
The following lemma shows a nice structural result on minimal paths in evil locally semicomplete digraph. The proof can be found in Section 1.2.1.

Lemma 1.14 ('New': Bang-Jensen and Christiansen)**.** Let D be an evil locally semicomplete digraph, then for every $x, y \in V(D)$ either D contains a (x, y)-path of length at most 3, or $yx \in A(D)$ and D contains a (x, y)-path of length 4 containing non-adjacent vertices. [2]

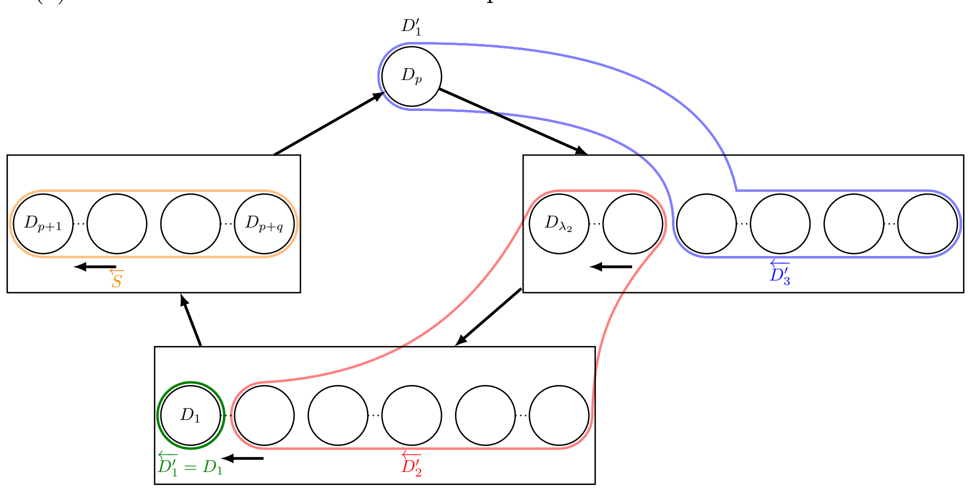
Notice the nice connection between Lemma 1.8 and 1.14. If we have a pair of vertices x, y where the minimal (x, y)-path has length 4 and D is 5-strong, then we have a (x, y)-path that induces a good separator of D.

Now a (x_1, x_2)***-out-switch*** S is a collection of at least 4 vertices x_1, x_2, y_1, y_2 such that in $D\langle S\rangle$ there are disjoint (x_i, y_i)- and (x_{3-i}, y_{3-j})-paths for both $i, j \in [2]$. A (x_1, x_2)***-out-switch of order*** l is a (x_1, x_2)-out-switch containing at most l vertices. We will call x_1, x_2 the start vertices and y_1, y_2 the end vertices of S. If we want to

[2] Corresponds to equivalent to Claim 3 and Claim 4

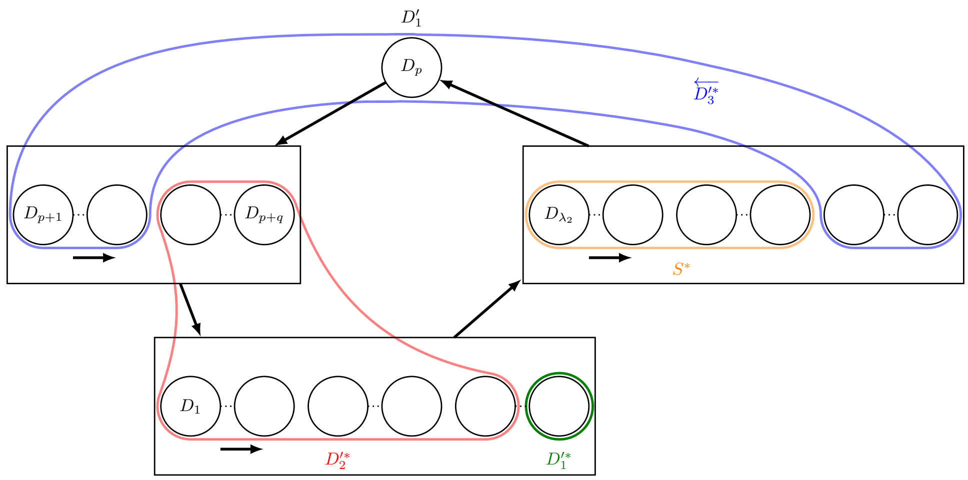


(a) D with the blocks marked and the pair of evil arcs showed as dotted arcs.

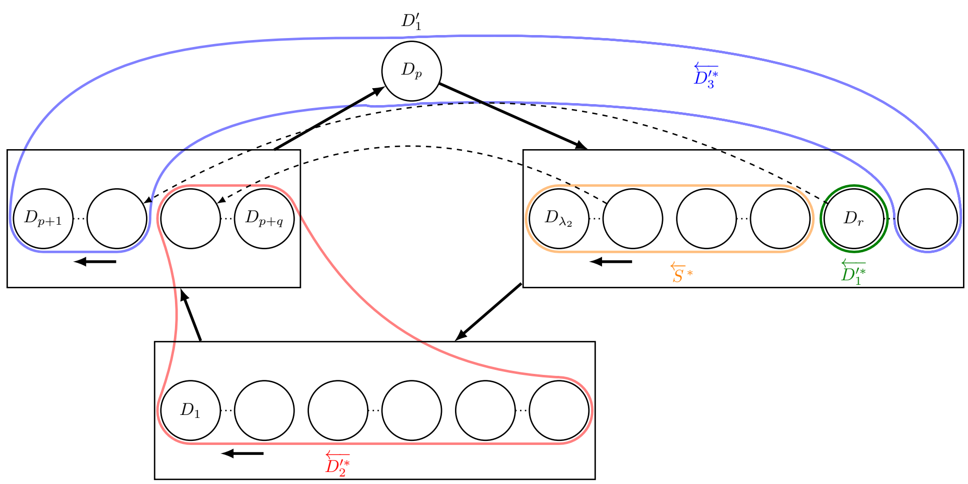


(b) The semicomplete decomposition of $\overleftarrow{D}$ with respect to the separator $\overleftarrow{S}$. The coloured sets indicate the four parts.

Figure 3.1



(a) The semicomplete decomposition of D with respect to S^*. The sets indicate the four parts.



(b) The semicomplete decomposition of $\overleftarrow{D}$ with respect to the separator $\overleftarrow{S^*}$. The colored sets indicate the four parts. The dotted arcs are in D and indicate the arcs with head y respectively y', where y,y' are as defined before Theorem 1.13 part 3.

Figure 3.2

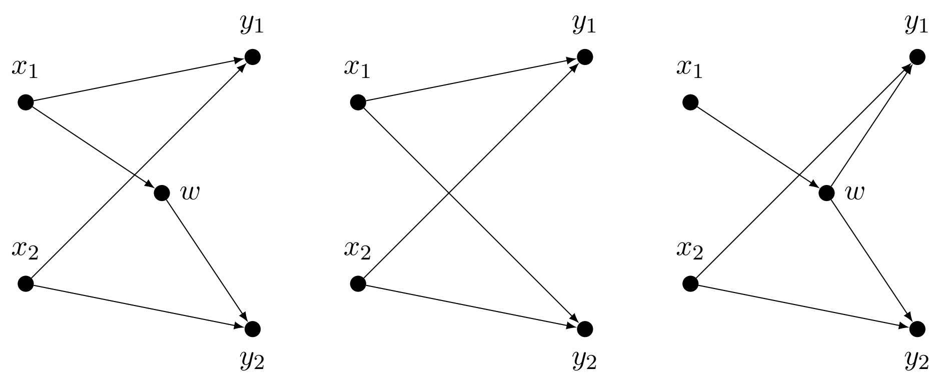


Figure 3.3: Three different (x_1, x_2)-out-switched of order 5.

specify the end vertices instead of the start vertices, we consider a (y_1, y_2)***-in-switch***[3]. In Figure 3.3 three kinds of (x_1, x_2)-out-switches of order 5 is shown. The following lemma shows that finding a (x_1, x_2)-switch of order k in a k-strong digraph directly implies that x_1, x_2 are linked with almost all other vertices of D.

Lemma 3.6 (NEW: Christiansen)**.** *Assume that D is a k-strong digraph and let S be a (x_1, x_2)-out-switch of order k in D. Then for every $v_1, v_2 \in V - S$ there exist disjoint (x_i, v_i)-paths in D for $i \in [2]$.*

Proof. This follows by Menger's theorem. As $D - (S - \{y_1, y_2\})$ is at least 2-strong and we disjoint (y_1, v_1)-path and (y_1, v_2)-path, call these P_1, P_2. Similarly we have disjoint (y_2, v_1)-path and (y_2, v_2)-path, call these Q_1, Q_2. Now if P_i is disjoint from Q_{3-i} for some $i \in [2]$, we are done. We need to show that for each $i \in [2]$ and hence we have disjoint (y_i, v_j)-path for $i, j \in [2]$. Now we use the property of the switch to link x_j to v_j for each $j \in [2]$. □

We are now ready to give the outline (of the remaining) of the proof of Theorem 3.5.

Outline of proof of Theorem 3.5. The proof is by contradiction and the approach is to consider four terminal vertices $\{s_1, t_1, s_2, t_2\}$ and prove that no matter where the four vertices are positioned in D, we can find vertex disjoint (s_1, t_1)- and (s_2, t_2)-paths.

Let $D^i = D - \{s_{3-i}, t_{3-i}\}$. If there is a (s_i, t_i)-path P_i of length at most 3 in D^i, then $D - P_i$ is strong and the (s_{3-i}, t_{3-i})-path can be found. Hence we may assume that every (s_i, t_i)-paths in D^i has length at least 4. Considering $\{s_i, t_i, \widehat{s_j}, \widehat{t_j}\}$-linkage in D^i, where $\widehat{s_j}$ is an out-neighbour of s_j and $\widehat{t_j}$ is an in-neighbour of t_j, we

[3]Kühn, Lapinskas, Osthus and Patel introduced the notion switch in [35] when proving that $10^4 k \log k$-strong semicomplete digraphs are k-linked

use Proposition 1.1 to conclude that D^i is not semicomplete and Theorem 2.16 to conclude that D^i is not round decomposable. Hence for each $i \in [2]$, D^i is evil locally semicomplete digraph. Lemma 1.14 then implies that we have a (s_i, t_i)-path P_i of length 4 such that $D\langle P_i \rangle$ is not semicomplete and that $t_i s_i \in A(D)$. Hence s_i, t_i is adjacent to all but at most 3 vertices in D^i for $i \in [2]$ and by Lemma 1.8 P_i is a good separator of D^i.

Through a number of claims, we eliminate positions of the terminals for any good separator of D. We will only prove the following claim, but many other positions are proved using the same type of arguments. The most important tool is to consider the 4 different semicomplete decompositions of D described in Section 1.2.1.

Claim 1. [4] *For every good separator S and semicomplete decomposition D'_1, D'_2, D'_3 of $D - S$, $|V(D'_3) \cap \{s_1, s_2\}| = 1$ and $|V(D'_1) \cap \{s_1, s_2\}| = 1$*

Proof of claim. We will repeatedly find a (s_1, s_2)-out-switch S of order 5 in $D - \{t_1, t_2\}$. Then it follows by Lemma 3.6. Suppose first that $s_1, s_2 \in D'_3$, then s_1, s_2 are adjacent as D'_3 is a semicomplete component, and we assume without loss of generality that $s_1 s_2 \in A(D)$. If $|N^+(s_1) \cap N^+(s_2)| \geq 2$ we have a (s_1, s_2)-out-switch of order 4 and as $t_i s_i \in A$ it is a switch disjoint from t_1, t_2. Hence we may assume that $|D'_{2,top}| = 1$. If we consider the semicomplete decomposition D'^*_1, D'^*_2, D'^*_3 with the good separator S^*, then the last component of $D'_{3,bot}$ is the D'^*_1 component. Hence as all other components of D'_3 dominates the last component we conclude that $s_2 \in D'_{3,bot}$. Also as $|S^*| = 5$ we conclude that $s_1 \notin D'_{3,bot}$, as this would again impose two common out-neighbours of s_1, s_2. Thus every out-neighbour of s_1 is in D'_3. But s_1 must dominate a vertex w in $D'_3 - \{s_2, t_2\}$ which dominates a vertex in $D'_{2,mid}$. This will induce a (s_1, s_2)-switch of order 5. This proves that $|V(D'_3) \cap \{s_1, s_2\}| = 1$.

Now to realize that $|V(D'_1) \cap \{s_1, s_2\}| = 1$, we simply consider the semicomplete decomposition D'^*_1, D'^*_2, D'^*_3 with the good separator S^*. Then it follows as $D_1 \subset D'^*_3$. ♦

After eliminating most of the position for the terminals for all good separator, we pick a good separator S and an orientation of D such that there are no terminals in D'_1. Then we prove that for this choice of separator $\{s_1, t_1\} \in D'_2$ and $\{s_2, t_2\} \in S$ and it remains to considering two cases, depending on the adjacency between D_1 and D'_2.

Case 1. *D_1 dominates D'_2*

[4] Corresponds to Claim 7 and 8 in the original proof

Proof of case. Pick $w \in D'_2 - \{s_1, t_1\}, s \in S - \{s_2, t_2\}, q \in D'_1$ and $p \in D'_{3,top}$. Then $P = s_2pwqt_2$ and $Q = s_1qspt_1$ are both paths that induce digraphs which is not semicomplete. Let P_1, P_2, P_3 be internally disjoint minimal (s_1, t_1)-paths in D^1. Then for $j \in [3]$ each P_j intersects P, and hence each contains exactly one of the vertices $\{p, w, q\}$. Let α_i, β_i denote, respectively, the successor of s_1 and the predecessor of t_1 on P_i. Then wlog $\alpha_1 = q$, $\beta_2 = p$. Now if $\alpha_2 \in D'_2$, then $p\alpha_2$ and $\alpha_2 q$ would give a (s_2, t_2)-path $s_2p\alpha_2Pt_2$ disjoint from P_3, a contradiction. Hence $\alpha_2 \in D'_1$ and D'_1 is non-trivial. Considering 3 internally disjoint (s_2, t_2)-paths Q_1, Q_2, Q_3 in D^2 we use similar arguments to conclude that D'_3 contains at least two vertices. But then both P and $D - V(P)$ is not semicomplete, a contradiction with Lemma 1.8. $\Diamond$

Case 2. *There exists $p \in D_1$, $x \in D'_2$ so that p and x are not adjacent*

By a similar approach considering 3 disjoint (s_i, t_i)-paths in D^i we also prove this case. This completes the proof of Theorem 3.5. $\square$

Chapter 4

Further results and open problems on the Linkage problem

In this chapter we will prove Conjecture 2.18 and formulate a few so far not mentioned problems that lies in natural extension to the problems considered in this part.

4.1 Linear function ensuring k-linked locally semicomplete digraphs

In Theorem 2.16 and Theorem 2.12 we saw that a linear bound on the strong connectivity of round decomposable respectively semicomplete digraphs are sufficient to ensure that they are k-linked. Naturally Bang-Jensen conjectured (Conjecture 2.18) that this is also true for locally semicomplete digraphs. In this section we will verify this conjecture by proving that every $454k$-strong locally semicomplete digraphs are k-linked. Remember that Theorem 1.9 implies that locally semicomplete digraphs are either semicomplete, round decomposable or evil. We will start by considering the evil class, and use Lemma 1.14 that says that between any pair of vertices in an evil locally semicomplete digraphs there is a path of length at most 4.

Theorem 4.1 (NEW: Christiansen). *Let $k > 1$ be a fixed integer. Then every $f(k)$-strong evil locally semicomplete digraph is k-linked. Furthermore $f(k) \leq 454k - 1$.*

Proof. The proof is by induction over k. First $k = 2$ follows by Theorem 3.5, so assume that the theorem is true for all $k - 1 > 1$ and consider a $f(k)$-strong evil locally semicomplete digraph. Let $\Pi = \{s_1, t_1, s_2, t_2, \dots, s_k, t_k\}$ be an arbitrary set of k pairs of disjoint terminals in D. Furthermore let $\Pi^i = \Pi - \{s_i, t_i\}$ and let $D^i = D - \Pi^i$. If $s_it_i \in A(D)$ for some $i \in [k]$, then it follows easily by considering $D - s_it_i$. So we may assume that for all $i \in [k]$, $s_it_i \notin A(D)$.

Assume first that there exists an $i \in [k]$ such that D^i is an evil locally semicomplete digraphs. Then Lemma 1.14 implies that D_i has a (s_i, t_i)-path P of length at most 4 and the theorem follows by induction as $D - P$ is at least $f(k) - 5 \geq f(k-1)$-strong.

Now consider D^i for any $i \in [k]$. Then as $s_j t_j \notin A(D)$ for each $j \in [k]$ we have

$$|N^+(s_j) \cap D^i| \geq f(k) - 2(k-1) + 1 \geq 3k - 2$$
$$|N^-(t_j) \cap D^i| \geq 3k - 2.$$

As $3k - 2 \geq 2(k-1)$, there exist a set of $2(k-1)$ disjoint vertices in D_i

$$\widehat{\Pi}^i = \{\widehat{s}_1, \widehat{t}_1, \ldots, \widehat{s}_{i-1}, \widehat{t}_{i-1}, \widehat{s}_{i+1}, \widehat{t}_{i+1}, \ldots, \widehat{s}_k, \widehat{t}_k\}$$

such that for each such $j \in [k]$, with $j \neq i$, $\widehat{s}_j$ is an out-neighbour of s_j and $\widehat{t}_j$ is an in-neighbour of t_j. Furthermore it is not hard to extend a $\widehat{\Pi}^i \cup \{s_i, t_i\}$-linkage in D^i to a Π-linkage in D. Now the theorem follows by Theorem 2.16 and Theorem 2.12: If D^i is round decomposable then as D^i is at least $2k-1$ strong it has a $\widehat{\Pi}^i \cup \{s_i, t_i\}$-linkage and if D^i is semicomplete then as D^i is at least $452k$-strong it has a $\widehat{\Pi}^i \cup \{s_i, t_i\}$-linkage. □

Looking at the proof we saw that it is the case where D^i is semicomplete for every $i \in [k]$ which bounds $f(k)$. For D^i evil and round decomposable it would be sufficient with $f(k) = 5(k-1)$. As Pokrovskiy conjectured (Conjecture 2.14) the correct bound for semicomplete digraphs is close to $2k$, and if so $f(k) = 5(k-1)$ will be sufficient. One could also try to improve directly on $f(k)$. When D^i is semicomplete for all $i \in [k]$, then one can realize that (a subset of) the terminals are positioned in a well-ordered way; If D^i is semicomplete, Π^i must contain all vertices of D'_3 or all vertices of D'_1, and this has to apply for all $i \in [k]$ and any semicomplete decomposition of D. This observation together with the refined structure defined in Section 1.2.1 might either give the existence of a short (s_i, t_i)-path in D^i for some $i \in [k]$, or give a contradiction with D^i semicomplete for all $i \in [k]$. This leads to the following conjecture.

Conjecture 4.2 (NEW: Christiansen)**.** *Every $5(k-1)$-strong evil locally semicomplete digraph is k-linked.*

Now the general bound on locally semicomplete digraph follows by the structure of the locally semicomplete digraphs (Theorem 1.9).

Corollary 4.3 (NEW: Christiansen)**.** *There exists a linear function $g(k)$, such that every $g(k)$-strong locally semicomplete digraphs are k-linked. Furthermore $g(k) \leq 454k - 1$.*

Proof. This follows by combining the results for semicomplete (Theorem 2.12), round decomposable (Theorem 2.16) and locally semicomplete digraphs (Theorem 4.1) and the bound is the maximum for the three $g(k) \leq \max\{452k, 2k-1, f(k)\}$. □

4.2 Other related results and conjectures

We will end with a small collection related conjectures and open problems not stated elsewhere. Remember that we in the previous have considered both the complexity version and setting up sufficient conditions for a digraph to be k-linked.

Starting with complexity related problems. Bang-Jensen conjectured that it is also polynomial to decide if locally in- (and out-) semicomplete digraphs are k-linked. Even for $k = 2$, this is unproven.

Conjecture 4.4. *[11] For fixed k, the k-linkage problem is polynomial on locally in-semicomplete digraphs.*

Now Chudnovsky, Seymour and Scott showed that the k-linkage problem is polynomial on digraphs consisting of union of p semicomplete digraphs (Theorem 2.3). This is a special type of digraphs with independence number p. They ask what the complexity for all digraphs with bounded independence number is.

Problem 4.5. *[20] Determine the complexity of the k-linkage problem in digraphs with bounded independence number*

Moving on to sufficient conditions in order of a digraph to be k-linked, we do not know of any general results on Φ-decomposable digraphs. Bang-Jensen proved [4] that every 5-strong quasi-transitive digraphs are 2-linked and we proved that $2k-1$-strong round decomposable digraphs are k-linked. It would be interesting if we could generalize these results in two ways; Finding a sufficient condition for a quasi-transitive digraph to be k-linked and prove that for some class of Φ-decomposable digraphs which are also $f(k)$-strong, they are all k-linked.

Finally we will mention two complexity results on the k-linkage problem, when k is part of the input. Notice that it follows directly from Fortune, Hopcroft and Wyllie [22] that this problem is NP complete for general digraphs, as it is when k is fixed.

Theorem 4.6. *[43] The k-linkage problem i $W[1]$-hard for acyclic digraphs when k is part of the input.*

Theorem 4.7. *[17] The k-linkage is NP complete in tournaments when k is part of the input.*

Part

II

Degree constrained partitions of digraphs

Chapter 5

Introduction to partition problems

Remember that a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of a digraph D is a partition (V_1, V_2) of the vertices of D such that V_i induces a graph with property $\mathcal{P}_i$. It is natural to study whether there exist partitions of the vertices (arcs) of a (di)graph such that some properties is 'maintained' in each partition. For example does there exist a function $f(k_1, k_2)$ such that every digraph with minimum out-degree $f(k_1, k_2)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition. For undirected graphs this and the similar problem for connectivity was proved by Thomassen [47] and the bound on these have later been improved by Hanjal [27], Stiebitz [45] and Kühn and Osthus [32].

In 1995-1996 Alon [1] and Stiebitz [44] independently posed the equivalent problem for minimum out-degree in digraphs. It is not hard to see that the answer is yes if and only if there exist a function $f'(k_1, k_2) \geq k_1 + k_2 + 1$ such that every digraph with minimum out-degree $f'(k_1, k_2)$ has a $([cycle, \delta^+ \geq k_1], [cycle, \delta^+ \geq k_2])$-partition. As high out-degree is sufficient to guarantee disjoint cycles (See Thomassen [46] and Alon [1]) and as out-degree 3 guarantees two disjoint cycles this implies that out-degree 3 is sufficient for partitions with out-degree 1 in each. The problem of finding partitions where we want the out-degree to be more than 2 is for general digraphs still open.

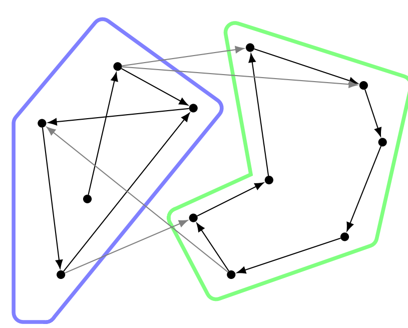


Figure 5.1: A $(\delta^+ \geq 1, strong)$-partition.

In 2012 Lichiardopol, proved this and the related result for minimum semi-degree in tournaments.

Theorem 5.1. *[37] Let k_1, k_2 be integers and let T be a tournament with minimum out-degree at least $\frac{k_1^2+3k_1+2}{2} + k_2$. Then T has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.*

Theorem 5.2. *[37] Let k_1, k_2 be integers and let T be a tournament with minimum semi-degree at least $k_1^2 + 3k_1 + 2 + k_2$. Then T has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition.*

Both results can be generalized to semicomplete digraphs. Also similar results on digraphs with ***bounded independence number*** can be found using the same techniques as seen in [37]. Alon, Bang-Jensen and Bessy have recently shown that for large constants the bound found in Theorem 5.1 and Theorem 5.2 are not the best possible.

Theorem 5.3. *[2] There exist an absolute constant c_1 such that every semicomplete digraph S with minimum out-degree at least $2k + c_1\sqrt{k}$ has a $(\delta^+ \geq k, \delta^+ \geq k)$-partition.*

Theorem 5.4. *[2] There exist an absolute constant c_2 such that every semicomplete digraph S with minimum semi-degree at least $2k + c_1\sqrt{k}$ has a $(\delta^0 \geq k, \delta^0 \geq k)$-partition.*

In the above we only consider 2-partitions, but it is not hard to generalize these results to larger partitions with corresponding larger functions.

Instead of looking for a function that guarantees the existence of a partition, one could ask for the complexity of the given partition problem. In [12] and [8] Bang-Jensen, Cohen[1] and Havet determined the complexity of 120 such 2-partition problems for general digraphs. Bang-Jensen and this author further investigated these problems [6], and focused on partitions with certain minimum degree properties. We proved, that apart for the 'trivial' $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition, the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition is NP complete for general digraphs.

Theorem 5.5. *[6] Let k_1, k_2 be fixed integers such that $k_1 + k_2 \geq 3$, then it is NP complete to decide if a semicomplete digraph contains a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.*

Together with the results of Bang-Jensen, Cohen and Havet, this implies that all variations (except $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition) of minimum degree partition problems are NP complete for general digraphs and naturally we considered these problems on

[1]Cohen is only co-author on [8]

semicomplete digraph. Here we found that the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition and the $(\delta^+ \geq 1, \delta^- \geq k_2)$-partition problem are polynomial for all k_1, k_2 and for the special case where $k_1 = k_2 = 1$ we produce constructive proofs for the $(\delta^+ \geq 1, \delta^- \geq 1)$-partition, the $(\delta^0 \geq 1, \delta^- \geq 1)$-partition and the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition problem. The complexity of remaining problems has not been found. We conjecture that all of these are polynomial. All results of [6] will be further described in Chapter 6.

Knowing that a partition exist, we could also ask if we can find the partition and if so, how fast. In [6] all proofs are constructive, but no work has been done in order to improve the runtime. In Chapter 7 we will improve the runtime on one of the problems. Here we describe an $O(n^2)$ algorithm that finds a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition in a semicomplete digraph, given the semicomplete digraph has minimum semi-degree $O(k_1^2)$. This is significantly better than the obvious runtime obtained by using Theorem 5.2.

We will end this part in Chapter 8 where we will discuss the issues and problems found in the work with degree constrained partition. Here we will also mention other partition results and give some open problems.

Chapter 6

Degree constrained 2-partitions of semicomplete digraphs

6.1 Introduction

In this chapter we will give an introduction to the results from the article 'Degree constrained 2-partitions of semicomplete digraphs', which is joined work between this author and Bang-Jensen [6]. The full paper can be found in Part IV. As mentioned in the introduction, Bang-Jensen, Havet and Cohen determined the complexity of 120 partition problems in [8] and [12]. The following three motivates the work done in the paper 'Degree constrained 2-partition of semicomplete digraphs'.

Theorem 6.1. *[8] The following three partition problems are NP-complete on general digraphs:*

- *The* $(\delta^+ \geq 1, \delta^- \geq 1)$*-partition problem.*
- *The* $(\delta^0 \geq 1, \delta^- \geq 1)$*-partition problem.*
- *The* $(\delta^0 \geq 1, \delta^0 \geq 1)$*-partition problem.*

Now for fixed integers k_1, k_2 we obtain the following results.

- If $k_1 + k_2 \geq 3$, then the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem is NP-complete for general digraphs (for $k_1 + k_2 = 2$ it is polynomial). See Section 6.2.
- If $k_1, k_2 \geq 1$, then the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem is polynomial for semicomplete digraphs and digraphs with bounded independence number. See Section 6.3.
- The $(\delta^+ \geq 1, \delta^- \geq 1)$-partition problem, $(\delta^0 \geq 1, \delta^- \geq 1)$-problem and $(\delta^0 \geq 1, \delta^0 \geq 1)$-problem is polynomial solvable for semicomplete digraphs. See Section 6.4.

- The $(\delta^+ \geq 1, \delta^- \geq k)$-partition problem is polynomially solvable for semicomplete digraphs. See Section 6.5

- The $([strong, tournament], [strong, tournament])$-partition problem is NP-complete for semicomplete digraphs, i.e the problem of finding a partition where both partitions induce a strong digraph without 2-cycles. See Section 6.6.

Notice that in each of these problems an obvious necessary condition is that the digraph has at least k_1+k_2+2 vertices and such a partition can only be obtained when the digraph is a biorientation of the complete graph. In the following we will assume that the digraph always has order at least $k_1 + k_2 + 2$. Also other trivial necessary condition such has $\delta^+(D) \geq k$ when looking for a $(\delta^+ \geq k, \delta^+ \geq k)$-partition will be assumed true below.

Before a more detailed presentation on the results obtained, we need a few extra definitions. Given a digraph D with minimum out-degree at least k and a subset X of its vertices, we say that a set $X' \subseteq V$ is X***-out-critical*** if $X \subseteq X'$, $\delta^+(D\langle X'\rangle) \geq k$ and $\delta^+(D\langle X' - Z\rangle) < k$ for every $\emptyset \neq Z \subseteq X' - X$. Notice that if $\delta^+(D\langle X\rangle) \geq k$, then X is the only X-out-critical set in D. Also every digraph of minimum out-degree at least k contains at least one X-out-critical set for every subset X of vertices (including the empty set). Furthermore considering a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem in a digraph D, a vertex v is said to be ***out-dangerous*** if $d^+(v) < (k_1 + k_2) - 1$.

6.2 Complexity for $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem

In [8] one result missing is the complexity of $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem when $k_1+k_2 \geq 3$. We prove that it is NP-complete using a reduction from monotone 1-IN-3-SAT [1]. The construction makes a digraph on the backbone of the 3-SAT instance that forces certain vertices in a certain set to get a valid $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition. A valid partition will then correspond to a yes-instance. With this construction we can prove that if there is a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition then we can find a satisfying assignment and conversely. As this construction can be made in polynomial time, the complexity then follows.

[1] 3-SAT problem where there are no negated literals and where a truth assignment has exactly one true literal in each clause.

6.3 The $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem of digraphs with bounded independence number

In [37] Lichiardopol proved Theorem 5.1. We noticed that the same technique can be used to prove a similar bound for semicomplete digraphs. We can also use the techniques to develop a polynomial algorithm to find such a partition. Actually, as also mentioned in our paper, similar approach can be used to find bound and polynomial algorithm for digraphs with bounded independence number. Below we will present this result for digraphs of bounded independence number. Notice that these proofs are not presented in [6] but are almost identical to the proofs for the corresponding results on semicomplete digraphs found in [6].

The first lemma is a corollary of Turan's theorem 1.3.

Lemma 6.2 ('NEW': Bang-Jensen and Christiansen)**.** *Let k_1, k_2, α be fixed integers and let D be a digraph with independence number at most α. Then the number of out-dangerous vertices of D is at most $\alpha(2(k_1 + k_2) - 3)$.*

Proof. Let X be the set of out-dangerous vertices of D. Then the number of arcs in $D\langle X\rangle$ is at most $|X|(k_1 + k_2 - 2)$. On the other hand Turan's theorem implies that the number of arcs in $D\langle X\rangle$ are at least

$$\frac{|X|(|X|-1)}{2} - \left(1 - \frac{1}{\alpha}\right)\frac{|X|^2}{2} = \frac{|X|}{2}\left(\frac{|X|}{\alpha} - 1\right).$$

Collecting these two we obtain the result

$$|X| \leq \alpha(2(k_1 + k_2) - 3)$$

□

The following lemma is closely related to the result of Lichiardopol. While Lichiardopol only considered critical sets, we consider critical sets containing some predefined vertices X, the X-out-critical set.

Lemma 6.3 ('NEW': Bang-Jensen and Christiansen)**.** *Let k, α be fixed integers and let D be a digraph with independence number at most α, and with minimum out-degree at least k. Furthermore let $X \subseteq V(D)$. Then every X-out-critical set X' of D will have size at most $\frac{\alpha}{2}((k+1)(k+2)) + |X|$.*

Proof. Suppose that for some set $X \subset V$ there is an X-out-critical set X' of size at least $\frac{\alpha}{2}((k+1)(k+2)) + |X| + 1$. Consider the digraph with independence number at most α induced by X', $D' = D\langle X'\rangle$. Let M be the set of vertices that have out-degree exactly k in D' and let $m = |M|$. As each $v \in M$ has out-degree k in D'

$$|N^+_{D'}[M]| \leq m + mk - \frac{m}{2}(\frac{m}{\alpha} - 1) = -\frac{1}{2\alpha}m^2 + \left(\frac{3}{2} + k\right) m =: P(m).$$

Now $P(m)$ has global maximum at $\alpha(3/2+k)$ with value $P(\alpha(3/2)+k) = \frac{\alpha}{2}(\frac{3}{2}+k)^2$ and integer maximum at $\frac{\alpha}{2}((k+1)(k+2))$. Hence as $|X'| > \frac{\alpha}{2}((k+1)(k+2)) + |X|$ there exists a vertex $u \in X' - (N^+_{D'}[M] \cup X)$ such that $\delta^+(D'\langle X' - u\rangle) \geq k$. But then the set $Z = \{u\}$ is contained in $X' - X$ and $\delta^+(D\langle X' - Z\rangle) \geq k$, contradicting the fact that X' is an X-out-critical set in D.

□

Theorem 6.4 ('NEW': Bang-Jensen and Christiansen). *For every fixed integers k_1,k_2,α there exist a polynomial algorithm that either construct a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition of a given digraph D with independence number at most α, or correctly outputs that none exist.*

Proof. Let (O_1, O_2) be a given partition of the out-dangerous vertices of D. Let $X \subseteq V - O_2$ be a set containing O_1 such that $|X| \leq \frac{\alpha}{2}((k+1)(k+2)) + |O_1|$ and $\delta^+(D\langle X\rangle) \geq k_1$ (if no such set exists, we stop considering the pair (O_1, O_2)). The following sub-algorithm $\mathcal{B}$ will decide whether there exists a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition (V_1, V_2) with $X \subseteq V_1, O_2 \subseteq V_2$: Starting from the partition $(V_1, V_2) = (X, V - X)$, and moving one vertex at a time, the algorithm will move vertices of $V_2 - O_2$ which have $d^+_{D\langle V_2\rangle}(v) < k_2$ to V_1. If, at any time, this results in a vertex $v \in O_2$ having $d^+_{D\langle V_2\rangle}(v) < k_2$, or $V_2 = \emptyset$, then there is no $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition with $X \subseteq V_1$ and $O_2 \subseteq V_2$ and the algorithm $\mathcal{B}$ terminates. Otherwise $\mathcal{B}$ will terminate with $O_2 \subseteq V_2 \neq \emptyset$ and hence it has found a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition (V_1, V_2) with $O_i \subseteq V_i$, $i = 1, 2$.

The correctness of $\mathcal{B}$ follows from the fact that we only move vertices that are not dangerous and each such vertex has at least $k_1 + k_2 - 1$ out-neighbours in D. Hence, as the vertex that we move does not have k_2 out-neighbours in V_2, it must have at least k_1 out-neighbours in V_1, so $\delta^+(S\langle V_1\rangle) \geq k_1$ will hold throughout the execution of $\mathcal{B}$.

By Lemma 6.2, the number of out-dangerous vertices is at most $\alpha(2(k_1 + k_2) - 3)$ and hence the number of (O_1, O_2)-partitions of the set of out-dangerous vertices is at most $2^{\alpha(2(k_1+k_2)-3)}$ which is a constant because k_1, k_2 are fixed. Furthermore, by Lemma 6.3, the size of every O_1-critical set is also bounded by a function of k_1 and hence for each (O_1, O_2)-partition there is only a polynomial number of O_1-critical sets that are disjoint from O_2. Thus we obtain the desired polynomial time algorithm by running the sub-algorithm $\mathcal{B}$ for all possible partitions (O_1, O_2) of the out-dangerous vertices and all possible choices of sets X with $O_1 \subseteq X$ and $|X| \leq \frac{\alpha}{2}((k+1)(k+2)) + |O_1|$. Note that we do not need to check whether X is O_1-out-critical, we just check all possible supersets of O_1 of size at most $\frac{\alpha}{2}((k+1)(k+2)) + |O_1|$. □

6.4 Partitions of semicomplete digraphs where both constants are 1

We prove that $(\delta^+ \geq 1, \delta^- \geq 1)$- $(\delta^0 \geq 1, \delta^- \geq 1)$- and the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition problems are all polynomial and that each of these can be found in polynomial time. Clearly disjoint cycles are necessary for all three problems, and for the first two we prove that two disjoint cycles are also sufficient. Polynomial then follows easily, by finding two disjoint cycles of length 3 in the semicomplete digraph.

Now for the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition problem, disjoint cycles is not sufficient but complementary cycles are. Reid [41] proved that every 2-strong tournament of at least 8 vertices contains complementary cycles, and Guo and Volkmann improved this to also apply to semicomplete digraphs [26]. As Bang-Jensen and Nielsen [16] found a polynomial algorithm to decide if a semicomplete digraph is 2-strong, we are left with analysing the following cases for the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition problem.

1. D is a semicomplete digraph of at most 7 vertices.
2. D is a semicomplete digraph which is not strong.
3. D is a semicomplete digraph that is strong but not 2-strong.

While 1 can be decided by brute force, 2 follows considering the first and last strong component of D and applying the corresponding algorithms for the $(\delta^+ \geq 1, \delta^+ \geq 1)$-, $(\delta^+ \geq 1, \delta^- \geq 1)$- and $(\delta^0 \geq 1, \delta^- \geq 1)$-partition problem. Now to prove 3 we consider a separator x of D and the semicomplete decomposition $D_1, D_2, \ldots, D_r$ of $D - x$. By case analysis we either find a partition, reduces the problem to a problem on a smaller semicomplete digraph or concludes that none exist. For example Figure 6.1 shows how a structure where only D_r is non-trivial and x dominates D_{r-1} can be reduced to the same problem on a smaller semicomplete digraph. Another example is seen in Figure 6.2. Here both D_1, D_r are non-trivial strong components and every vertex y of D_1 (z of D_r) that x dominates (is dominated by) is a separator of D_1 (D_r) such that $D_1 - y$ ($D_r - z$) is a transitive tournament. In this case there is only a solution, if x belongs to a cycle disjoint from D_1 and D_r.

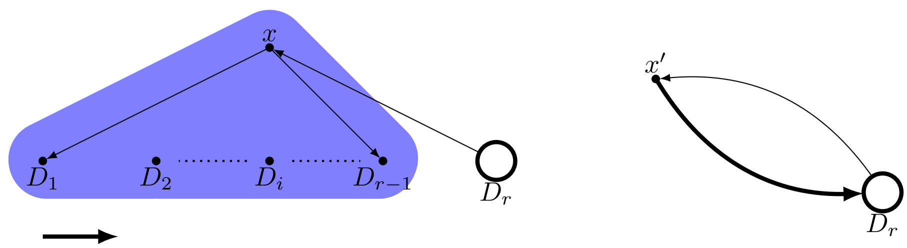


Figure 6.1: One of the cases in the analysis of 3. In the figure to the left the semicomplete digraph D is seen and all vertices except D_r is marked with blue. On the right the blue vertices is replaced by just one vertex x'. Here x' is dominated by the same vertices of D_r that x where, and x' dominates all vertices of D_r (notice that it is a semicomplete digraph and hence 2-cycles is allowed).

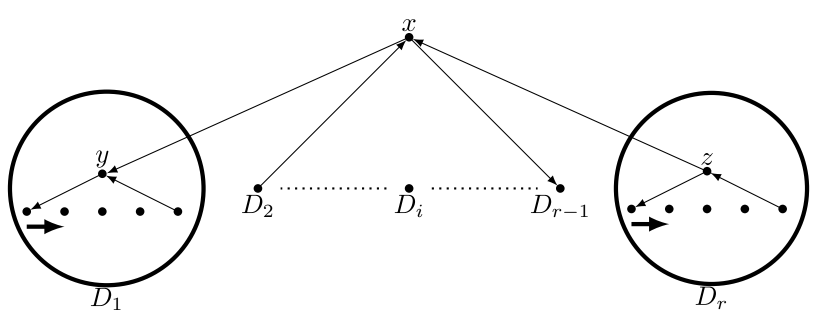


Figure 6.2: Another of the cases in the analysis of 3. Here we see that $\{x, y, z\}$ is a feedback vertex set and there is only a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if x belongs to a cycle disjoint from D_1 and D_r.

Notice that Lichiardopol in Theorem 5.2 proved that all tournaments (semicomplete digraphs) with minimum semi-degree 7 has an $(\delta^+ \geq 1, \delta^- \geq 1)$- $(\delta^0 \geq 1, \delta^- \geq 1)$ and $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. As we proved that two disjoint cycles are sufficient for $(\delta^+ \geq 1, \delta^- \geq 1)$- and $(\delta^0 \geq 1, \delta^- \geq 1)$-partition [2] minimum out-degree 3 (minimum in-degree 3) is sufficient for these problems. Now by inspection of the construction of the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition one realizes that minimum semi-degree at least 3 will always be sufficient. This follows as every no-instance has vertices with in- or out-degree less that 3. Consider for example the case where D is not strong and $\delta^0(D) \geq 3$. Then the initial strong component D_1 will have minimum in-degree at least 3 and the terminal strong component D_r will have minimum out-degree at least 3. But then D_1 has a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition, (P_1, P_2) and D_r has a $(\delta^0 \geq 1, \delta^+ \geq 1)$-partition, (Q_1, Q_2) and it follows that $(P_i \cup Q_j, P_{3-i} \cup Q_{3-j} \cup D_2 \cup \cdots \cup D_{r-1})$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition for every $i, j \in [2]$. If D is strong and x is a separator, then considering the strong components of $D - x$, D_1 and D_r will always be non-trivial, and we will never end up in the case illustrated in Figure 6.2.

Corollary 6.5. *['NEW': Bang-Jensen and Christiansen] Every semicomplete digraph with minimum semi-degree* 3 *contains a* $(\delta^0 \geq 1, \delta^0 \geq 1)$*-partition.*

6.5 The $(\delta^+ \geq 1, \delta^- \geq k)$-partition problem for semicomplete digraphs

We start by proving the following lemma.

Lemma 6.6. *Let k be a fixed integer and D a semicomplete digraph. Furthermore let X_1, X_2 be disjoint subsets of V such that*

(a) $V - X_1 - X_2$ induces a transitive tournament.

(b) If there is a vertex v of X_1 such that $d^+_{X_1}(v) = 0$ then v is dominated by at most $k - 1$ vertices of $V - X_1 - X_2$.

Then there exist a polynomial algorithm that decide whether D has a $(\delta^+ \geq 1, \delta^- \geq k)$-partition (V_1, V_2) with $X_i \subset V_i$ for $i \in [2]$ and find such a partition when it exists.

Then using this lemma, the following two are proved using similar techniques.

Theorem 6.7. *[6] There exists a polynomial algorithm that either finds a $(\delta^+ \geq 1, \delta^- \geq 2)$-partition of a semicomplete digraph D or correctly outputs that none exist.*

[2] actually also $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition

Theorem 6.8. *[6] For every fixed integer $k \geq 1$ there exists a polynomial algorithm that either constructs a $(\delta^+ \geq 1, \delta^- \geq k)$-partition of a semicomplete digraph D or correctly outputs that none exist.*

The following is a short sketch of the proofs. For each 3-cycle C_i of the semicomplete digraph D we want to check if D has a partition (V_1, V_2), where $C_i \subset V_1$, $\delta^+(V_1) \geq 1$ and $\delta^-(V_2) \geq 2$ ($\delta^-(V_2) \geq k$). We call such a partition a ***good partition*** of D. We start by letting $V_1 = C$ and $V_2 = V - C_1$. Then vertices of V_2 not having 1 (k) in-neighbours in V_2 are moved to V_1. If we do not terminate in this process, then V_2 induces a semicomplete digraph with minimum in-degree at least 2 (k), while there is a vertex $v \in V_1$ that is dominated by all other vertices in V_1. Now we prove that either

- V_2 contains a collection of short cycles $\mathcal{C}$ that can be moved from V_2 to V_1 giving a good partition.
- or we find a feedback vertex set F of $S\langle V_2 \rangle$.

To do this notice that $v \notin C_i$ and hence was moved to V_1 because it did not have sufficiently many in-neighbours, meaning that v is dominated by at most 1 ($k-1$) vertices of V_2. Hence if the collection of short cycles $\mathcal{C}$ has size least 2 (k) vertices, then $V_1 \cup \mathcal{C}$ will induce a semicomplete digraph with minimum out-degree 1. We look for $\mathcal{C}$ amongst vertices that will not decrease the out-degree of V_2 below the threshold of 1 (k). If $\mathcal{C}$ does not exist, then all such collections of short cycles is dominated by vertices of low out-degree in V_2. Together with some additional vertices this will lead to a feedback vertex set of V_2.

Now if D has a good partition $(\widehat{V_1}, \widehat{V_2})$ then some subset (possible empty) of F belongs to $\widehat{V_1}$. Considering a partition of F, (F_1, F_2), we check if D has a good partition with $F_1 \cup C_i$ using Lemma 6.6.

To realize that this can be done in polynomial time notice that we can list and find all cycles of length at most 3 in D. Also as F is bounded the number of partitions of F is bounded, and we can check each of these in polynomial time using Lemma 6.6.

6.6 The $(\delta^+ \geq k, \delta^+ \geq k)$-partition of semicomplete digraphs such that each partition is a tournament

Like the first NP-complete proof of the paper, this is also done by a reduction from 3-SAT, this time NAE-3-SAT. Due to the fact that the digraph now has to be a semicomplete digraph, this construction is even more complicated, and we will not go into detail with the proof here.

Chapter 7

An efficient algorithm for finding a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition in tournaments with high minimum semi-degree

In Theorem 5.2 Lichiardopol proved that every tournament with minimum semi-degree at least $k_1^2 + 3k_1 + 2 + k_2$ contains a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition. To prove this he first considered tournaments with minimum semi-degree k and proved that there is a subset X of V of size at most $k^2 + 3k + 2$, such that $T\langle X \rangle$ has minimum semi-degree at least k. This will then imply that every tournament with minimum semi-degree at least $k_1^2 + 3k_1 + 2 + k_2$ contains a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition. The proof directly leads to an algorithm that finds such a partition; Let $X_1, X_2, \ldots, X_m$ be every subset of $V(T)$ of size at most $k_1^2 + 3k_1 + 2$. For each $i \in [m]$, we let $V_1 = X_i, V_2 = V - X_i$ and move vertices from V_2 to V_1 until $T\langle V_2 \rangle$ has minimum semi-degree k_2 or V_2 is empty. If (V_1, V_2) is a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition, then the algorithm outputs this, otherwise it moves to the next X_i set. The runtime of this algorithm is dominated by the time it takes to find the X_i set, which is $O(n^{k_1^2})$.

In 2014 Bang-Jensen and this author was inspired by a structural result on tournaments obtained by Kühn, Lapinskas, Osthus and Patel [35, 34] to consider the $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition problem for tournaments, unaware that Lichiardopol had already proven Theorem 5.2. In the following we will present our result. Here we show that increasing the minimum semi-degree slightly compared to Theorem 5.2 we obtain an algorithm that find a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition of a tournament in just $O(kn^2)$ time.

Theorem 7.1 (NEW: Bang-Jensen and Christiansen). *Let k_1, k_2 be fixed integers and define*

$$q(k_1, k_2) = 4(k_1^2 + k_2^2) + 4(k_1 + k_2) + 4\lceil \log_2(4(k_1 + k_2)^2)\rceil(k_1 + k_2).$$

Then every tournament T with minimum semi-degree $\delta^0(T) \geq q(k_1, k_2)$ contains a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition and such a partition can be found in $O(kn^2)$ time.

7.1 A few extra definitions and two easy results

Let $D = (V, A)$ and k be a fixed integer non-negative. We say that a set $X \subseteq V$ is ***in-happy*** (***out-happy***) if each vertex $v \in X$ has k in-neighbours (out-neighbours) in D. X is ***happy*** if it is both in- and out-happy. A vertex that is not happy is said to be ***sad*** and it is i***-in-sad*** (i***-out-sad***) in D if the in-degree (out-degree) of this vertex is $k - i$ (for $i \leq k$). Let I_i be the set of vertices in D that are i-in-sad. Then the ***in-sadness*** of D is

$$\sigma^{\mathrm{in}}(D) = \sum_{i=1}^{k} i|\mathrm{I}_i|.$$

Similarly if O_i is the set of i-out-sad vertices then the ***out-sadness*** of D is

$$\sigma^{\mathrm{out}}(D) = \sum_{i=1}^{k} i|\mathrm{O}_i|.$$

Finally the ***sadness*** of D is $\sigma(D) = \sigma(D)^{\mathrm{out}} + \sigma(D)^{\mathrm{in}}$. Now consider the subdigraph $D' = D\langle X\rangle$. A vertex $v \in V - X$ is said to be ***safe*** with respect to D' if adding v to this subdigraph will not increase the sadness of D'. This means that a vertex is safe if it dominates at least k vertices of X and is dominated by at least k vertices of X. In the above definitions we assume that k is already specified, but when it is not clear from the context which integer they refer to, we may add k explicitly in the definitions. For example, we may write k-happy, k-safe and $\sigma_k(D)$.

Lemma 7.2 (NEW: Bang-Jensen and Christiansen). *Given a tournament T on n vertices and an integer k with $k \leq n$. Then the out-sadness (in-sadness) of T is at most $\frac{k^2+k}{2}$. Equality is obtained for transitive tournaments on at least k vertices.*

Proof. We will only prove the result for out-sadness as the proof for in-sadness is analogous. Let Z be the collection of out-sad vertices in T. Notice that every arc of $T\langle Z\rangle$ leaves a vertex whose out-degree is less than k. Hence we have

$$\sigma^{\mathrm{out}}(T) \leq |Z|k - \binom{|Z|}{2} \leq \frac{k^2 + k}{2},$$

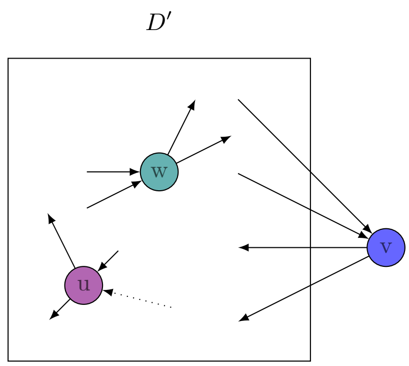


Figure 7.1: Illustration of safe, sad and happy vertices with $k = 2$. Here v is a safe, u is 1-in-sad and $\{w\}$ is happy.

where the last inequality is attained with equality for $|Z| \in \{k, k+1\}$. Considering the vertices $v_{n-k+1}, v_{n-k+2}, \dots, v_n$ in an acyclic ordering of the vertices of a transitive tournament T, we see that the set consisting of these vertices has k-out-sadness exactly $\frac{k^2+k}{2}$ in T. □

Corollary 7.3 (NEW: Bang-Jensen and Christiansen)**.** *Let T be a tournament on n vertices and k an integer with $k \leq n$. Then $\sigma(T) \leq k^2 + k$.*

7.2 Proof of Theorem 7.9

Before proving the theorem we will give a short outline of the proof. The proof is inspired by an approach used with great success in [35, 34]. We will start with the empty sets $V_1^{(0)} = V_2^{(0)} = \emptyset$ and will add vertices to the current sets $V_1^{(i)}$ respectively $V_2^{(i)}$ in three stages until the resulting sets $V_1^{(3)}$ and $V_2^{(3)}$ form a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition of T. First we find $2k_1 + 2k_2$ vertex disjoint sets each inducing transitive tournaments and add the first $2k_1$ of these to $V_1^{(0)}$ to obtain $V_1^{(1)}$ and the remaining $2k_2$ to $V_2^{(0)}$, obtaining $V_2^{(1)}$. These $2k_1 + 2k_2$ sets have the property that after adding them to $V_1^{(0)}$ and $V_2^{(0)}$ most of the vertices of $V - (V_1^{(1)} \cup V_2^{(1)})$ will be k_1-safe with respect to $V_1^{(1)}$ and k_2-safe with respect to $V_2^{(1)}$. In the next stage we will add vertices to $V_1^{(1)}$ and $V_2^{(1)}$ such that the resulting sets $V_1^{(2)}, V_2^{(2)}$ become k_1-happy, respectively k_2-happy sets. This is mostly done by adding safe vertices from $V - (V_1^{(1)} \cup V_2^{(1)})$ that have the connections to sad vertices of $T\langle V_1^{(1)} \rangle$ and $T\langle V_2^{(1)} \rangle$. Finally we will add each of the remaining vertices of T to either $V_1^{(2)}$ or $V_2^{(2)}$ (building $V_1^{(3)}, V_2^{(3)}$). In this stage, if every vertex left would be either k_1-safe with respect to $V_1^{(2)}$ or k_2-safe with respect to $V_2^{(2)}$ this would be easy as we could just add these to $V_1^{(2)}$ respectively $V_2^{(2)}$. This is though not necessarily the case, and we need to handle the non-safe vertices by adding them together with safe vertices in a specific order.

Now let T be a given tournament, k_1 and k_2 fixed integers and let $k = k_1 + k_2$.

For convenience we define the following sets, where $V_1^{(i)}, V_2^{(i)}$ denote the current sets $V_1^{(i)}, V_2^{(i)}$, for $i \in [0,3]$ (they change during the algorithm). The set E will be defined in the next section.

$$V_{rest}^{(i)} = V - (V_1^{(i)} \cup V_2^{(i)}) \tag{7.2.1}$$

$$\hat{V}^{(i)} = V_{rest}^{(i)} - E \tag{7.2.2}$$

7.2.1 Constructing $V_1^{(1)}$ and $V_2^{(1)}$ by adding disjoint transitive tournaments

Let $X = \{x_1, x_2, \ldots, x_k\}$ be the k vertices of lowest in-degree in T and let $Y = \{y_1, y_2 \ldots, y_k\}$ be the k vertices of $V - X$ of lowest out-degree in T. Furthermore let

$$\hat{\delta}^-(T) = \min_{v \in V-X} d^-(v), \quad \hat{\delta}^+(T) = \min_{v \in V-Y} d^+(v), \quad \hat{\delta}^0(T) = \min\{\hat{\delta}^+(T), \hat{\delta}^-(T)\}$$

Now the $2k = 2k_1 + 2k_2$ sets that we describe below will each induce a transitive tournament either with sink $x_i \in X$ or source $y_i \in Y$. The following results are found in [35, 34]. For completeness, we will give the proof.

Lemma 7.4. *[34] Let T be a tournament on n vertices, let $v \in V(T)$ and suppose that $c \in \mathbb{N}$. Then, in $O(n^2)$, we can construct disjoint sets $A, E_A \subseteq V(T)$ such that the following properties hold:*

1. *$1 \le |A| \le c$ and $T\langle A\rangle$ is a transitive tournament with sink v*
2. *A out-dominates $V(T) - (A \cup E_A)$*
3. *$|E_A| \le (1/2)^{c-1} d^-(v)$*
4. *either $E_A = \emptyset$ or E_A is the common in-neighbourhood of all vertices in A*

Proof. We will build A in such a way that in each step the size of the common in-neighbourhood of A, E_A, is decreased to at least half its size. First we let $A_1 = \{v\} = \{v_1\}$. Then clearly A_1 is a transitive tournament with sink v_1. If E_{A_1} fulfills 3, then there is nothing to prove. So assume it does not, and that we have added the vertices $v_1, v_2, \ldots, v_i$ with $i < c$ to obtain the set A_i in such a way that

$$|E_{A_i}| \le \left(\frac{1}{2}\right)^{i-1} d^-(v) \tag{7.2.3}$$

If (7.2.3) is at most $(1/2)^{c-1}d^-(v)$, then we are done and let $A = A_i$ and $E_A = E_{A_i}$. Otherwise if (7.2.3) is strictly greater than $(1/2)^{c-1}d^-(v)$, then consider the set E_{A_i}. As E_{A_i} induces a tournament, there is a vertex v_{i+1} in E_{A_i} that are dominated by at most $|E_{A_i}|/2$ vertices. Hence adding this vertex to A_{i+1} we obtain

$$|E_{A_{i+1}}| \leq \frac{1}{2}|E_{A_i}|.$$

Now either A_{i+1} fulfills 3, or we repeat until at latest in step c we terminate with sets A_c and E_{A_c}. Notice that $|E_{A_c}| \leq (1/2)^{c-1}d^-(v)$ and hence 3 will be fulfilled in step c.

Now for the runtime, notice that at step i, the size of E_{A_i} is at most $\frac{n}{2^{i-2}}$ and hence finding the vertex v_{i+1} can be done in $O(\left(\frac{n}{2^{i-2}}\right)^2)$. As we repeat at most c times and c is a fixed integer, this implies that the runtime is $O(n^2)$. $\square$

The following lemma can be proven similarly.

Lemma 7.5. *[34] Let T be a tournament on n vertices, let $v \in V(T)$ and suppose that $c \in \mathbb{N}$. Then, in time $O(n^2)$, we can construct disjoint sets $B, E_B \subseteq V(T)$ such that the following properties hold:*

1. *$1 \leq |B| \leq c$ and $T\langle B\rangle$ is a transitive tournament with source v*
2. *B in-dominates $V(T) - (B \cup E_B)$*
3. *$|E_B| \leq (1/2)^{c-1}d^+(v)$*
4. *either $E_B = \emptyset$ or E_B is the common out-neighbourhood of all vertices in B*

By applying each of these two lemmas k times, each time removing the vertices of the transitive tournament just constructed and each time ensuring that the vertices of X are only used as sinks (one for each A_i) and the vertices of Y are only used as sources (one for each B_i) we have the following corollary (which can also be found in [34])

Corollary 7.6. *[34] Let c be a fixed integer. Then, in time $O(kn^2)$, we can construct disjoint vertex sets A_i, B_i, E_{A_i} and E_{B_i} for $i \in [k]$ fulfilling the following*

1. *$1 \leq |A_i| \leq c$ and $T\langle A_i\rangle$ is a transitive tournament with sink x_i*
2. *A_i out-dominates $V(T) - (D \cup E_{A_i})$*
3. *$|E_{A_i}| \leq (1/2)^{c-1}\hat{\delta}^-(T)$*
4. *either $E_{A_i} = \emptyset$ or E_{A_i} is common in-neighbourhood of all vertices in A_i*
5. *$1 \leq |B_i| \leq c$ and $T\langle B_i\rangle$ is a transitive tournament with source y_i*
6. *B_i in-dominates $V(T) - (D \cup E_{B_i})$*

7. $|E_{B_i}| \leq (1/2)^{c-1}\hat{\delta}^+(T)$

8. *either* $E = \emptyset$ *or* E *common out-neighbourhood of all vertices in* B_i

Furthermore by letting $E_A = \bigcup_{j=1}^k (E_{A_i})$, $E_B = \bigcup_{j=1}^k (E_{B_i})$ *and* $E = E_A \cup E_B$ *we have*

$$|E_A| \leq k\left(\frac{1}{2}\right)^{c-1}\hat{\delta}^-(T)$$

$$|E_B| \leq k\left(\frac{1}{2}\right)^{c-1}\hat{\delta}^+(T)$$

Now let $c := \lceil \log_2(4(k_1+k_2)^2) \rceil$ and use Corollary 7.6 to obtain $2k_1 + 2k_2$ vertex disjoint transitive tournaments. We will then let $V_1^{(1)} = \bigcup_{i=1}^{k_1}(A_i \cup B_i)$ and $V_2^{(1)} = \bigcup_{i=k_1+1}^{k}(A_i \cup B_i)$. Assume below that $|E_A| \leq |E_B|$. The proof in the opposite case is analogous, and we leave it to the reader. Notice that by the choice of c we have

$$|E_A| \leq \frac{1}{2(k_1+k_2)}\hat{\delta}^-(T) \tag{7.2.4}$$

$$|E_B| \leq \frac{1}{2(k_1+k_2)}\hat{\delta}^+(T) \tag{7.2.5}$$

$$|E| \leq 2|E_B| \leq \frac{1}{(k_1+k_2)}\hat{\delta}^+(T) \tag{7.2.6}$$

Combining (7.2.6) with $\hat{\delta}^+(T) < \frac{n}{2}$, we have

$$|E| < \frac{n}{2(k_1+k_2)} \tag{7.2.7}$$

Now, given that the transitive tournaments with sinks x_i, $i \in [k]$ out-dominate most of the vertices of $V_{rest}^{(1)}$ and the transitive tournaments with sources y_i, $i \in [k]$ in-dominate most of the vertices of $V_{rest}^{(1)}$, we have the safeness structure described in the first step of the proof sketch.

Proposition 7.7. *After constructing the transitive tournaments using Corollary 7.6 we have, for each* $j \in [2]$

(1) *All vertices in* $V_{rest}^{(1)} - (E_A)$ *are* k_j*-in-safe with respect to* $V_j^{(1)}$ *and all vertices in* $V_{rest}^{(1)} - (E_B)$ *are* k_j*-out-safe with respect to* $V_j^{(1)}$.

(2) *All vertices in* $\widehat{V}^{(1)}$ *are* k_j*-safe with respect to* $V_j^{(1)}$.

(3) *All vertices of* $\bigcap_{i\in[k]}(E_{A_i} \cap E_{B_i})$ *are* k_j*-safe with respect to* $V_j^{(1)}$.

(4) *Let* $E_1 = \bigcup_{i=1}^{k_1}(E_{A_i} \cup E_{B_i})$ *and* $E_2 = \bigcup_{i=k_1+1}^{k}(E_{A_i} \cup E_{B_i})$. *Then all vertices of* $E_{3-j} - E_j$ *are* k_j*-safe with respect to* $V_j^{(1)}$

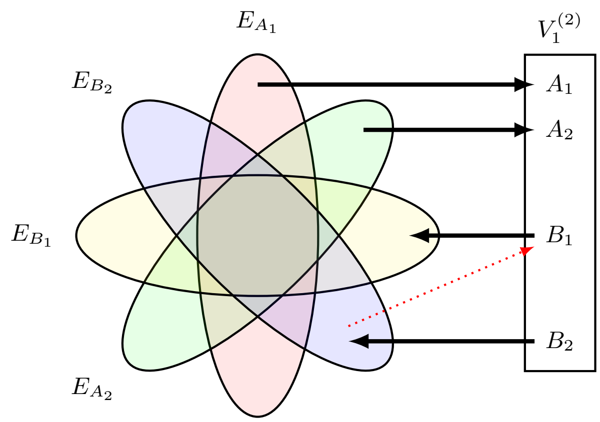


Figure 7.2: The dominance structure between the E sets and the A_i and B_i sets in $V_1^{(1)}$ when $k_1 = 2$. Every arc missing between E vertices and the A_i and B_i sets is an arc in the direction opposite to the direction of the corresponding black arcs. An example is the red dotted arc between a $E_{B_1} - (E_{A_1} \cup E_{A_2} \cup E_{B_2}))$ vertex and B_1.

Proof. First notice that, by definition, there are k_j A_i sets in $V_j^{(1)}$ and k_j B_i sets in $V_j^{(1)}$. This immediately implies that (1) and (2) hold. To prove (3) let $v \in \bigcap_{i\in[k]}(E_{A_i} \cap E_{B_i})$. Then v dominates every vertex of $\bigcup_{i\in[k]} A_i$ and is dominated by every vertex of $\bigcup_{i\in[k]} B_i$. Hence as $|A_i|, |B_i| \geq 1$ and as there are k_1 A_i and k_1 B_i sets in $V_1^{(1)}$ v is k_1-safe with respect to $V_1^{(1)}$ and similarly v is k_2-safe with respect to $V_2^{(1)}$. The correctness of (4) follows by similar arguments as the ones above. □

7.2.2 Adding more vertices to ensure that $V_1^{(2)}$ and $V_2^{(2)}$ are happy

We will now add vertices to $V_1^{(1)}$ and $V_2^{(1)}$ such that the resulting set $V_i^{(2)}$ induce a k_i-happy set for $i \in [2]$. To simplify notation we let $T_1^{(j)} = T\langle V_1^{(j)}\rangle$ and $T_2^{(j)} = T\langle V_2^{(j)}\rangle$ for $j \in [0, 3]$. Notice that, by Corollary 7.3, we have

$$\begin{aligned} q(k_1, k_2) &= 4(k_1^2 + k_2^2) + 4(k_1 + k_2) + 4c(k_1 + k_2) \\ &\geq 4\sigma(T_1^{(1)}) + 4\sigma(T_2^{(1)}) + 4ck_1 + 4ck_2 \end{aligned}$$

Now let $X_i = X \cap V_i^{(1)}$ and $Y_i = Y \cap V_i^{(1)}$, $i \in [2]$. These vertices are of special nature since they may have in- respectively out-degree smaller than $\hat{\delta}^0(T)$. For this reason we will start by ensuring that these vertices become happy. As the total sadness of the vertices in $X \cup Y$ is at most $\sigma(T_1^{(1)}) + \sigma(T_2^{(1)})$ this is an upper bound on the number of neighbour vertices we need to pick from $V_{rest}^{(1)}$. Now since

$$\begin{aligned}\delta^0(T) - |V_1^{(1)}| - |V_2^{(1)}| &\geq q(k_1,k_2) - |V_1^{(1)}| - |V_2^{(1)}| \\ &\geq 4\sigma(T_1^{(1)}) + 4\sigma(T_2^{(1)}) + 4ck_1 + 4ck_2 - 2ck_1 - 2ck_2 \\ &= 4\sigma(T_1^{(1)}) + 4\sigma(T_2^{(1)}) + 2ck_1 + 2ck_2,\end{aligned}$$

even if no two in-sad (out-sad) vertices of $X_i \cup Y_i$, $i \in [2]$ have a common in-neighbour (out-neighbour) in $V_{rest}^{(1)}$, there are a sufficient number of vertices in $V_{rest}^{(1)}$ to choose from. Add such a set of (greedily) chosen vertices to $V_1^{(1)}$ respectively $V_2^{(1)}$ to make the vertices of X_i and Y_i safe in the new induced subtournament $T_i^{(1)}$. By Corollary 7.3, this will increase the size of $V_i^{(1)}$ by (at most) $k_i^2 + k_i$ and again, by Corollary 7.3, after this we also have that the sadness of the resulting set $V_i^{(1)}$ is at most $k_i^2 + k_i$.

We are now ready to add further vertices so that each of the resulting sets $V_i^{(2)}$ induces a k_i-happy tournament. We start by ensuring that every vertex in $V_i^{(1)}$ (including the new ones added above) will be k_i-in-happy in $T_i^{(1)}$. Recall that, by Proposition 7.7, all vertices of $V_{rest}^{(1)} - (E_A \cup E_B)$ are safe and that all the vertices in $E_B - E_A$ are k_i-in-safe for $i \in [2]$ since these vertices receive at least one arc from each A_j, $j \in [k_1 + k_2]$.

Now by (7.2.4) we have

$$\begin{aligned}\hat{\delta}^-(T) - |E_A| &\geq \hat{\delta}^-(T) - \frac{1}{2k}\hat{\delta}^-(T) \\ &\geq (1/2)\hat{\delta}^-(T) \\ &\geq (1/2)\delta^-(T) \\ &\geq (1/2)q(k_1,k_2) && (7.2.8) \\ &\geq 2\sigma(T_1^{(1)}) + 2\sigma(T_2^{(1)}) + 2ck_1 + 2ck_2, && (7.2.9)\end{aligned}$$

and hence every vertex $v \in V_1^{(1)} \cup V_2^{(1)}$ satisfies

$$\begin{aligned}d^-_{T\langle v \cup (\widehat{V}_{rest}^{(1)} - E_A)\rangle}(v) &\geq \hat{\delta}^-(T) - |E_A| - |V_1^{(1)}| - |V_2^{(1)}| \\ &= 2\sigma(T_1^{(1)}) + 2\sigma(T_2^{(1)}) + 2ck_1 + 2ck_2 - |V_1^{(1)}| - |V_2^{(1)}| \\ &\geq \sigma(T_1^{(1)}) + \sigma(T_2^{(1)})\end{aligned}$$

This implies that starting with the current $V_1^{(1)}$ and after adding some vertices to this set, proceeding to the current $V_2^{(1)}$, we may greedily pick distinct in-neighbours in $V_{rest}^{(1)} - E_A$ for each in-sad vertex in $V_i^{(1)}$ and add these in-neighbours to $V_i^{(1)}$ so that after this we have added at most $\sigma^{in}(T_i^{(1)}) \leq k_i^2 + k_i$ new vertices to $V_i^{(1)}$ for $i \in [2]$. Now the resulting subtournaments $T_i^{(1)}$, $i \in [2]$ contain no in-sad vertices.

By similar calculations as in (7.2.9) we obtain

$$\begin{aligned}\hat{\delta}^+(T) - |E| &\geq (1/2)q(k_1,k_2) \\ &\geq 2\sigma(T_1^{(1)}) + 2\sigma(T_2^{(1)}) + 2ck_1 + 2ck_2,\end{aligned} \tag{7.2.10}$$

and as adding the vertices above will increase $|V_i^{(1)}|$ by at most $\sigma^{\mathrm{in}}(T_i^{(1)})$ for $i = 1, 2$, this implies that every vertex $v \in V_1^{(1)} \cup V_2^{(1)}$ satisfies

$$\begin{aligned}d^+_{T\langle v \cup \widehat{V}^{(1)}\rangle}(v) &\geq \hat{\delta}^+(T) - |E| - |V_1^{(1)}| - |V_2^{(1)}| \\ &\geq \sigma(T_1^{(1)}) + \sigma(T_2^{(1)}) - \sigma^{\mathrm{in}}(T_1^{(1)}) - \sigma^{\mathrm{in}}(T_2^{(1)}) \\ &\geq \sigma^{\mathrm{out}}(T_1^{(1)}) + \sigma^{\mathrm{out}}(T_2^{(1)})\end{aligned}$$

Hence, as above, we can greedily pick distinct out-neighbours of every out-sad vertex in $V_1^{(1)} \cup V_2^{(1)}$ from $\widehat{V}^{(1)}$ such that after adding all these safe vertices, every vertex in the resulting set $V_i^{(2)}$ is k_i-happy in $T_i^{(2)}$ for $i \in [2]$.

Notice that each of the three steps above can be done in $O(k^2)$ time. This follows as at most $k^2 + k$ vertices are chosen greedily from $V_{rest}^{(1)}$ respectively $V_{rest}^{(1)} - E_A$ respectively $V_{rest}^{(1)} - E$. Hence in $O(k^2)$ time we have added vertices to $V_i^{(1)}$ for $i = 1, 2$ obtaining $V_i^{(2)}$.

7.2.3 Distributing the last vertices

Now that we have ensured that $T_i^{(2)}$ is happy for $i \in [2]$, we will add each of the remaining vertices of T to either $V_1^{(2)}$ or $V_2^{(2)}$ such that we obtain a real partition. As noted earlier, if every remaining vertex is either k_1-safe with respect to $T_1^{(2)}$ or k_2-safe with respect to $T_2^{(2)}$ then we would just add these vertices to their respective sets, and we were done [1]. This may though not be the case as we might still have vertices of E left. As we will use the safe vertices of $\widehat{V}^{(2)}$ to distribute the vertices of E we first find a lower bound on the size of $\widehat{V}^{(2)}$. First we have

$$|V_1^{(2)}| + |V_2^{(2)}| \leq 2\sigma(T_1^{(1)}) + 2\sigma(T_2^{(1)}) + 2ck_1 + 2ck_2 = \frac{1}{2}q(k_1,k_2).$$

Furthermore by (7.2.7)

$$|\widehat{V}^{(2)}| - |E| > n - \frac{n}{2(k_1+k_2)} \geq \frac{n}{2} + \frac{n}{4}.$$

[1] Note the special case where $E_j = \emptyset$ for some $j \in [2]$, then all vertices of E can be added to $V_j^{(2)}$ by Proposition 7.7 (4) and we are left with vertices that are safe with respect to both sets.

Hence, by (7.2.7) and the fact that $n > 2q(k_1, k_2)$

$$
\begin{aligned}
|\widehat{V}^{(2)}| &\geq |V| - |E| - |V_1^{(2)}| - |V_2^{(2)}| \\
&\geq \frac{n}{2} + \frac{n}{4} - |V_1^{(2)}| - |V_2^{(2)}| \\
&\geq (k_1 + k_2)|E| + \frac{1}{2}q(k_1, k_2) - \frac{1}{2}q(k_1, k_2) \\
&= (k_1 + k_2)|E| \qquad (7.2.11)
\end{aligned}
$$

We are now ready to distribute those vertices of E that have not already been added to $V_1^{(2)} \cup V_2^{(2)}$. The following theorem is similar to Claim 5 of [34].

Theorem 7.8 (NEW: Bang-Jensen and Christiansen)**.** *In time $O(n)$ we can add the vertices of $E^{(2)} = E - (V_1^{(2)} \cup V_2^{(2)})$ together with some additional $\widehat{V}^{(2)}$ vertices to obtain new sets $V_1^{(3)}, V_2^{(3)}$ satisfying that the resulting tournaments $T_1^{(3)}$ and $T_2^{(3)}$ are happy.*

Proof. We will assign vertices of $E^{(2)}$ to $V_1^{(2)}$ or $V_2^{(2)}$ by dividing the vertices into three types, and handling each type in separate cases. Before describing these cases notice the following property of vertices of E. For $v \in E$

(a) If $|\{i \in [k_1] : v \in E_{A_i}\}| \leq |\{i \in [k_1] : v \in E_{B_i}\}|$ then v has at least k_1 in-neighbours in $V_1^{(2)}$. Similarly, if $|\{i \in [k_1+1, k], v \in E_{A_i}\}| \leq |\{i \in [k_1+1, k], v \in E_{B_i}\}|$ then v has at least k_2 in-neighbours in $V_2^{(2)}$. We give the argument for $V_1^{(2)}$: as $k_1 = |\{i \in [k_1] : v \notin E_{A_i}\}| + |\{i \in [k_1] : v \in E_{A_i}\}|$ and this implies that $k_1 \leq |\{i \in [k_1] : v \in E_{B_i}\}| + |\{i \in [k_1] : v \notin E_{A_i}\}|$.

(b) If $|\{i \in [k_1] : v \in E_{A_i}\}| \geq |\{i \in [k_1] : v \in E_{B_i}\}|$ then v has at least k_1 out-neighbours in V_1. If $|\{i \in [k_1+1, k], v \in E_{A_i}\}| \geq |\{i \in [k_1+1, k], v \in E_{B_i}\}|$ then v has at least k_2 out-neighbours in $V_2^{(2)}$.

In Figure 7.3 $|\{i \in [k_1] : v \in E_{A_i}\}| \leq |\{i \in [k_1] : v \in E_{B_i}\}|$ is illustrated for $k_1 = 2$. On the left a specific $v \in E_{A_1} \cap E_{B_1} \cap E_{B_2}$ and the adjacency that give the k_1 in-neighbours. On the right an indication of which intersections of the E sets this inequality is fulfilled.

We will now process the vertices of $E^{(2)}$ in the order corresponding to Case 1 to 3 below, that is, we only proceed to Case $i + 1$ if no more vertices of $E^{(2)}$ satisfy the condition of Case i.

Case 1: There is a vertex $v \in E^{(2)}$ for which there exists $j_1, j_2 \in [2]$ such that j_1 satisfies (a) and j_2 satisfies (b).

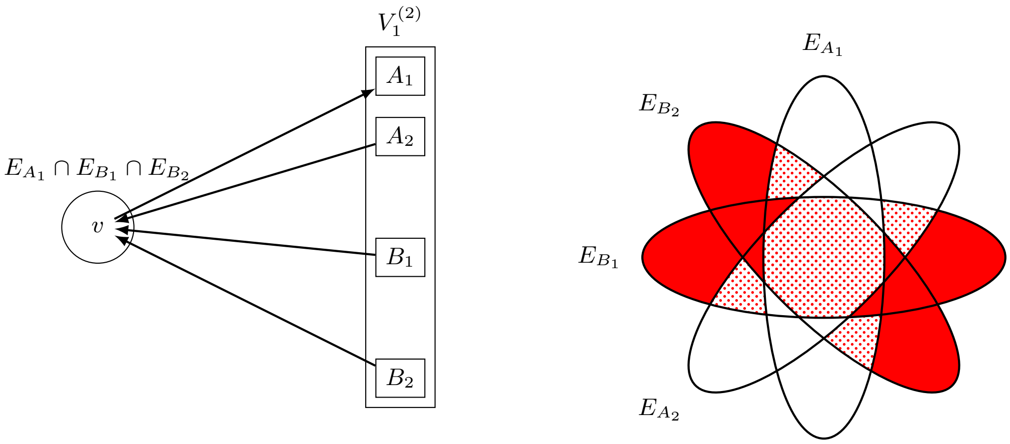


Figure 7.3: Illustration of the adjacency obtained by $|\{i \in [k_1] : v \in E_{A_i}\}| \leq |\{i \in [k_1] : v \in E_{B_i}\}|$. On the left a specific intersection and on the right all sets fulfilling this marked in red, where semi-filled indicate the places where equality holds.

1. If $j_1 = j_2$ then add v to $V_{j_1}^{(2)}$. Otherwise assume below that $j_1 \neq j_2$.
2. If v has k_{j_1} out-neighbours in $\widehat{V}^{(2)}$ then assign v and these k_{j_1} out-neighbours to V_{j_1} (Recall that $\widehat{V}^{(2)}$ vertices are k_1-safe with respect to $V_1^{(2)}$ and k_2-safe with respect to $V_2^{(2)}$). Then v is V_{j_1}-safe by (a).
3. If v has k_{j_2} in-neighbours in $\widehat{V}$ then assign v and these k_{j_2} in-neighbours to V_{j_2}. Then v is V_{j_2}-safe by (b).

Note that one of the three conditions above will be fulfilled as (7.2.11) implies that there are at least $(k_1 + k_2)$ vertices in $\widehat{V}^{(2)}$ to reserve for each vertex in E which still has not been added to one of the sets $V_1^{(2)}, V_2^{(2)}$. Hence if 1. and 2. is not fulfilled then v must be adjacent from at least $(k_{j_1} + k_{j_2}) - (k_{j_1} - 1) > k_{j_2}$ ($k_{j_1} \neq k_{j_2}$ as 1. not fulfilled) of these $k_1 + k_2$ vertices and hence 3. is fulfilled.

Case 2: There is a vertex v where $j = 1$ and $j = 2$ satisfies (a). If v has at least k_1 out-neighbours in $V_1^{(2)}$, then add v to $V_1^{(2)}$. Otherwise if v has at least k_2 out-neighbours in $V_1^{(2)}$, then add v to $V_2^{(2)}$. Suppose none of these hold. We have by (7.2.10)

$$\begin{aligned}
d^+(v) - |E| &\geq (1/2)q(k_1, k_2) \\
&= 2(k_1^2 + k_2^2) + 2(k_1 + k_2) + 2c(k_1 + k_2) \\
&\geq 6(k_1 + k_2)
\end{aligned}$$

so v must have has at least $5(k_1 + k_2)$ out-neighbours in $\widehat{V}^{(2)}$ and we can add v together with the required number of out-neighbours from $\widehat{V}^{(2)}$ to $V_1^{(2)}$.

Note that when we are done with Case 2, we have that $E_B - E_A = \emptyset$. Indeed suppose that $v \in E_B - E_A$. Then $|\{i \in [k] : v \in E_{A_i}\}| = 0$ and $|\{i \in [k] : v \in E_{B_i}\}| \geq 1$ and hence each such vertex would have been assigned to $V_1^{(2)}$ or $V_2^{(2)}$ in either Case 1 or Case 2.

Case 3: There is a vertex v left in $E^{(2)}$ such that (b) holds for $V_1^{(2)}$ and $V_2^{(2)}$. Now, by (7.2.8) and by similar arguments as in Case 2, v has at least

$$d(v)^- - |E_A| \geq 6(k_1 + k_2)$$

in-neighbours not in E_A. Hence we either find sufficiently many in-neighbours in $\widehat{V}^{(2)}$ or conclude that v is already $V_j^{(2)}$-safe for some $j \in [2]$.

Now for the complexity, each of these preforms the following two operations. For each $v \in E$, it is decided which of (a) and (b) v belongs to and v is assigned to a partition together that a constant number of greedily chosen safe neighbours. Both of these can be done in constant time, and the only time-consuming factor is that Case i has to be completed before Case i+1. This is done by checking all vertices of E for Case i, before Case i+1, i.e each case takes $O(|E|) = O(n)$ time. □

Let $V_1^{(3)}, V_2^{(3)}$ be the resulting sets after the process above. If $V_1^{(3)}$ and $V_2^{(3)}$ is not a 2-partition of $V(T)$ yet, the only vertices of V that are still not contained in $V_1^{(3)} \cup V_2^{(3)}$ are $\widehat{V}^{(3)}$ vertices. We may add these to $V_1^{(3)}$ or $V_2^{(3)}$ arbitrarily since they are k_j-safe with respect $V_j^{(3)}$ for $j \in [2]$. Hence we obtain a partition of V into $V_1^{(3)}$ and $V_2^{(3)}$ and $T_1^{(3)}$ is k_1-happy and $T_2^{(3)}$ is k_2-happy.

This completes the proof of Theorem 7.9 where the most time-consuming job is the construction of $V_1^{(1)}$ and $V_2^{(1)}$ which takes $O(kn^2)$.

7.3 Concluding remarks

One thing to notice is that we cannot prove the existence of a minimum out-degree function using similar techniques. This follows from the way we handled the E vertices in Subsection 7.2.3. This depends strongly on the fact that the number of A_i and B_i sets that are in each partition is the same. If we would use similar techniques to find such an out-degree function, we would only construct k_1 B_i sets for the first partition and k_2 B_i sets for the second partition, as these will be sufficient to make most vertices of V happy with respect to the out-degree constraint. But then when trying to distribute the E vertices (indeed E_B vertices) we do not have the adjacency to A_i sets that helped us in our proof above.

In this chapter we have considered the $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition problem for tournaments, and it would be natural to extend this to semicomplete digraphs. We can do this by increasing the minimum semi-degree in Theorem 7.9 by a factor 2. Then the result follows as a semicomplete digraph with minimum semi-degree $2k$ contains a spanning tournament with minimum semi-degree at least k. It is though not hard to prove Theorem 7.9 for semicomplete digraphs with the same bound on the minimum semi-degree. This follows as Lemma 7.4, Lemma 7.5 and Corollary 7.6, can be extended to semicomplete digraph. In [11], Bang-Jensen and Havet did this in order to extend Pokrovskiys linear bound on linkage in tournaments (Theorem 2.12) to semicomplete digraphs. We will give the intuition (which is also most the proof) for Lemma 7.4: In the original proof of this Lemma the main argument is that every tournament of order n contains a vertex with in-degree at most $n/2$. This is generally not true for semicomplete digraphs, but it is true that semicomplete digraphs contains a vertex that is completely dominated by at most $n/2$ vertices of D. This is sufficient to prove the corresponding results for semicomplete digraphs.

Lemma 7.4. Let D be a semicomplete digraph on n vertices, let $v \in V(D)$ and suppose that $c \in \mathbb{N}$. Then in time $O(n^2)$, we can construct disjoint sets $A, E_A \subseteq V(D)$ such that the following properties hold:

1. $1 \leq |A| \leq c$ and $D\langle A\rangle$ is a transitive tournament with sink v
2. A out-dominates $V(D) - (A \cup E_A)$
3. $|E_A| \leq (1/2)^{c-1} d^-(v)$
4. either $E_A = \emptyset$ or E_A is the common in-neighbourhood of all vertices in A

Now we can prove the following theorem by similar arguments as in proof of Theorem 7.9.

Theorem 7.9 (NEW: Bang-Jensen and Christiansen). *Let k_1, k_2 be fixed integers. Then every semicomplete digraph D with minimum semi-degree $\delta^0(T) \geq q(k_1, k_2)$ contains a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition and such a partition can be found in $O(kn^2)$ time.*

Chapter 8

Discussion on degree constrained partitions

In this chapter we will mention and discuss some related partition problems. Notice however that many problems on graphs can be described as partitioning problems, and covering all of these would be to comprehensive compared to the aim of this dissertation. To mention a few different problems, the three below were all considered by Bang-Jensen, Cohen and Havet

- Deciding if a digraph has a connected feedback vertex set, i.e finding a connected set such that removal the of this set leaves, the remaining digraph acyclic. This is the $(acyclic, connected)$-partition problem and it is polynomial [12].
- Deciding if a digraph is bipartite. This is the $(independent, independent)$-partition problem and it is polynomial [8].
- Deciding if a digraph has disjoint out- and in-branchings. This is the $(out-branching, in-branching)$-partition problem at it is NP complete [12].

8.1 Other related partitioning problems

The most obvious partition problem to consider is probably the $(strong, strong)$-partition. Bang-Jensen, Cohen and Havets proved that it is NP complete for general digraphs [8]. For semicomplete digraphs finding complementary cycles is sufficient since strong semicomplete digraphs are hamiltonian. As also noted in Section 6.4, 2-strong connectivity of semicomplete digraphs implies the existence of complementary cycles (Reid, Guo and Volkmann [41, 26]) and deciding whether semicomplete digraphs is 2-strong is polynomial (Bang-Jensen and Nielsen [16]). Hence just as the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition problem in Section 6.4, we may assume that the semicomplete digraph D is not 2-strong. Now Li and Shu found a sufficient condition for the existence of complementary cycles.

Theorem 8.1. *[36] Let T be a tournament of at least 6 vertices. If T is not the Paley tournament $\mathbb{P}_7$*[1] *and $\max\{\delta^-(T), \delta^+(T)\} \geq 3$, then T has a $(strong, strong)$-partition.*

Not surprisingly $(strong, strong)$-partition demands more on the semicomplete digraphs than the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition does. This is also seen by comparing with the equivalent result from section 6.4.

Corollary 6.5 ('NEW': Bang-Jensen and Christiansen)**.** Every semicomplete digraph D with $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\} \geq 3$ contains a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition.

Kühn, Osthus and Townsend proved the strong connectivity of tournaments, that implies the existence of a $(k - strong, k - strong)$-partition. Multiplying this bound on the strong connectivity by 3 (every $3k - 2$ strong semicomplete digraph contains a k-strong spanning tournament, See Part III) we can obtain similar results for semi-complete digraphs.

Theorem 8.2. *[34] There exists a constant c such that every ck^7-strong semicomplete digraph as a $(k - strong, k - strong)$-partition.*

Combining conditions of strong connectivity and degree constrained conditions, we might ask the following. Notice that Theorem 8.1 implies that $g(1) = 3$.

Problem 8.3. *[6] Does there exist a function $g(k)$ such that every strong semicomplete digraph S with $\delta^+(S) \geq g(k)$ has a $([strong, \delta^+ \geq k], [strong, \delta^+ \geq k])$-partition.*

Another variation of the degree constrained partition could be to also demand properties between partitions. For example, in Section 6.6, we demanded that all 2-cycles of the semicomplete digraph where between the two set. Alon, Bang-Jensen and Bessy recently showed the following.

Theorem 8.4. *[2] Let D be a semicomplete digraph which is not $\mathbb{P}_7$ and assume that $\delta^+(D) \geq 3$. Then D has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition (V_1, V_2) such that the bipartite digraph (V_1, V_2) also has minimum out-degree at least 1* [2]

Theorem 8.5. *[2] There exist two absolute positive constants c_1, c_2 such that the following holds.*

[1] The Paley tournament $\mathbb{P}_7$, is a tournament of 7 vertices $v_1, v_2, \ldots, v_7$ where there is an arc v_iv_j if and only if $((j - i)\ mod\ 7) \in \{1, 2, 4\}$

[2] They also shows that V_1, V_2 can be chosen such that $|V_1| - |V_2| \leq 1$

- *Let S be a semicomplete digraph with minimum out-degree at least $2k + c_1\sqrt{k}$. Then D contains a $(\delta^+ \geq k, \delta^+ \geq k)$-partition (V_1, V_2) such that the bipartite digraph (V_1, V_2) also has minimum out-degree at least k* [3]
- *For infinitely many values of k there is a tournament T with minimum out-degree at least $2k + c_2\sqrt{k}$ such that for any 2-partition (V_1, V_2) of V, either $\delta^+(T\langle V1\rangle) < k$ or $\delta^+(T\langle V2\rangle) < k$ or $\delta^+(T\langle[V_1, V_2]\rangle) < k$.*

8.2 Finding the complexity of the remaining problems

Lichiardopol showed that there exist functions of minimum out- in- and semi-degree such that semicomplete digraphs are insured to contain a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-, $(\delta^+ \geq k_1, \delta^- \geq k_2)$-, $(\delta^0 \geq k_1, \delta^- \geq k_2)$- and $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition. In our paper [6] we proved that for a number of these problems, there is also a polynomial algorithm that not only decides if such a partition exists, but also finds it if it does. We were not able to find the complexity of all the related partitioning problems, but believe that there exist polynomial algorithms for all of these.

Conjecture 8.6. *[6] For every fixed integer $k_1, k_2 \geq 2$, the following partitioning problems are polynomial on semicomplete digraphs.*

- *$(\delta^+ \geq k_1, \delta^- \geq k_2)$-partition problem*
- *$(\delta^0 \geq k_1, \delta^- \geq k_2)$-partition problem*
- *$(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition problem*

Consider the two polynomial algorithms found in Chapter 6. While the algorithm for the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition uses the nice symmetry in the problem, roughly saying that if the vertex does not have the property (out-degree k_i) in one part of the partition, it will have the property (out-degree k_{3-i}) in the other part. The algorithm for the $(\delta^+ \geq 1, \delta^- \geq k)$-partition uses the nice structure and control of disjoint cycles. None of these approaches can be generalized to prove the three problems in the conjecture, and different approaches must be found.

8.3 Similar partitioning problems on other graph classes

Apart from finding the complexity for the remaining problems for tournaments and semicomplete digraphs, it is natural to continue by considering generalizations of tournaments. As locally semicomplete digraphs often inherit the same structural results

[3] Again V_1, V_2 can be chosen balanced

as semicomplete digraphs, this is an obvious class to consider next. Remember that Theorem 1.9 and Corollary 1.11 says that locally semicomplete digraphs are either semicomplete, round decomposable or digraphs with independence number at most 2. In Section 6.3 we proved that $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition is polynomial for digraphs with bounded independence number. So in order prove that the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem is polynomial, we only need to prove that it is polynomial on round decomposable digraphs.

Conjecture 8.7. *[NEW: Christiansen] The $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem is polynomial on round decomposable digraphs.*

Conjecture 8.8. *[NEW: Christiansen] Let k_1, k_2 be fixed integers. There exists a function $g(k_1, k_2)$ such that every locally semicomplete digraph D with $\delta^+ \geq g(k_1, k_2)$ contains a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.*

Let us end this section by considering the $(\delta^+ \geq 1, \delta^- \geq 1)$-partition problem. Clearly disjoint cycles are necessary for an $(\delta^+ \geq 1, \delta^- \geq 1)$-partition for any digraph and in Section 6.4 we saw that disjoint cycles was sufficient when we consider semicomplete digraphs. This is not the case for round decomposable digraphs. To see this consider the following round decomposable digraph $D = R[S_1, S_2, S_3, S_4, S_5]$, where S_1, S_3 and S_5 are trivial components and S_2, S_4 are 3-cycles. It is not hard to see that D does not have a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition.

8.4 FPT

Another and very relevant issue to consider is the fixation of the constants k, k_1, k_2. What would happen to the complexity if these become part of the input.

Problem 8.9. *[6] Is the problem of deciding whether a semicomplete digraph has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition in FPT?*

Problem 8.10. *[6] Is the problem of deciding whether a semicomplete digraph has a $(\delta^+ \geq 1, \delta^- \geq k)$-partition in FPT?*

Part

III

Semicomplete digraphs containing k-strong spanning tournaments

Chapter 9

Introduction to the k-strong spanning tournament problem

In 1989, after private talks with Jackson, Thomassen conjectured the following:

Conjecture 9.1. *[49] For k integer, every $2k$-strong digraph contains a k-strong spanning oriented digraph.*

Considering the k'th power of a cycle of at least $2k+1$ vertices and replacing every arc with a 2-cycle, it is not hard to realize that this is best possible. Now even for $k = 2$, this has not been proven, and it seems very hard to do so. Indeed restricting the conjecture to semicomplete digraphs or generalizations of semicomplete digraphs, the problem remains open for $k \geq 3$. In 1990 [3], Bang-Jensen found an upper bound on the strong connectivity of a locally semicomplete digraph containing a spanning local tournament and posed the corresponding extension of Conjecture 9.1.

Theorem 9.2. *[3] For every integer k every $f(k)$-strong locally semicomplete digraph contains a spanning k-strong local tournament. Furthermore $f(k) \leq 5k$.* [1]

Conjecture 9.3. *[3] For every integer k, if D is a $2k$-strong locally semicomplete digraph, then D contains a spanning k-strong local tournament.*

Later Bang-Jensen and Jordán [14] posed a slightly modified conjecture and proved this for $k = 3$. They also gave a family of $2k-2$ strong digraphs that does not contain a k-strong spanning tournament and hereby proving that this is best possible.

Conjecture 9.4. *[14] For every integer k, every $(2k-1)$-strong semicomplete digraph of at least $2k+1$ vertices contains a k-strong spanning tournament.*

[1] A remark states that the result for semicomplete digraphs was found together with Thomassen

Theorem 9.5. *[14] Every* 3*-strong semicomplete digraph of at least* 5 *vertices contains a* 2*-strong spanning tournament.*

Now both Huang [29] and Guo [24] considered the problem for locally semicomplete digraphs and while Huang[2] proved that k-strong round decomposable locally semicomplete digraphs contains a spanning k-strong local tournament, Guo [24] found a better bound on the function $f(k)$ in Theorem 9.2.

Theorem 9.6. *[29] Every round decomposable* k*-strong locally semicomplete digraph can be oriented as a* k*-strong local tournament.*

Theorem 9.7. *[24] For every integer* k*, if* D *is a* $(3k-2)$*-strong semicomplete digraph, then* D *contains a* k*-strong spanning tournament.*

For $k = 2$ this implies Conjecture 9.3 and it is the best possible. Modifying a wrong proof of Guo [24], Bang-Jensen [5] proved the following theorem, leaving 'only' the semicomplete version of Conjecture 9.3 (Conjecture 9.4) unanswered.

Theorem 9.8 ([5])**.** *Let* $f(k)$ *be an integer function such that* $f(1) = 1$ *and* $f(k) \geq f(k-1) + 2$ *for every* $k \geq 1$*. Suppose that every* $f(k)$*-strong semicomplete digraph contains a spanning* k*-strong tournament. Then every* $f(k)$*-strong locally semicomplete digraph contains a* k*-strong spanning local tournament.*

Combined with Theorem 9.7 this gives the best known result for locally semicomplete digraph:

Theorem 9.9 ([3])**.** *Every* $(3k-2)$*-strong locally semicomplete digraph contains a spanning* k*-strong spanning local tournament.*

In the next chapter we will in an attempt to prove Conjecture 9.4, obtain three subresults. All of these improves the current best known result by Guo (Theorem 9.7), which says that 7-strong semicomplete digraphs contains a 3-strong spanning tournament. We will also discuss how one might prove Conjecture 9.4 on the basis of these three results and their proofs.

In Chapter 11 we will conjecture that the correct bound on strong connectivity for which a semicomplete digraph contains a k-strong spanning tournament is $2k+1$ and give three conjectures that follows naturally from the results obtained in Chapter 10. In this chapter we will also mention a few related problems and results.

[2]with corrections from Bang-Jensen [5]

Chapter 10

Semicomplete digraphs containing 3-strong spanning tournaments

In this chapter we will show that a semicomplete digraph D contains a spanning 3-strong tournament if D is 5-strong and satisfying one of the following conditions:

- D contains an induced 3-strong subtournament T'.
- D is 6-strong.
- D has minimum semi-degree at least 7.

These are all improvements of the previous best known bound by Guo, saying that every 7-strong semicomplete digraph contains a 3-strong spanning tournament. With the knowledge obtained throughout these proofs we will in the end of the chapter discuss if and how one might prove Conjecture 9.4, i.e prove that every 5-strong semicomplete digraph contains a 3-strong spanning tournament.

10.1 A 3-strong subtournament

In this section we will prove that the existence of an induced 3-strong subtournament T in a 5-strong semicomplete digraph D is sufficient to guarantee a spanning 3-strong tournament. This will be done by cleverly orientating 2-cycles in D step wise and moving vertices to a set G with $T \subseteq G$ in such a way that two properties are preserved: Every vertex in G can reach every other vertex in G by 3 vertex disjoint paths each of which avoiding 2-cycles not yet oriented. Secondly vertices in $V - T$ must remain connected to all vertices other vertices in $V - G$ by 5 vertex disjoint paths and to vertices in G by at least 3 vertex disjoint paths.

Lemma 10.1 (NEW: Christiansen)**.** *Let D be a digraph, and A be a subset of $V(D)$ that induces a k-strong digraph. If there for all $v \in V(D) - A$ exist k vertex disjoint (v, A)-paths and k vertex disjoint (A, v)-paths, then D is k-strong.*

Proof. This follows by Menger's theorem. For any $u, v \in V(D)$ we will prove that we cannot separate u, v with a set of fewer than k vertices. First if $u, v \in A$ then it follows directly as A is k-strong and hence do not have a separator of size $k-1$. Now assume that $u \in A$ and $v \notin A$ and let S be an (u, v)-separator in D of size $k-1$. In $D - S$ there are at least one (u, A)-path P_1 ending in a vertex $u_a \in A$ and as A is k-strong there is an (u_a, v)-path P_2 in $A - S$. Hence P_1P_2 is an (u, v)-path in D. Finally let $u, v \notin A$ and assume again that S is an (u, v)-separator in D of size at most $k-1$. Then by assumption we have an (u, u_a)-path P_1 and a (v_a, v)-path P_2 in $D - S$ for some $u_a, v_a \in A - S$ and together with (u_a, v_a)-path P inside $A - S$ we obtain an (u, v)-path P_1PP_2. □

Brualdi and Kiernan [18] use Rado's theorem to prove the following generalization of Landau's theorem.

Theorem 10.2. *[18] Let D be an oriented digraph with out-degree sequence s_1, $s_2, \dots,$ s_n. Furthermore, let r_1, $r_2, \dots r_n$ be a sequence of non-negative integers with $s_i \leq r_i$ for $i \in [n]$. Then D can be completed to a tournament with score sequence $r_1, r_2, \dots, r_n$ if and only if*

$$\binom{|X|}{2} + |X|(n - |X|) - \gamma(X) \geq \sum_{i \in X}(r_i - s_i) \quad (X \subseteq \{1, 2, \dots, n\}) \tag{10.1.1}$$

where $\gamma(X)$ is the number of arcs that has at least one end in X.

The following corollary is proved by similar techniques as Corollary 3.2 of [18]. The only difference is the extra information on maximal in- and out-degree which give us a tight bound.

Corollary 10.3 (NEW: Christiansen)**.** *Let k be a fixed integer. Furthermore, let D be an oriented digraph of $n = 2k+1$ vertices, with $\Delta^+(D) \leq k$ and $\Delta^-(D) \leq k-1$. Then D can be completed to a k-regular tournament if and only if every vertex of D with out-degree k are adjacent.*

Proof. Let D be an oriented digraph as described above and let $s_1, s_2, \dots, s_n$ be a degree-sequence of the vertices of D. Clearly if D has two non-adjacent vertices of out-degree k, then completing D to a tournament, at least one vertex will have out-degree at least $k+1$ and cannot be k-regular.

Now assume that D do not have non-adjacent vertices with out-degree k. We will prove that equation (10.1.1) of Theorem 10.2 holds when $r_1, r_2, \ldots, r_n = k$. Clearly $s_i \leq k = r_i$ for all $i \in [n]$. By simple algebraic manipulations it can be shown that this equation is equivalent to the following:

$$k|X| - \binom{|X|}{2} \geq \gamma^*(X) \quad (X \subseteq \{1, 2, \ldots, n\})$$

where $\gamma^*(X)$ is the number of arcs into the set X in D. To prove that this equations holds, we start by bounding the size of $\gamma^*(X)$. First each vertex in $V(D) - X$ has at most k out-neighbours and hence contributes with at most k arcs into X. But each non-adjacent pair of vertices in $V(D) - X$ with out-degree k can only contribute with $2k - 1$ arcs into X, and we need to subtract one for each such pair. Hence

$$\gamma^*(X) \leq k(n - |X|) - \binom{n - |X|}{2}$$

Now the result follows by further simple algebraic manipulations.

$$\begin{aligned}
\gamma^*(X) &\leq k(n - |X|) - \binom{n - |X|}{2} \\
&= \frac{-n^2 + n(2k+1) + 2|X|n - 2k|X| - |X|^2 - |X|}{2} \\
&= \frac{2|X|(2k+1) - 2k|X| - |X|^2 - |X|}{2} \\
&= \frac{2k|X| - |X|^2 + |X|}{2} \\
&= k|X| - \binom{|X|}{2}
\end{aligned}$$

□

Theorem 10.4 (NEW: Christiansen)**.** *Let D be a 5-strong semicomplete digraph and T a 3-strong induced subtournament of D, then D contains a 3-strong spanning tournament.*

Proof. Let T be a 3-strong subtournament of D and define $\widehat{D} = D - \overleftrightarrow{A}$, where $\overleftrightarrow{A}$ is the set of arcs in 2-cycles of D. Also let G be the set of vertices of $V(D)$ such that for each vertex $v \in G$, there exist 3 vertex disjoint (v, T)-paths and 3 vertex disjoint (T, v)- paths in $\widehat{D}$. If $V(D) = G$, then by Lemma 10.1, $\widehat{D}$ induces a 3-strong digraph, and $\widehat{D}$ can be completed to a 3-strong spanning tournament by adding an arbitrary arc between non-adjacent vertices in $\widehat{D}$. We will prove that while $V(D) \neq G$ there exist a set $A' \subseteq \overleftrightarrow{A}$, where A' does not contain both arcs of any 2-cycle, such that updating G for the semicomplete digraph $D - A'$, the following remains true.

(I) For every $v \in V(D) - G$ there are at least 5 vertex disjoint (G, v)-paths in $D - A'$ and 5 vertex disjoint (v, G)-paths in $D - A'$.

Case 1. *There is a vertex $v \in V(D) - G$ that dominates at most $|G| - 3$ in vertices of G and is dominated by at most $|G| - 3$ vertices of G in $D - A'$.*

Proof of case. It is not hard to see that we can delete arcs in 2-cycles between v and G such that v completely dominates at least 3 vertices of G and is completely dominated by 3 vertices of G. After deleting these arcs (adding them to A'), v will become a vertex of G and (I) remains true. $\diamondsuit$

Now all vertices of $V(D) - G$ belong to one of the following two sets.

- Vertices that have at least $|G| - 2$ out-neighbours (at most 2 in-neighbours) in G in $D - A'$. We call the set of these vertices for G_{in}.
- Vertices that have at least $|G| - 2$ in-neighbours (at most 2 out-neighbours) in G in $D - A'$. We call the set of these vertices for G_{out}.

Case 2. *There is a 2-cycle uv in $D - A'$ with $u \in G$ and $v \in G_{out}$ or there is a 2-cycle uv in $D - A'$ with $u \in G_{in}$ and $v \in G$.*

Proof of case. Assume uv is a 2-cycle with $v \in G_{out}$ and $u \in G_{in}$. As v has at most 2 out-neighbours in G, u must be one of these. Then v is dominated by at least $|G| - 2 + 1 \geq 6$ vertices of G. Now by deleting uv from D, we delete a path from G to v, but v will still have 5 disjoint in-neighbours in G and hence 5 vertex disjoint (G, v)-paths. As all other vertices of $V(D) - G$ can only use v once in the 5 vertex disjoint paths, (I) remains true. Hence for every such 2-cycle we add uv to A'.

The argument for 2-cycles between G and G_{in} is analogous.

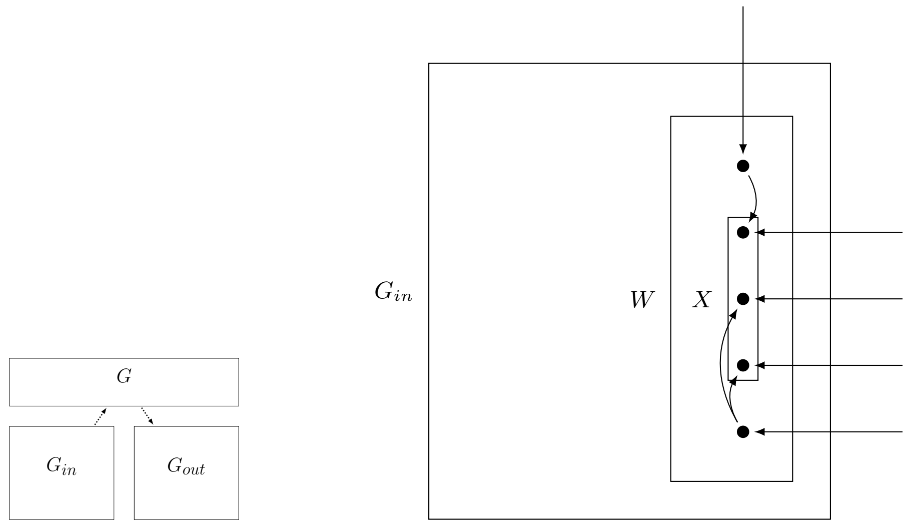
$\diamondsuit$

Case 3. *There is a 2-cycle uv in $D - A'$ with $v \in G_{out}$ and $u \in G_{in}$.*

Proof of case. We will delete uv (add uv to A'). This will possibly delete a path from G to v and a path from u to G. But $v \in G_{out}$ and is dominated by at least $|G| - 2 \geq 5$ vertices of G and $u \in G_{in}$ and dominates at least 5 vertices of G. So after deletion (I) remains true. $\diamondsuit$

Repeating the three cases, we may now assume that all remaining 2-cycles of D belong to either G_{in} or G_{out}.

Case 4. *$G_{in} \neq \emptyset$ ($G_{out} \neq \emptyset$).*



(a) Structure of D. (b) Observation in Case 4. W contains a set of at least 3 vertices with in-degree 2 in $\widehat{D}$.

Figure 10.1

Proof of case. Now by (I) every vertex of G_{in} can be reached by 5 vertex disjoint paths from G in $D - A'$. Menger's theorem implies that we can obtain $\min\{5, |G_{in}|\}$ vertex disjoint (G, G_{in})-paths using these paths. Let $P_1, P_2, \ldots, P_5$ be such 5 minimal (G, G_{in})-paths, where duplication of $5 - |G_{in}|$ heads is allowed. By minimality, it is not hard to see that $|V(P_i)| \leq 3$: First P_i will only contain one vertex of G and one vertex of G_{in} and if $P_i = v_1 v_2 \ldots v_r$ for some $i \in [5]$ and $r > 3$, then $v_2, \ldots, v_{r-1} \in G_{out}$. But v_{r-1} is dominated by at least 5 vertices $v_1^1, v_1^2, \ldots, v_1^5$ of G, and hence for each $j \in [5]$, $P_i^j = v_1^j v_{r-1} v_r$ is also a (G, G_{in})-path. Now pick $j \in [5]$ such that v_1^j is disjoint from $t(P_l)$ for all $l \in [5]$ with $l \neq i$ and obtain a contradiction by replacing P_i by P_i^j. Hence $|V(P_i)| \geq 3$. Let

$$W = \{h(P_1), h(P_2), h(P_3), h(P_4), h(P_5)\} = \{w_1, w_2, \ldots, w_5\} \subseteq G_{in}$$

and let

$$\widehat{D}' = \widehat{D}\langle W \cup G \cup G_{out}\rangle.$$

Notice that each vertex $v \in G_{in}$ is dominated by at most 2 vertices in $G \cup G_{out}$. This follows as each vertex in G_{out} is dominated by 5 disjoint vertices in G and hence three in-neighbours in $G \cup G_{out}$ would induce 3 vertex disjoint (G, v)-paths in $D - A'$. This directly implies that $|W| \geq 4$. Furthermore if $|W| = 4$, then we see that W contains only 2-cycles as all vertices in G_{in} must have in-degree at least 5 in $D - A'$ but they have only 2 in-neighbours outside G_{in}. Completing W to a 1-regular tournament (adding the opposite to A'), we obtain an orientation where all vertices of W can be reached be 3 vertex disjoint paths in $\widehat{D}$. Hence W is moved to G.

Finally consider the case where $|W| = 5$. If w_i has 2 in-neighbours, say $w_j, w_{j'}$ in W then $P_i, P_j wi, P_{j'} w_i$ are three disjoint (G, w_i)-paths in $\widehat{D}'$. Hence every vertex in W is dominated by at most one other vertex in W. Now either W can be completed to a 2-regular tournament, or there is a vertex in W with out-degree 3. But then these three vertices can be completed to a 1-regular tournament, and we obtain the three paths for each of these. Hence adding the corresponding opposite arcs to A', we move this 1-regular tournament of W to G.

Similarly arguments can be made if $G_{out} \neq \emptyset$. ◊

By repeating Case 1-4 we see that we can eliminate all 2-cycles of D while maintaining 3 vertex disjoint paths between any pair of vertices inside G, and 5 vertex disjoint path between vertices of $V(D) - G$ and G. Then, as noted in the start, when $G = V(D)$, we have obtained a 3-strong spanning digraph $\widehat{D}\langle G\rangle = D\langle G\rangle$ and by completing this arbitrarily to a tournament we obtain a 3-strong spanning tournament. This completes the proof. □

We can adapt the proof above to obtain a polynomial algorithm that finds a 3-strong spanning tournament.

Corollary 10.5 (NEW: Christiansen)**.** *If D is a 5-strong semicomplete digraph and T is a given induces 3-strong subtournament of D, then we can find a 3-strong spanning tournament in $O(nm^2)$ time, where $m = |A(D)|$.*

Proof. We will describe a high level pseudo-algorithm. First the subroutine UPDATE G(D,G) runs through all vertices of $v \in V(D) - G$ and checks if there are 5 vertex disjoint (v, G)- and 5 vertex disjoint (G, v)-paths in D avoiding every 2-cycle of D. If so it moves the vertex v to G. Now the algorithm that finds the 3-strong spanning tournament is the following:

1. Input: A semicomplete digraph D and a 3-strong subtournament T.
2. Let $G = V(T)$ and run UPDATE G. Then find the sets G_{in}, G_{out} and G_{mid}, where G_{mid} are the vertices not in $G \cup G_{in} \cup G_{out}$,
3. While there is a 2-cycle with ends in two different sets G, G_{in}, G_{out} and G_{mid}[1], then delete an arc in D according to case 1-3.
4. UPDATE G and if there are 2-cycles between sets G, G_{in} and G_{out} then go to step 3.

[1] G_{mid} will only exist in the first iteration

5. If $|G_{in}| \neq \emptyset$ find the set W described in Case 4. and delete arcs of 2-cycles accordant to case 4. Move all vertices incident to deleted 2-cycles to G and UPDATE G. If there are 2-cycles between sets G, G_{in} and G_{out} go to 3, otherwise repeat this step
6. If $G = V(D)$ then delete arbitrarily arcs in the remaining 2-cycles and output D.

To see that the algorithm terminates, we only need to notice that in Step 3 and 5 at least one arc is deleted in each run and Step 4 is always followed by 3 or 5. Hence after at most $|A(D)| = m$ steps all 2-cycles are deleted and all vertices of V has been moved to G. To see the runtime, consider first step 5. We can find W and direct the respective arcs in constant time (as $|W| \leq 5$). Now every time we run UPDATE G, at least one 2-cycle has been deleted and UPDATE G runs in $O(nm)$, using a flow-algorithm n times. This gives a runtime on $O(nm^2)$. □

10.2 A 2-strong spanning tournament

We will now show that if we either ensure that the 5-strong semicomplete digraph has minimum semi-degree at least 7, or increase the demand to 6-strong connectivity, then we can also find a spanning 3-strong tournament. To do this we find a 2-strong spanning tournament, chosen optimal (will be clear later) and assuming by contradiction that the semicomplete digraph do not have a 3-strong spanning tournament.

We will start with a few lemmas.

Lemma 10.6. *[14] Let D be a k-strong digraph and xy be an arc of D. If D contains at least $k+1$-internally disjoint (x, y)-paths of length at least 2, then $D' = D - xy + yx$ is k-strong. Furthermore, if D' is not $k+1$ strong, then every minimum separator S' of D' is also a minimum separator of D.*

The following is a well-known fact, but is added to simplify formulation in the proof of Theorem 10.8.

Lemma 10.7. *Let T be a strong tournament of at least 4 vertices, then T has at most 2 vertices with out-degree (in-degree) 1.*

Proof. Let T be a strong tournament and consider 3 vertices of T. If they all have out-degree 1, then they must form a 3-cycle in T.

But there are at least 4 vertices in T and T is strong. Hence there must be an arc from $\{v_1, v_2, v_3\}$ to $V(T) - \{v_1, v_2, v_3\}$ and hence for some $i \in [3]$, $d^+(v_i) \geq 2$. □

We are now ready to prove the main result of this chapter. The approach of taking a 2-strong spanning tournament with certain properties are similar to that used in the proof of Theorem 9.5 and Theorem 9.7.

Theorem 10.8 (NEW: Christiansen)**.** *Every* 6*-strong semicomplete digraph D of at least* 7 *vertices contains a* 3*-strong spanning tournament.*

Proof. First if $n = 7$ then D is the biorientation of the complete graph and clearly contains a 3-strong spanning tournament. So we may assume that $n > 7$. Assume by contradiction that D is a 6-strong semicomplete digraph that does not contain a 3-strong spanning tournament. By Theorem 9.5 we know that D contains a 2-strong spanning tournament. Choose such a tournament T such that

(T.1) T has the fewest number of minimum separators.

(T.2) Among the tournaments with the fewest number of minimum separator choose T and a separator $S = \{s_1, s_2\}$ of T that leaves the fewest number of strong components in T.

We may assume without loss of generality that s_1s_2 is an arc of T. Let $T_1, T_2, \ldots,$ T_r be the strong components of $T - S$ for some integer $r \geq 2$ and denote a trivial strong component T_i by t_i. Also we denote the semicomplete digraph induced by the vertices of T_i by D_i, i.e $D_i = D\langle V(T_i)\rangle$. Observe first that we by Theorem 10.4 may assume that all strong components $T_1, T_2, \ldots, T_r$ is a most 2-strong. Now we will call arcs of T that are 2-cycles in D, for ***important arcs***. Through a number of cases we find one or more important arcs that can be reversed such that reversing this/these will give a better choice of T, and hereby contradicting the current one. The better choice are i many cases obtained by reducing the number of strong components in the tournament $T - S$, while ensuring that the number of separators in T do not increase.

Now by Lemma 10.6 we see that an important arc uv can be reversed if T contains three internally disjoint (u, v)-paths of length at least 2. Such triplets of (u, v)-path are often found in one of the three following forms (for $uv \in T - S$).

- ***Type 0*** (u, v)***-paths***, three (u, v)-paths of length at least 2 in $T - S$. Typically these are found as $|N^+(u) \cap N^-(v)| \geq 3$.
- ***Type 1*** (u, v)***-paths***, two (u, v)-paths in $T - S$ and one using a vertex of S.
- ***Type 2*** (u, v)***-paths***, one (u, v)-path in $T - S$ and two using the vertices of S.

Though D is 6-strong we will in the following, until otherwise stated, only use that D is 5-strong.

Case 1. $r = 2$ *and* T_1, T_2 *are both non-trivial strong components.*

Proof of case. As D is 5-strong there are at least 3 disjoint important arcs between T_1 and T_2, say $u_i v_i$ for $i \in [3]$. If $n > 8$ then there either are at least 4 vertices in T_1 or at least 4 vertices in T_2, say in T_1. Hence by Lemma 10.7 there exist u_i with out-degree at least 2 in T_1. Then $|N^+(u_i) \cap N^-(v_i)| \geq 3$ as u_i also dominates the in-neighbour of v_i in T_2. Hence we find Type 0 (u_i, v_i)-paths in T, and we can reverse $u_i v_i$ contradicting the choice of T.

Assume now that $n = 8$. Then T_1 and T_2 are 3-cycles and $|N^+(u_i) \cap N^-(v_i)| = 2$ for all $i \in [3]$. Hence we need prove that $T - (N^+(u_i) \cap N^-(v_i))$ contains an (u_i, v_i)-path for some $i \in [3]$. As T is 2-strong there exists a $j \in [3]$ such that $v_j s_1$ is an arc of T. Also all u_i are dominated by at least one vertex of S. But then

$$u_i v_j s_1 (s_2) u_{j'} v_i$$

is an (u_i, v_i)-path of T disjoint from $N^+(u_i) \cap N^-(v_i)$, where v_j is the out-neighbour of v_i in T_2 and $u_{j'}$ the in-neighbour of u_i in T_1. $\diamondsuit$

Case 2. $r \geq 5$ *and there exists an* $i \in [3, r-2]$ *such that* T_i *is trivial a trivial strong component.*

Proof of case. As D is 5 strong $D - S - t_i$ is at least 2-strong and hence there must be two important arcs, $u_1 v_1$ and $u_2 v_2$ with $u_1, u_2 \in T_1 \cup \cdots \cup T_{i-1}$ and $v_1, v_2 \in T_{i+1} \cup \cdots \cup T_r$. Furthermore as $|T_1 \cup \cdots \cup T_{i-1}|, |T_{i+1} \cup \cdots \cup T_r| > 1$ we can assume $u_1 \neq u_2$ and $v_1 \neq v_2$, for otherwise we find a separator of D of size less than 5. Assume without loss of generality that u_1 dominates u_2 in T. If v_2 dominates v_1 then $|N^+(u_1) \cap N^-(v_1)| = 3$, and we find Type 0 (u_1, v_1)-paths. Hence we may assume that v_1 dominates v_2. Notice that this implies that $|N^+(u_j) \cap N^-(v_j)| \geq 2$ for both $j \in [2]$ as t_i is a trivial strong component between the strong components containing u_1, u_2 respectively v_1, v_2. Now as $n \geq 8$ there is a vertex $w \in T - S$. Let T_{u_1} (T_{v_2}) denote the strong component containing u_1 (v_2). If $w \in T_{u_1} \cup \cdots \cup T_{v_2}$, one can easily realize that $|N^+(u_i) \cap N^-(v_i)| \geq 3$ for some $i \in [2]$. Hence we may assume that w belongs to a strong component before T_{u_1} or after T_{v_2}, i.e $T_{u_1} \neq T_1$ or $T_{v_2} \neq T_r$. Assume without loss of generality that $w \in T_1$. Then we obtain Type 1 (u_1, v_1)-paths by the following three paths

$$u_1 u_2 v_1$$

$$u_1 T_i v_1$$

$$u_1 T_r S w v_1$$

Where T_r might be the vertex v_2. $\diamondsuit$

Case 3. *$r \geq 4$ and there exist $i, j \in [2, r-1]$ with $i < j$ such that T_i and T_j are non-trivial strong components.*

Proof of case. Assume first that $i = j - 1$. Then as D is 5-strong there are at least 3 important arcs $u_j v_j$ over the cut $(T_i \cup \cdots \cup T_i, T_{i+1} \cup \cdots \cup T_r)$. If $u_j \notin V(T_i)$ for some $j \in [3]$, then $|N^+(u_j) \cap N^-(v_j)| \geq |T_i| \geq 3$, and we have Type 0 (u_j, v_j)-paths. So we may assume that $u_j \in V(T_i)$ for all $j \in [3]$. Similarly $v_j \in T_{i+1}$ for all $j \in [3]$. Now it is not hard to find Type 1 (u_j, v_j)-paths for any $j \in [3]$, by combining $|N^+(u_j) \cap N^-(v_j)| \geq 2$ with the path $u_j T_r S T_1 t_{i+1}$.

Now if $i < j - 1$ then there is a $l \in [i+1, j-1]$ such that T_l is trivial. Also $i + 1 \geq 3$, $j - 1 \leq r - 2$ and $r \geq 4 + 1 = 5$. Hence this is covered by the previous case. $\Diamond$

Case 4. *$r \geq 3$ and $|T_1| \geq 3$ ($|T_r| \geq 3$).*

Proof of case. There are three important arcs out of T_1, and they are disjoint in both ends as otherwise we find a separator of D of size less than 5. Let these be $u_j v_j$ for $j \in [3]$. If $|N^+(u_j) \cap N^-(v_j)| \geq 3$ we are done. This implies that $|T_3| \geq 3$ and $v_1, v_2, v_3 \in T_2$. By the same argument we conclude that $|T_1|, |T_2| = 3$ for otherwise at least one vertex v_j has out-degree 2 in T_1 or one vertex u_j has in-degree two T_2 also giving $|N^+(u_j) \cap N^-(v_j)| \geq 3$. Now a vertex of $s' \in S$ dominates a vertex $z \in T_1$ which is an in-neighbour of u_j for some $j \in [3]$. Hence we find Type 1 (u_j, v_j)-paths by using $N^+(u_j) \cap N^-(v_j)$ and the path $u_j T_r s' z v_j$. $\Diamond$

Case 5. *$r = 4$ and $|T_3| \geq 3$ ($r = 4$ and $|T_2| \geq 3$) and all other are trivial strong components.*

Proof of case. Assume that T_3 is the only non-trivial strong component (if T_2 then reverse all arcs and obtain the same problem).

Now for any vertex $z \in T_3$ with at least 2 in-neighbours inside T_3, $t_1 z$ respectively $t_2 z$ is not an important arc. This follows as otherwise we can find Type 0 (t_1, z)-paths respectively Type 1 (t_2, z)-paths. Using Lemma 10.7 and the fact that D is 5-strong we can conclude that $t_1 t_2$, $t_1 v_1$, $t_1 v_2$, $t_2 u_1$ and $t_2 u_2$ are important arcs of T, where $v_1, v_2, u_1, u_2 \in T_3$. Furthermore, $|T_3| = 3$ for otherwise either one of the vertices $\{v_1, v_2, u_1, u_2\}$ has in-degree 2 or $\{S, u_1, u_2\}$ is a separator of D of size 4. If there is an arc $s_j v_i$ for some $i, j \in [2]$ we find Type 1 (t_1, v_i)-paths by using t_4. Hence v_1, v_2 dominates S. But now, as there are three important arcs in $T - S$ with head in t_4, at least one has tail v_i for some $i \in [2]$. Then the paths $v_i s_1 t_1 t_4$ and $v_i s_2 t_2 t_4$ are disjoint from $N^+(v_i) \cap N^-(t_4)$, and we find Type 2 (v_i, t_4)-paths. $\Diamond$

The remaining cases are $r = 2$ and $r = 3$, where T_2 is the only component which is non-trivial. Before considering these cases, let us make a few observations and assumptions. Notice that for every important arc t_1v, $|N^-(v) \cap T_2| \leq 2$ for otherwise we find Type 0 (t_1, v)-paths. We will call the head v an ***in-critical vertex*** if $|N^-(v) \cap T_2| = 2$ and will denote it v_{crit}. Similarly for important arcs wt_3 we have $|N^+(w) \cap T_2| \leq 2$ and the tail w is an ***out-critical vertex*** if $|N^+(w) \cap T_2| = 2$. In this case w is denoted w_{crit}. As D is 5-strong there are at least three important arcs t_1v_1, t_1v_2 and t_1v_3 (w_1t_3, w_2t_3, w_3t_3) and by Lemma 10.7 and the fact that $n \geq 8$ there exist at least one $i \in [3]$ $(j \in [3])$ such that v_i (w_j) is an in-critical (out-critical) vertex. In the following v_i (w_i) will always denote the head (tail) of an important arc t_1v_i (w_it_3).

Before considering T_2 in more detail and looking at the two cases, observe the following that implies Theorem 10.10 which is stated after this proof.

Claim 1. *There are at most four arcs from t_1 to T_2 which are important arcs.*

Proof of Claim 1. Assume that there are 5. If $|T_2| \geq 6$, then at least one of the v_i vertices has in-degree 3, implying the existence of Type 0 (t_1, v_i)-paths. If $|T_2| = 5$ then as each v_i vertex has in-degree at most 2, we see that T_2 is a 2-regular tournament. Label the vertices $v_1, v_2, \ldots, v_5$ in T_2, such that $N^+(v_i) = \{v_{i+1}, v_{i+2}\}$ (mod 5) for all $i \in [5]$. As T is 2-strong there is a vertex v_i that dominates s_1. If v_{i-1} is dominated by s_1, then the path $t_1v_is_1v_{i-1}$ together with $N^+(t_1) \cap N^-(v_i)$ gives Type 1 (t_1, v_{i-1})-paths. Hence v_{i-1} dominates s_1. Repeating this argument we conclude that $T_2 \rightarrow s_1$. But then as T is 5-strong, there is an arc s_2v_i' for some $i' \in [5]$. Now $t_1v_{i'+1}s_1s_2v_{i'}$ together with $N^+(t_1) \cap N^-(v_{i'})$ will give Type 1 $(t_1, v_{i'})$-paths. ♦

Now as T_2 contains most of the vertices of T, it is worth considering this subtournament in more detail. We will pick T_2 such that, while maintaining the property of T:

- T2.1) The number of in-critical and out-critical vertices (in T_2) are maximal.
- T2.2) T_2 has the fewest number of separators and S' is the separator that leaves the fewest number of strong components in $T_2 - S'$.

For any separator S' of T_2 we denote the strong components of $T_2 - S'$ by $T_{2,1}$, $T_{2,2}, \ldots, T_{2,r'}$ and again if we know $T_{2,i}$ is a trivial component it is denoted $t_{2,i}$.

Case 6. $r = 3$ *and T_2 is the only non-trivial strong component.*

Proof of case.

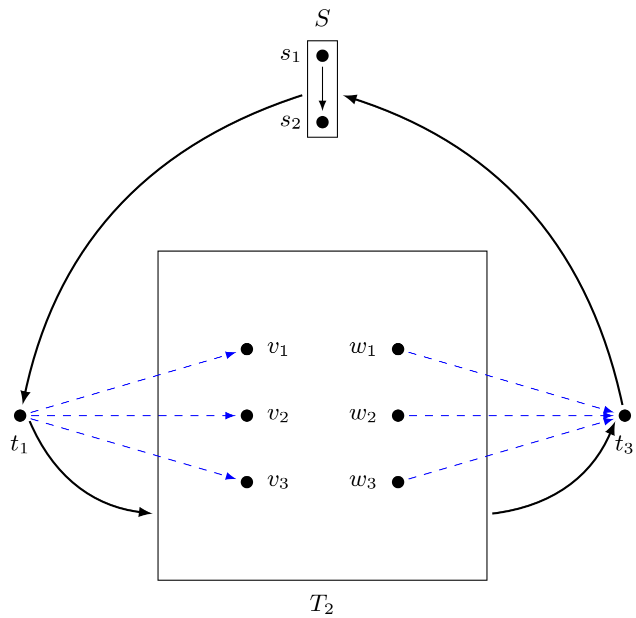


Figure 10.2: Structure of T in Case 6. Arcs between S and T_2 and arcs inside T_2 are left out. In dashed blue the important arcs with ends in t_1 or t_3.

Claim 2. *Every in-critical vertex v_{crit} dominates S and every out-critical vertex w_{crit} is dominated by S.*

Proof of Claim 2. Otherwise we find Type 1 (t_1, v_{crit})-paths using t_3 or find Type 1 (w_{crit}, t_2)-paths using t_1. ♦

Claim 3. *If v_i is not an in-critical vertex, then v_i dominates at least one vertex of S. Similarly if w_i is not an out-critical vertex, then w_i is dominated by at least one vertex of S.*

Proof of Claim 3. Assume wlog that v_1 is not an in-critical vertex and that S dominates v_1. Then if v_1 dominates a vertex $z \in T_2$ that dominates a vertex of S, then it is not hard to find Type 2 (t_1, v_1)-paths using z, S, t_3 and the in-neighbour of v_1. As v_{crit} dominates S, this implies that v_{crit} is the only in-neighbour of v_1, S dominates $T_2 - v_{crit}$ and that the last v_i vertex (say v_2) is not an in-critical vertex. But then consider v_2. This is dominated by v_1, dominates v_{crit} and, we find Type 2 (t_1, v_2)-paths using v_1, v_{crit}, t_3, S, a contradiction.

Similar can be proved for w_i vertices. ♦

Claim 4. *$v_i \neq w_j$ for all $i, j \in [3]$ and $n \geq 10$.*

Proof of Claim 4. Claim 2 and 3 implies that $v_i \neq w_{crit}$ for any $i \in [3]$ and $w_j \neq v_{crit}$ for any $j \in [3]$. But as $|T_2| \geq 4$ ($n \geq 8$), every vertex of T_2 has out- or in-degree at least 2. This implies that $v_i \neq w_j$ for any $i, j \in [3]$ and $|T_2| \geq 6$. ♦

Claim 5. *There is no important arcs $v_i w_j$ for $i, j \in [3]$.*

Proof of Claim 5. Assume that $v_i w_j$ is an important arc for some $i, j \in [3]$. Using t_1, t_3 and S together with Claim 2 and 3 it is not hard to see that T contains at two (v_i, w_j)-paths of length at least 2 internally disjoint from T_2. So if we find a (v_i, w_j)-path of length at least 2 in T_2 then $v_i w_j$ can be reversed, and we have a contradiction of T2.1). Now if $|T_2| > 6$, then it follows as $|N^+(v_i) \cap N^-(w_j)| \geq 1$. Otherwise by Claim 4 $|T_2| = 6$, in which case we find a path of length 3 using the fact that T_2 is strong. ♦

Claim 6. *For each $i \in [2]$, we may assume that s_i has at least two out-neighbours and at least two in-neighbours in T_2.*

Proof of Claim 6. Assume that there is an $i \in [2]$ such that s_i only has one out-neighbour in T_2. Then there is only one out-critical vertex w_{crit}. By Claim 3, we see that the two vertices w_1, w_2 that are not out-critical dominates s_i and are dominated by s_{3-i}. As D is 5-strong, there must be an important arc zs_i with $z \in T_2$. If z has two out-neighbours in T_2 then we find three (z, s_i)-paths of length at least 2

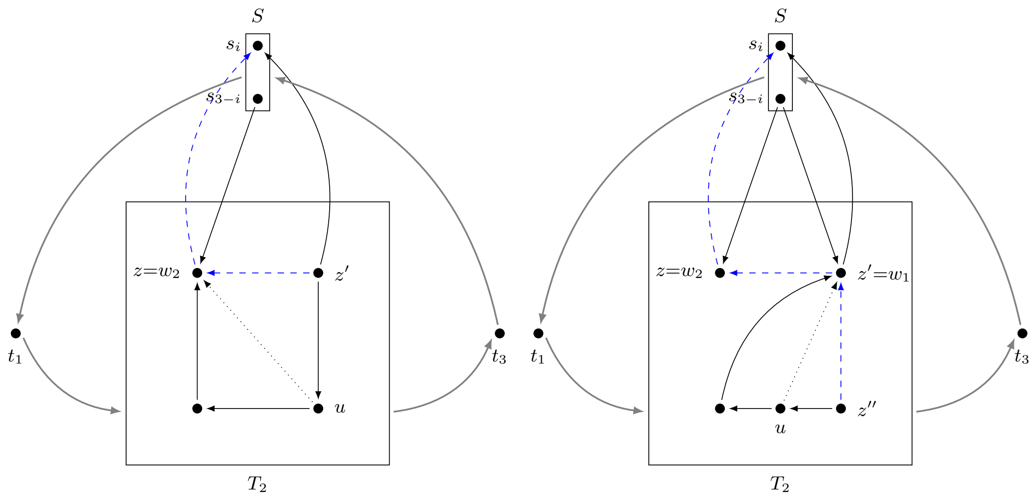
$$zx_1s_i$$

$$z(w_{crit})x_2s_i$$

$$zt_3s_i$$

where $x_1 \neq x_2 \in T_2$ and by Lemma 10.6 we may reverse the arc zs_i giving a tournament where s_i dominates and is dominated by at least 2 vertices of T_2. Notice that both property T2.1) and T2.2) of T_2 is maintained. First z is not a w_i vertex(out-degree 2 in T_2 but not w_{crit}), so we do not decrease the out-degree of such a vertex. Secondly we reverse an arc with only one end in T_2 and hence do not change the nature of T_2.

If z only has one out-neighbour in T_2, then $z = w_j$ for some $j \in [2]$ by Lemma 10.7. Assume wlog that $z = w_2$. But then there is another important arc $z'z$ of T_2 (notice that t_1 is not an out-neighbour of z in D). Assume first that z' has another out-neighbour u in T_2. Then either u dominates z or every out-neighbour of u dominates z. (See Figure 10.3a). In any case we find Type 2 (z', z)-path ((z', w_j)-paths) contradiction T2.1). Hence z' has exactly one out-neighbour and must be w_1. Again



(a) The case where z' has at least 2 out-neighbours in T_2.

(b) Then case where z is the only out-neighbour of z' in T_2.

Figure 10.3: Claim 6. Dashed blue arcs are important arcs and dotted arc corresponds to a possibly shorter (u, w_2)-path in T_2.

w_1 has an in-neighbour z'' that is an important arc and for this arc we are guaranteed one internal path in T_2 of length at least 2, as this in-neighbour has out-degree at least 2 by Lemma 10.7 (See Figure 10.3b). Hence we have found Type 2 (w_1, z'')-paths and after reversing this w_1 becomes an out-critical vertex that dominates s_i, a contradicting the choice of T as w_1t_3 can now be reversed.[2].

Similarly, by considering the v_j vertices, we can prove that both vertices of S is dominated by at least two vertices of T_2. ♦

Claim 7. *w_{crit} (v_{crit}) is completely dominated by (dominates) s_1 (s_2) in D and at least one of the important arcs incident to w_{crit} (incident with v_{crit}) belongs to T_2.*

Proof of Claim 7. Assume first that s_1w_{crit} is an important arc in T. Then the following three (s_1, w_{crit})-paths implies that s_1w_{crit} can be reversed giving a contradiction with the choice of T, as w_{crit} then have three out-neighbours in $T - t_3$.

$$s_1t_1w_{crit}$$

$$s_1s_2w_{crit}$$

$$s_1P(T_2)w_{crit}$$

[2] z'' might be w_{crit}, but this does not matter

Secondly w_{crit} has at least 5 out-neighbours in D and s_1, t_1 are not among them. This implies at least three of these out-neighbours are in T_2. As w_{crit} only has 2 out-neighbours in T_2, this implies that there is at least one important arc incident to w_{crit} in T_2. ♦

Claim 8. *T_2 is 2-strong.*

Proof of Claim 8. Assume by contradiction that T_2 is only 1-strong and let $S' = s'$ be the minimal separator of T_2 leaving the fewest number of strong components in $T_2 - s'$. Furthermore, let $j \in [r']$ be the maximal index such that for some $i \in [3]$, $v_i \in V(T_{2,j})$. We define the set

$$X = \bigcup_{m \in [j]} T_{2,m}$$

As D is 5-strong there are at least two disjoint paths from t_3 to X in $D - \{s_1, s_2, s'\}$ and hence at least two important arcs incident to X in $T_2 - s'$. Call these important arcs ab and $a'b'$. Notice that as X contains all v_i vertices that belongs to $T_2 - s'$, $|X| \geq 2$. This implies that $a \neq a'$, and we assume wlog that $aa' \in A(T)$.

Assume first that $|V(T_2) - \{X, s'\}| \geq 2$. Then neither a nor a' is a w_{crit} vertex (and hence $b, b' \in T_2$). This follows as either both has out-degree at least 3 in T_2 or if a' only has out-degree 2, then $a' = T_{2,j} = v_i$ (Claim 4). As we have the path t_3St_1 either $|N^+(a) \cap N^-(b)| \geq 2$ or $|N^+(a') \cap N^-(b')| \geq 2$ would imply Type 1 paths, and therefore we can conclude:

- $bb' \in A(T)$.
- The last (second last) component of X is the vertex a' (a)
- The first (second) component of X is the vertex b' (b).

As $|V(T_2)| \geq 6$ (Claim 4) we have $r' \geq 5$ and either $a \neq T_{2,1}$ or $b' \neq T_{2,r'}$. Assume wlog $a \neq T_{2,1}$. Then we find Type 1 (a, b)-paths contradiction T2.2) (See Figure 10.4):

$$aa'b$$

$$aT_{2,r'}s'T_{2,1}b$$

$$at_3St_1b.$$

Now consider the case where $|V(T_2) - \{X, s'\}| \leq 1$ and notice that since $|V(T_2)| \geq 6$ we have $|X| \geq 4$. But then as there exist $i \in [3]$, such that v_i belongs to the last strong component of X, $T_{2,j}$, we can conclude that $j \leq 2$ and $T_{2,j}$ is the only non-trivial strong component of X. As $r' \geq 2$ this implies one of the following two:

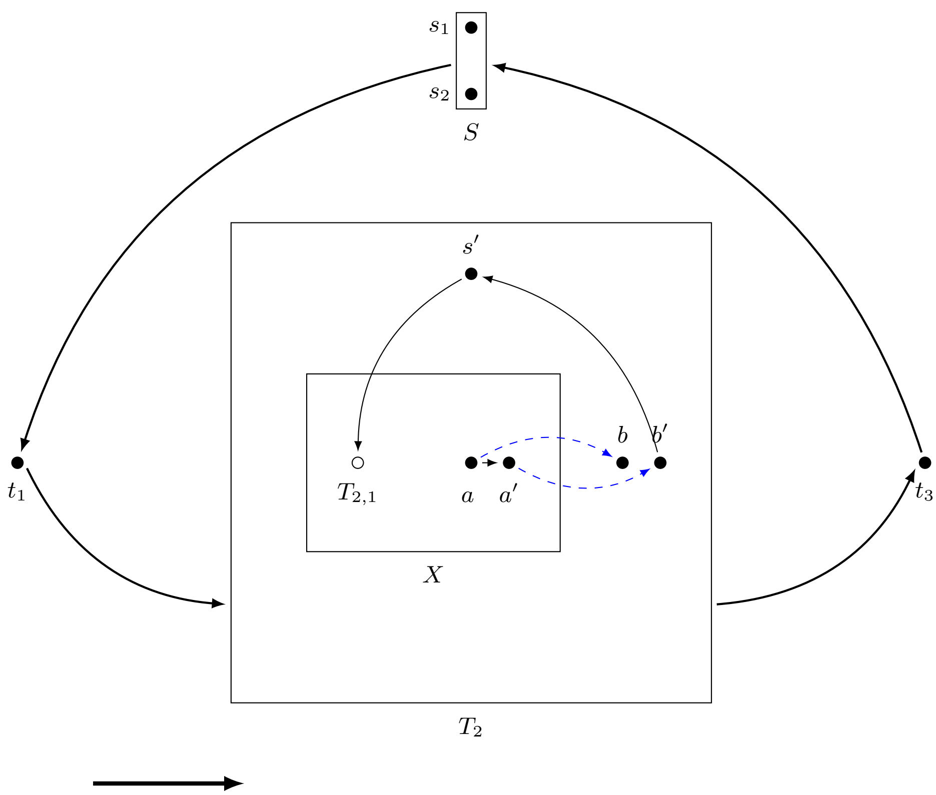


Figure 10.4: The case where ab and $a'b'$ are contained in T_2, $b' \in T_{2,r'}$ and $a \notin T_{2,1}$.

1. $r' = 3$, $|V(T_{2,1})| = |V(T_{2,3})| = 1$ and $T_{2,2}$ is a non-trivial strong component.
2. $r' = 2$, $|V(T_{2,2})| = 1$ ($|V(T_{2,1})| = 1$) and $T_{2,1}$ ($T_{2,2}$) is a non-trivial strong component.

Notice that the non-trivial strong component is just 1-strong. For the first case this follows as $T_{2,2}$ contains a w_i vertex, and if this has out-degree 2 in $T_{2,2}$ it has out-degree 3 in T, as $t_{2,3}$ is also an out-neighbour of w_i. In the following z is the out-neighbour of a w_{crit} vertex in $T_{2,2}$ and hence a separator of $T_{2,2}$.

Proof of 1: See Figure 10.5. Assume first that there is no important arc inside $T_{2,2}$ with head w_{crit}. Then by Claim 7 either $t_{2,1}w_{crit}$ or $s'w_{crit}$ is an important arc. But as w_{crit} is dominated by a vertex of $T_{2,2}$ and s' we find Type 1 $(t_{2,1}, w_{crit})$-path contradicting T2.2). On the other hand if $s'w_{crit}$ is the only important arc incident to w_{crit} in T_2, then as w_{crit} has out-degree at least 5 in D, also s_2w_{crit} is an important arc. If $s_2s' \in A(T)$ then

$$s_2t_1w_{crit}$$

$$s_2s'w_{crit}$$

$$s_2P(T_2)w_{crit}$$

are three disjoint (s_2, w_{crit})-paths of length at least 2. Here we use that s_2 dominates at least two vertices of T_2 (Claim 6). Similar if $s's_2 \in A(T)$, then

$$s's_2w_{crit}$$

$$s't_{2,1}w_{crit}$$

$$s't_3s_1w_{crit}$$

are three disjoint (s', w_{crit})-paths of length at least 2. So in each case we find an arc that can be reversed such that w_{crit} has three out-neighbours in $T - t_3$, a contradiction with the choice of T. Hence we may conclude that there is at least one important arc u_1w_{crit} in $T_{2,2}$.

Notice that there are two disjoint paths of length 4 between any pair of vertices in $T_{2,2}$ by using $\{t_{2,3}, s', t_{2,1}, t_3, s_1, s_2, t_1\}$ and that such two path can be constructed avoiding s_i for specified $i \in [2]$. To avoid Type 1 (u_1, w_{crit})-path for the important arc u_1w_{crit}, we can conclude that in $T_{2,2}$, w_{crit} is dominated by $T_{2,2} - \{z\}$ and u_1 is dominated by $T_{2,2} - \{w_{crit}\}$. If there exist $u_2 \in T_{2,2}$ such that u_2u_1 is an important arc, then $u_2w_{crit}zu_1$ is an (u_2, u_1)-path of length at least 2 in $T_{2,2}$. Hence we may reverse u_2u_1. But then $u_1u_2w_{crit}$ together with the two external paths will give Type 1 (u_1, w_{crit})-paths and reversing u_1w_{crit} we obtain a contradiction with the choice of T. Hence the 5 out-neighbours of u_1 in D must be in the set $\{t_1, t_{2,1}, w_{crit}, t_{2,3}, t_3, s_1, s_2, s'\}$.

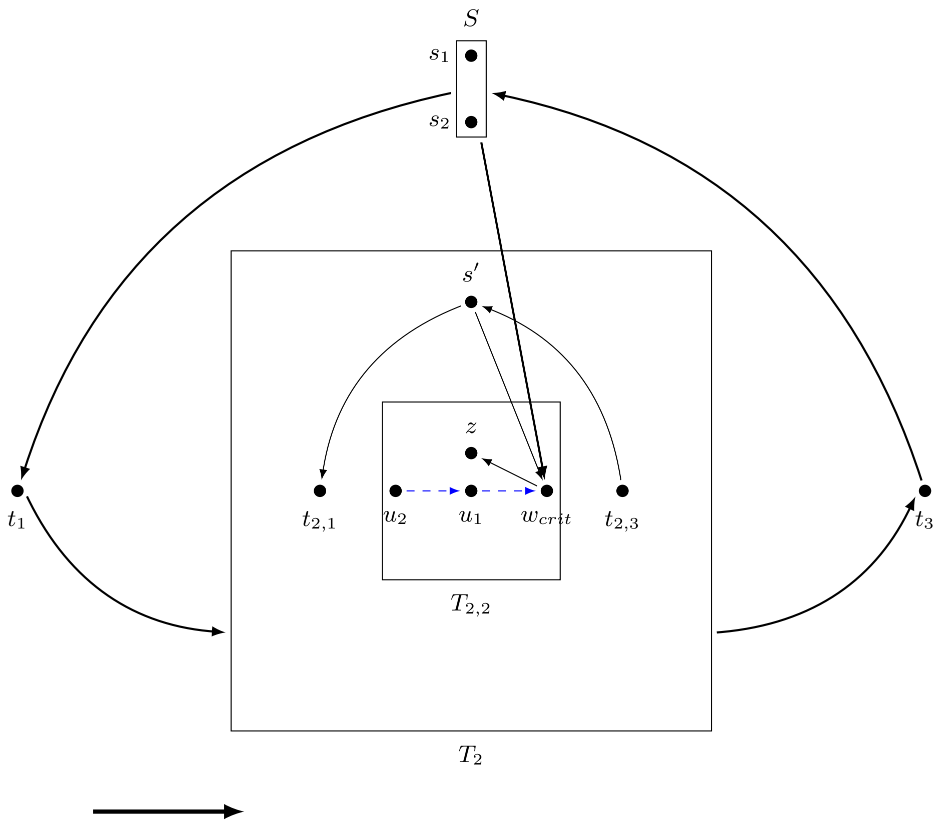


Figure 10.5: The main structure used in the argument of 1.

As u_1 has at least two in-neighbours in $T_{2,2}$ and one in $t_{2,1}$ we know t_1u_1 is not an important arc. Furthermore, as u_1 is dominated by S in T, we may use similar arguments that concluded that $t_{2,1}w_{crit}$ and s_1w_{crit} were not important arcs to conclude that $t_{2,1}u_1$ and s_1u_1 are not important arcs. Hence $\{w_{crit}, t_{2,3}, t_3, s_2, s'\}$ is the set of 5 out-neighbours of u_1 in D. But then either the path of important arc $s_2u_1w_{crit}$ (when s_2s) or $s'u_1w_{crit}$ (when $s's_2$) can be reversed using same arguments as above.

Proof of 2: Now consider $T_{2,1} - z$ and let $T_{2,1,1}, T_{2,1,2}, \dots, T_{2,1,r''}$ be the strong components of $T_{2,1} - z$. Then $T_{2,1,r''} = t_{2,1,r''} = w_{crit}$. Furthermore, there is at least one v_i vertex in $T_{2,1} - z$, and we let $j \in [r'' - 1]$ be the maximal index such that there exist $i \in [3]$ with $v_i \in V(T_{2,1,j})$. Define

$$X' = \bigcup_{m \in [j]} T_{2,1,m}$$

As there is at least one (t_3, X')-path in $D - \{s_1, s_2, s', z\}$, there is an important

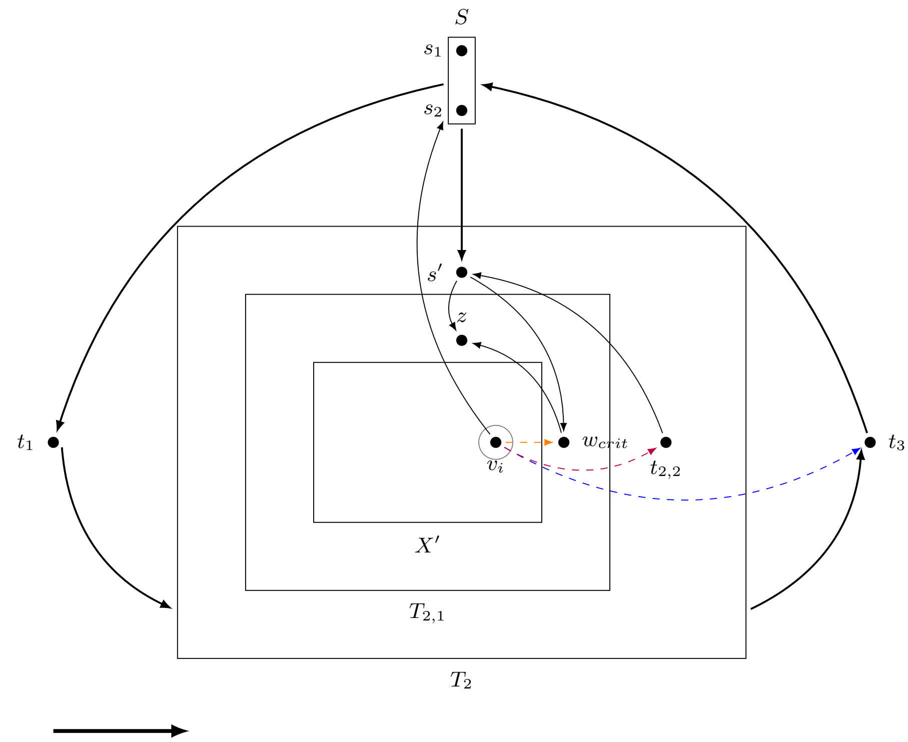


Figure 10.6: The main structure in the proof of 2. The coloured dashed arcs illustrate the three different possibilities of the important arc ab.

arc ab incident to X'. See Figure 10.6.

If $b = t_3$, then $a = w_{i'}$ for some $i' \in [3]$. But as $v_i \neq w_{i'}$ this implies that $T_{2,1,j}$ is a non-trivial strong component. Then $|N^+(a) \cap N^-(t_3)| \geq 3$, and we have Type 1 $(w_{i'}, t_3)$-paths.

If $b = t_{2,2}$ and $T_{2,1,j}$ is non-trivial or $a \notin T_{2,1,j}$, then we find Type 1 (a, b)-paths. Hence $a = v_i$ and $b = t_{2,2}$. By Claim 4 $b \neq w_{i'}$ for any $i' \in [3]$ and as all vertices of $X' - a$ has at least 3 out-neighbours in T_2, $\{w_1, w_2, w_3\} = \{s', z, w_{crit}\}$. Claim 2 and Claim 3 gives a $v_i s_h$ arc for some $h \in [2]$ and as s' is an out-critical vertex (dominates S and w_{crit}) $s's_h$ is an arc of T, and we find Type 2 (a, b)-paths.

The final case is that $b = w_{crit}$. As $a \neq v_i$ by Claim 5, there is a vertex $w \in N^+(a) \cap N^-(b) \cap X'$, and we find Type 1 (a, b)-paths

$$awb$$

$$at_{2,2}s'w_{crit}$$

$$at_3s_it_1w_{crit}$$

◆

Claim 9. *When T_2 is 2-strong any important arc between S and T_2 can be reversed.*

Proof of Claim 9. As T_2 is 2-strong all v_i vertices have in-degree exactly 2 in T_2 and hence are all in-critical and dominate S. Using $v_i s_j$ for $i \in [3]$, $j \in [2]$, that T_2 is 2-strong and the vertex t_3 we find at least 3 disjoint (u, s_j)-paths of length at least 2 for every $u \in T_2$. Hence an important arc us_j can be reversed. Notice that this especially implies that v_i completely dominates S in D.

Similarly it can be proved that w_j is completely dominated by S in D and that all other important arc with tail in S and head in T_2 can be reversed. ♦

Now consider T_2. As $W = \{w_1, w_2, w_3\}$ is completely dominated by S in D, there must be 4 out-neighbours to each w_i in D_2, though only 2 in T_2. We will show that one of these important arcs in T_2 can be reversed. Denote by w_i^1, w_i^2 the tail of the two important arcs of T_2 with head in w_i and when both are specified $w_i^1 w_i^2$ is an arc of T.

Assume first that $T_2\langle W\rangle$ is a transitive tournament where w_1 is the source and consider w_1^1, w_1^2. Then $w_1^1 w_1^2 w_1$ and $w_1^1 t_3 s_1 w_1$ are disjoint paths of length at least 2, and we will find a third (w_1^1, w_1)-path contradicting the choice of T. Let $Z = T_2 - \{W, w_1^1, w_1^2\}$. Then Z dominates w_1, and we may assume that Z dominates w_1^1, for otherwise we find the third path. But then w_1^1 either dominates w_2 or w_3, and as w_3 dominates a vertex of Z this gives a (w_1^1, w_1)-path of length 3 or 4 depending on whether w_3 or w_2 is an out-neighbour of w_1^1. See Figure 10.7a.

Assume now that $T_2\langle W\rangle$ is a 3-cycle with arcs $w_i w_{i+1}$ (mod 3), let $Q = N^+(w_1) = \{q, w_2\}$ and consider $T_2 - Q$ with the strong components $T_{2,1}, T_{2,2}, \ldots T_{2,r'}$ for some $r' \geq 2$. Now $T_{2,r'} = w_1$ and as T is 2-strong, $\{w_1, q\}$ are the two out-neighbours of w_3 and $T_{2,r'-1} = w_3$. Finally as $|T_2| \geq 6$ we see that $r' \geq 3$. Consider the important arc $w_3^1 w_3$ with $w_3^1 \in T_2 - Q$ (w_3 only has one in-neighbour in Q, so there must be an important arc). If $|N^+(w_3^1) \cap N^-(w_3)| \geq 1$ then together with the paths

$$w_3^1 w_1 w_2 w_3,$$

$$w_3^1 t_3 S t_1 w_3$$

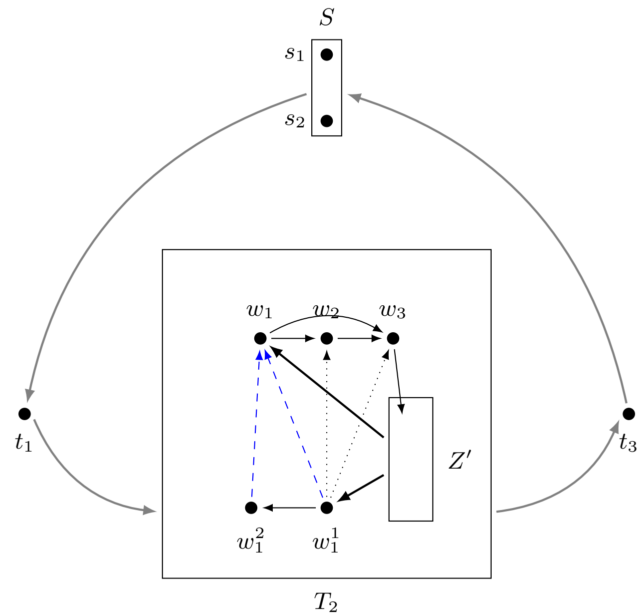
we have Type 1 (w_3^1, w_3)-paths. Hence $w_3^1 = T_{2,r'-2}$, $r' > 3$ and w_3^1 dominates w_2. Now

$$w_3^1 w_2 w_3$$

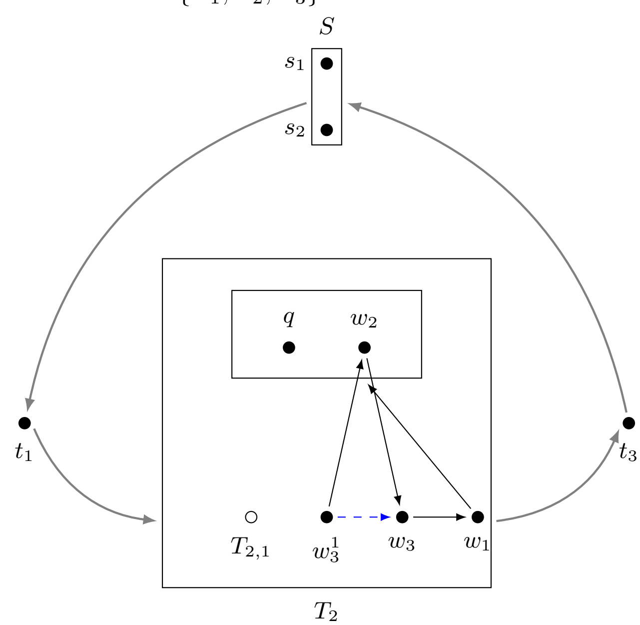
$$w_3^1 w_1 q T_{2,1}$$

$$w_3^1 t_3 S t_1 w_3$$

are three (w_3^1, w_3)-paths, contradiction the choice of T. See Figure 10.7b. ◇



(a) The case where $\{w_1, w_2, w_3\}$ induces a transitive tournament.



(b) The case where $\{w_1, w_2, w_3\}$ induces a 3-cycle.

Figure 10.7

Now we consider the last case $r = 2$. From now on we will use that D is 6-strong.

Case 7. *$r = 2$ and T_1 (T_2) is a trivial strong component.*

Proof of case. First as D is 6-strong, $D_2 = D - \{S, t_1\}$ is a 3-strong semicomplete digraph and by Theorem 9.5 it contains a 2-strong spanning tournament. So let T_2 be such a tournament and similar the previous case we let v_1, v_2, v_3, v_4 be the four heads of the important arcs with tail in t_1. Notice, that there are four of these vertices as D is 6-strong.

We will use the following trivial lemma.

Lemma 10.9. *Let D be a 2-strong digraph. Then for any choice of 3 distinct vertices $x_1, x_2, x_3 \in V(D)$ there exist internally disjoint paths P, Q, such that P is an (x_1, x_2)-path ((x_2, x_1)-path) and Q is an (x_1, x_3)-path ((x_3, x_1)-path).*

Proof. Let P_1, P_2 be the two disjoint (x_1, x_2)-paths and Q_1, Q_2 be the two disjoint (x_1, x_3)-paths in D. We may assume that for each $i \in [2]$, Q_i intersects both P_1 and P_2. Assume wlog that $x_2 \notin Q_1$ and let v be the last vertex of Q_1 that intersects a vertex of P_1 or P_2, say P_2. Then $P = P_1$ and $Q = P_2[x_1, v]Q_1[vx_3]$ are the two disjoint paths. $\diamondsuit$

Claim 10. *If there exists a $j \in [2]$ such that s_j dominates three vertices of T_2, then we may reverse any important arc $s_j x$ with $x \in T_2$.*

Proof of Claim 10. Let x_1, x_2 be two out-neighbours of s_j in $T_2 - x$. Then by Lemma 10.9 we have disjoint (x_1, x)-path P and (x_2, x)-path Q in T_2. Hence

$$s_j t_1 x$$

$$s_j P x$$

$$s_j Q x$$

are three (s_j, x)-paths and by Lemma 10.6 $s_j x$ may be reversed. ♦

Claim 11. *v_i dominates S for all $i \in [4]$.*

Proof of Claim 11. Assume by contradiction that $s_h v_i$ is an arc of T for some $h \in [2]$ and $i \in [3]$. If there exist a vertex $x \in N^+(v_i)$ such that $x s_h$ is an arc of T, then it is easy to find Type 1 (t_1, v_i)-paths. Hence s_h dominates X. As $n > 7$, $|X| \geq 2$, and we see that s_h has three out-neighbours in T_2. This implies that s_h completely dominates X in D, for otherwise we can use Claim 10 to reverse an arc $s_h x'$ for $x' \in X$, and after this have Type 1 (t_1, v_i)-paths. But as D is 6-strong (5 is sufficient here) the vertices $N^-(v_i) \cup v_i$ dominates s_h in D, and we may assume (using Claim 10 three times) that this is also true in T. Then considering the vertex $v_{i'} \in X$ we find Type 1 $(t_1, v_{i'})$-paths, a contradiction. ♦

Claim 12. *v_i completely dominates S in D*

Proof of Claim 12. Consider three distinct vertices v_i, v_j, v_z for $i, j, z \in [4]$. We will show that if $v_i s_h$ is an important arc for some $h \in [2]$ then we can reverse it giving a contradiction with the choice of T. Notice that, by Lemma 10.9, T_2 contains (v_i, v_j)-path P and $(v_i v_z)$-path Q such that P and Q are disjoint. This gives two (v_j, s_h)-paths in T, and we need only find a third path R disjoint from these. If $h = 1$ then $R = v_1 s_1 s_2$ and if $h = 2$ then $R = v_i s_2 t_1 v_q$, where $v_q = \{v_1, v_2, v_3, v_4\} - \{v_i, v_j, v_z\}$. ♦

Now as D is 6-strong (5 is sufficient), Claim 12 implies that for each $i \in [4]$ there are two important arcs $v_i v_i^1$ and $v_i v_i^2$ in T_2. We will show that for some $i \in [4]$ and $j \in [2]$ the arc $v_i v_i^j$ can be reversed, giving the final contradiction. As $v_i S t_1 v_i^j$ is a (v_i, v_i^j)-path disjoint from T_2 it is sufficient to find two (v_i, v_i^j)-paths inside T_2 of length at least two.

Let $W = \{v_1, v_2, v_3, v_4\}$ and assume we have labelled the vertices of W such that $v_1 v_2 v_3 v_4$ is a cycle and v_1, v_2 is the two vertices with one in-neighbour in W. First if there exist $i \in [2]$, $j \in [2]$ such that $v_i^j \notin W$, then as W contains disjoint (v_i, v_3)- and (v_i, v_4)-paths and v_3, v_4 dominates all vertices in $T_2 - W$ we find two disjoint (v_i, v_i^j)-paths. Hence the two important arcs with tail in v_1 $((v_2))$ belongs to $T_2\langle W\rangle$. Similarly for $i \in [3, 4]$, if both $v_i^1, v_i^2 \notin W$, then using a path in $T_2\langle W\rangle$ and $v_i^1 v_i^2$ we can reverse $v_i v_i^2$. Finally let v_3^2 and v_4^2 be the vertices not in W. If $v_3^2 \neq v_4^2$, then again it is easy to find two paths. Hence $v_3^2 = v_4^2$ and is the only important arc in T_2 with tail in W. Now $T_2 - v_3^2$ is at least 1-strong and assume that uv_1 is an arc of T_2. Then

$$v_3 v_4 v_3^2$$

$$v_3 u v_1 v_3^2$$

are the two disjoint (v_3, v_3^2)-paths in T_2.

◊

This completes the proof. □

As indicated in the proof above, we obtained a partial result.

Theorem 10.10 (Christiansen)**.** *Every* 5*-strong semicomplete digraph with* $\delta^0 \geq 7$ *contains a* 3*-strong spanning tournament.*

Proof. This follows by proof of Theorem 10.8 ending after Claim 1. If $\delta^0 \geq 7$, then t_1 is adjacent to at least 5 important arc with head in T_2. □

10.3 Closing the gap

In the previous two sections we proved that every member of the following three classes of semicomplete digraphs contains a 3-strong spanning tournament.

- 5-strong semicomplete digraphs with a 3-strong subtournament.
- 5-strong semicomplete digraphs with minimum semi-degree at least 7.
- 6-strong semicomplete digraphs.

These are all improvements of the previous best known result by Guo [24] on 7-strong semicomplete digraphs. Given the elaborate proof of the last of these, one might wonder whether the bound on 6-strong is the best we can obtain, or if there might be a better way to prove this and Conjecture 9.4. In the following we will continue with the assumption that Conjecture 9.4 is true for $k = 3$, then in the next Chapter we will take this into revised considerations. In the proof of Theorem 10.8 we only used that the semicomplete digraph was 5-strong until the last Case 7, so let us start by recalling the structure obtained just before Case 7:

- T is a 2-strong spanning tournament of D with the fewest number of separators.
- S is a separator of T such that $T - S$ has exactly two strong components, a trivial component t_1 and a non-trivial T_2.
- T_2 is a most 2-strong.
- There are at least 8 vertices in T and at least 5 in T_2.
- The minimum semi-degree of D is at most 6.

If we consider the proof the Case 7, two key structures are used, both of which are deduced from the fact that D is 6-strong.

- T_2 is a 2-strong tournament.
- There are least 4 important arcs t_1v_i.

This leads to two obvious problems to consider. First prove that D_2 can be oriented as a 2-strong tournament without contradicting the choice of T. Remember that $D - \{u_1, u_2\}$ is 3-strong and has a 2-strong orientation for all $u_1, u_2 \in V(D)$. If $D - \{S, t_1\}$ is only 2-strong then all three vertices plays a key role in the connectivity of D. How does this effect the arcs incident to $\{S, t_1\}$? Secondly assume that T_2 is 2-strong, but D is just 5-strong and prove that one of the three important arcs t_1v_1, t_1v_2, t_1v_3 can be reversed. Using the 2-strong connectivity of T_2 and the existence of important arc, it seems possible to do this by hard work using case analysis. Also looking at the structure of T at Case 7 it would be interesting to find all the separators of T and again remember that for any pair of vertices $u_1, u_2 \in V(D)$, $D - \{u_1, u_2\}$ is 3-strong and has a 2-strong orientation.

On a more general note the following might both prove Case 7 and improve some previous cases by:

1. Consider different sets of important arcs and prove that all vertices in such sets can be reverses together without contradiction the choice of T.
2. Understanding how the semicomplete digraph D looks like when we assume that D is minimal with respect to connectivity (both for vertices and arcs).
3. Consider 2-strong tournaments in $D - \{u_1, u_2\}$ for some $u_1, u_2 \in V(D)$. Is it even possible to coexist when we cannot find a 3-strong spanning tournament? For example, is it possible that $D - \{s_1, s_1\}$, $D - \{s_1, t_1\}$ and $D - \{s_2, t_1\}$ all has a 2-strong orientation, but D do not have a 3-strong orientation.
4. Find a 3-strong subtournament in D.

An interesting observation when attempting to prove Case 7 for 5-strong semicomplete digraphs is that we often ended in a situation where the important arcs were dense in the set $\{s_1, s_2, t_1, v_1, v_2, v_3\}$. Notice that this give a nice understanding on why the extra vertex v_4 helped. If there are many 2-cycles in the set $W = \{s_1, s_2, t_1, v_1, v_2, v_3, v_4\}$ then in many cases Corollary 10.3 implies that the arcs of W can be reversed such that W forms a 3-regular tournament. Then the theorem follows by Theorem 10.4, using this 3-regular 3-strong tournament to obtain a 3-strong spanning tournament.

Chapter 11

Discussion on the k-strong spanning tournament problem

In the previous chapter we attempted to prove Conjecture 9.4 for $k = 3$, and three partial results has been found. The last chapter also ended with a discussion on how one might prove this conjecture. In this chapter we will continue this discussion by considering the conjecture for general k. We will describe what issues there are when increasing the size of k and indicate why $(2k + 1)$-strong semicomplete digraphs may be the correct bound.

11.1 Conjecture 9.4 for general k

Going back to the proof of Theorem 10.8 we saw that the main tool to prove the theorem was Lemma 10.6. Informally this says that if we can find k disjoint paths between a pair of vertices, then we can safely reverse an arc between the two of them. If we were to prove Conjecture 9.4 for general k using a similar approach by finding a $k - 1$ strong spanning tournament T of 'optimal' structure, then clearly Lemma 10.6 will also play a key role here. We would expect that (as we did in the proof of Theorem 10.8) it is easy to deduce that $T - S$ contains few strong components. Notice however that for larger k, we do not have the same correlation between the size of a strong component and the existence of an important arc. For example, as a strong component has size at least 3, we might have an important arc from T_i to T_{i+2} even though T_{i+1} is non-trivial. Reducing to an instance with few strong components, we might be able to prove the problem corresponding to Theorem 10.10.

Problem 11.1 (NEW: Christiansen)**.** *Let k be a fixed integer. Does there exist a function $h(k)$ such that every $(2k - 1)$-strong semicomplete digraph with $\delta^0 \geq h(k)$*

contains a k-strong spanning tournament?

What is more troublesome is the amount of work that had to be done in the last two cases of Theorem 10.8. For a general proof this does not seem like a feasible approach, and we expect that new structural results has to be found. In Section 10.3, a few suggestions on subproblems were given and some, but not all extends, naturally to the problem for general k. One that does not extend to general k is 3. Here we remove two vertices from D, and are left with a semicomplete digraph that can be oriented to a 2-strong tournament using Theorem 9.5. This is not true for larger k. Removing $(k-1)$ vertices form a $(2k-1)$-strong semicomplete digraph, leaves a k-strong semicomplete digraph and not a $2(k-1)-1 = (2k-3)$-strong semicomplete digraph. Hence for $k > 3$ we can say even less about the structure of the $(k-1)$-strong spanning tournament.

These problems might suggest that the bound in Conjecture 9.4 is not correct, though no counterexamples has (yet) been found. The following conjecture will leave a small gap between the known best possible.

Conjecture 11.2 (NEW: Christiansen)**.** *Every $(2k+1)$-semicomplete digraph contains a k-strong spanning tournament.*

Notice the well-pleasing fact, that this conjecture do not have the extra 'of 2k+1 vertices' as Conjecture 9.4 does. Also proving this will improve the known best possible result by Guo as $2k+1 \leq 3k-2$ for all $k \geq 3$.

To support Conjecture 11.2 even more, consider the extension of Theorem 10.4; that a $(2k-1)$-strong semicomplete digraph containing a k-strong subtournament can be oriented to a k-strong tournament. We tried to prove this using similar techniques and had the following two issues:

Problem 1:

Eliminating 2-cycles between G, G_{in} and G_{out} do no longer follow directly as it depends on the size of G. If we assume that G (or the tournament T) has size at least $3k-2$, then such 2-cycles can be eliminated. It might also be possible to solve this 'problem' by moving a set of vertices with the right connectivity.

Problem 2:

We no longer have the same control on the W-set. In the proof of Theorem 10.4, a vertex v with out-degree 3 in W would naturally define a subset $N_W^+(v)$ where we delete arcs of 2-cycles. This is not true for larger k. Consider for example $k = 4$ and $|W| = 7$. A vertex with out-degree 4 in W implies that W cannot be oriented to a 3-regular tournament, but its out-neighbourhood is too small to contain a 2-regular tournament. Increasing the connectivity to $2k+1$ might give sufficient slack in such sets W in order for us to orient 2-cycles and move vertices to G.

Conjecture 11.3 (NEW: Christiansen). *Let k be a fixed integer and let D be a $2k+1$-strong semicomplete digraph with a k-strong subtournament of D, then D contains a k-strong spanning tournament.*

11.2 Locally semicomplete digraphs containing k-strong local tournaments

Remember that Bang-Jensen proved the following:

Theorem 9.8 ([5]). Let $f(k)$ be an integer function such that $f(1) = 1$ and $f(k) \geq f(k-1)+2$ for every $k \geq 1$. Suppose that every $f(k)$-strong semicomplete digraph contains a spanning k-strong tournament. Then every $f(k)$-strong locally semicomplete digraph contains a k-strong spanning local tournament.

Together with Theorem 10.8 this implies

Theorem 11.4 (NEW: Christiansen). *Every 6-strong locally semicomplete digraph contains a 3-strong spanning local tournament.*

It would be interesting to see if the relation between semicomplete digraphs and locally semicomplete digraphs seen in Theorem 9.8 also extends if we condition on more than just the strong connectivity of the digraph, i.e if there exist corresponding results on locally semicomplete digraphs as Theorem 9.5 and Theorem 10.10.

Conjecture 11.5 (NEW: Christiansen). *Let D be a 5-strong locally semicomplete digraph containing a 3-strong local tournament. Then D can be oriented to a 3-strong local tournament*

Conjecture 11.6 (NEW: Christiansen). *Let D be a 5-strong locally semicomplete digraph with minimum semi-degree at least 7. Then D can be oriented to a 3-strong local tournament.*

We could also state the obvious extensions to general k, but unsure whether the correct bound is $2k-1$ and $2k+1$ for these problems, we leave out the formal conjectures.

11.3 Concluding remarks

In this part of the dissertation we have considered sufficient conditions for a semicomplete digraph to contain a k-strong spanning tournament.

We could also have asked for a polynomial algorithm to find such (when they exist). Lichiardopol conjectured that if T is a k-strong tournament, then it contains a small set inducing the strong connectivity of T.

Conjecture 11.7. *[37] For every $k \geq 1$ there exists a function $f(k)$ such that every minimally k-strong tournament has at most $f(k)$ vertices.*

Proving this and Conjecture 11.3 will probably lead to a polynomial algorithm that both decides and finds a k-strong spanning tournament. Indeed consider the case $k = 3$. Then the algorithm will do the following: For every set of $f(3)$ vertices, try if it has a 3-strong orientation (try all possibilities). If one is found, then use the algorithm of Corollary 10.5 to obtain a 3-strong spanning tournament.

If conjecture 11.7 is not true, then we have to describe an algorithm that finds the spanning k-strong directly. It is not clear whether we can turn the proof of Theorem 10.8 into an algorithm finding such tournament. The proof starts with a $(k-1)$ strong spanning tournament and shows that this is not best possible. If we were to use the approach of this proof, we need a way to find a $(k-1)$-strong tournament. This might be done recursively: Start from a 1-strong tournament, and use the algorithm induced by the proof of Theorem 10.8 to find a 2-strong tournament. Then repeat.

Part

IV

Future work, papers and bibliography

Future work

This chapter is a brief summery of the problems that this author would consider next, had time permitted it.

Linkage in digraphs

In Part I we considered the k-linkage problem for fixed k and generalizations of semicomplete digraphs. As also noted in Chapter 4, using the refined structure described in Section 1.2.1 one might prove the following conjecture:

Conjecture 4.2 (NEW: Christiansen)**.** For k fixed integer, every $5(k-1)$-strong evil locally semicomplete digraph is k-linked.

Degree constrained partitions of digraphs

In Part II we considered the problem of partitioning a digraph in (two) parts such that each partition induced a digraph with certain properties. While the problem $(\delta^+ \geq k, \delta^+ \geq k)$-partition was proved to be NP complete for general digraphs, we found a polynomial algorithm for semicomplete digraphs and digraphs with bounded independence number. It is natural to consider round decomposable and locally semicomplete digraphs next.

Conjecture 8.7 (NEW: Christiansen)**.** The $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem is polynomial on round decomposable digraphs.

Conjecture 8.8 (NEW: Christiansen)**.** Let k_1, k_2 be fixed integers. There exists a function $g(k_1, k_2)$ such that every locally semicomplete digraph D with $\delta^+ \geq g(k_1, k_2)$ contains a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.

Semicomplete digraphs containing k-strong spanning tournaments

In Part III we considered the problem of finding the correct bound on the strong connectivity of a semicomplete digraph such that we were ensured that it contained a k-strong spanning tournament. The part ended with many related conjectures and problems, both for semicomplete digraphs and locally semicomplete digraphs. This author would start by considering the following two conjectures for semicomplete digraphs:

Conjecture 11.3 (NEW: Christiansen)**.** Let k be a fixed integer and let D be a $2k+1$-strong semicomplete digraph with a k-strong subtournament of D, then D contains a k-strong spanning tournament.

Conjecture 11.2 (NEW: Christiansen)**.** Every $(2k+1)$-semicomplete digraph contains a k-strong spanning tournament.

Paper: Disjoint paths in decomposable digraphs

Joint work with Jørgen Bang-Jensen and Alessandro Maddaloni. Published in Journal of Graph Theory 2016 [7]

Disjoint paths in decomposable digraphs

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Abstract

The k-linkage problem is as follows: given a digraph $D = (V, A)$ and a collection of k terminal pairs $(s_1, t_1), \ldots, (s_k, t_k)$ such that all these vertices are distinct; decide whether D has a collection of vertex disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is from s_i to t_i for $i \in [k]$. A digraph is k-linked if it has a k-linkage for every choice of $2k$ distinct vertices and every choice of k pairs as above. The k-linkage problem is NP-complete already for $k = 2$ [11] and there exists no function $f(k)$ such that every $f(k)$-strong digraph has a k-linkage for every choice of $2k$ distinct vertices of D [17]. Recently Chudnovsky, Scott and Seymour [9] gave a polynomial algorithm for the k-linkage problem for any fixed k in (a generalization of) semicomplete multipartite digraphs. In this paper we use their result as well as the classical polynomial algorithm for the case of acyclic digraphs by Fortune, Hopcroft and Wyllie [11] to develop polynomial algorithms for the k-linkage problem in locally semicomplete digraphs and several classes of decomposable digraphs, including quasi-transitive digraphs and directed cographs. We also prove that the necessary condition of being $(2k-1)$-strong is also sufficient for round-decomposable digraphs to be k-linked, obtaining thus a best possible bound that improves a previous one of $(3k-2)$. Finally we settle a conjecture from [3] by proving that every 5-strong locally semicomplete digraph is 2-linked. This bound is also best possible (already for tournaments) [1].

Keywords: disjoint paths, locally semicomplete digraph, quasi-transitive digraph, k-linkage problem, (round-)decomposable digraphs, polynomial algorithm.

1 Introduction

Let $s_1, t_1, \ldots\ldots, s_k, t_k$ be distinct vertices of a (di)graph D. The k-linkage problem is to determine whether there exists vertex-disjoint (directed) paths $P_1, \ldots, P_k$ such that P_i is from s_i to t_i for $1 \leq i \leq k$. Robertson and Seymour [15] showed that the problem is solvable in polynomial time for any fixed k in the case of undirected graphs. Fortune, Hopcroft and Wyllie [11] showed that if we impose no restriction on the input, the directed version is NP-complete already for $k = 2$. This motivates the study of subclasses of digraphs for which the problem is polynomial-time solvable.

In this paper we use the terminology and notation from [5], all digraphs are finite and without loops or parallel edges. By a path or a cycle in a digraph we always mean a directed path or cycle. A digraph is ***semicomplete*** if for all distinct vertices u, v, at least one of uv, vu is an arc. A ***tournament*** is a semicomplete digraph without 2-cycles (precisely one of the arcs uv, vu is present for all distinct u, v.). A digraph D is ***locally semicomplete*** if for every choice of 3 distinct vertices x, y, z, the presence of the arcs xz, yz or zx, zy implies that x and y are adjacent (have an arc between them). This is the same as saying that the subdigraphs $D\langle N^-(z)\rangle, D\langle N^+(z)\rangle$ induced by the in-neighbours respectively, the out-neighbours of every vertex z is a semicomplete digraph. If these two neighbourhoods are tournaments for every vertex z, then the digraph is a ***local tournament***.

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A digraph is ***quasi-transitive*** if the presence of the arcs xy, yz implies that x and z are adjacent. Finally, a digraph is ***acyclic*** if it contains no directed cycle.

Let $D_1, ..., D_h$ be a set of h disjoint digraphs. The disjoint union of $D_1, ..., D_h$ is the digraph $(\bigcup_i V(D_i), \bigcup_i A(D_i))$. The **series composition** of $D_1, ..., D_h$ is the disjoint union of these h digraphs plus all possible arcs between vertices of different D_i. The **order composition** is the disjoint union of the digraphs plus all possible arcs from D_i to D_j for every $1 \leq i < j \leq h$.
The class of digraphs recursively defined from the single vertex under the closure of these three operations is called ***directed cographs***.

The following is a partial list of known results on the k-linkage problem for digraphs:

- The problem is polynomial-time solvable for any fixed k in the class of acyclic digraphs [11].
- The problem is NP-complete already for tournaments when k is not fixed [8].
- The problem is polynomial-time solvable for any fixed k in the class of semicomplete digraphs[1] [9].
- The problem is polynomial-time solvable for any fixed k in the class of digraphs of bounded directed tree-width [14].
- Every 5-strong semicomplete digraph is 2-linked and this is best possible even for tournaments [1].

A digraph on at least $p+1$ vertices is **p-*strong*** if it remains strongly connected after deleting any set of at most $p-1$ of its vertices. Thomassen [17] proved that there is no natural number k such that every k-strong digraph D contains vertex disjoint paths P_1, P_2 such that P_i is an (s_i, t_i)-path for $i = 1, 2$ for every choice of 4 distinct vertices of D. He also proved [16] that for tournaments there exists a function $f(k)$ such that every $f(k)$-strong tournament is k-linked. This result was generalized by the first author to locally semicomplete digraphs and quasi-transitive digraphs in [3].

The purpose of this paper is twofold: Using the algorithms of [9] for semicomplete digraphs and [11] for acyclic digraphs, as well as a very recent result due to Chudnovsky, Scott and Seymour [10] we prove that the k-linkage problem is polynomially solvable for any fixed k in the classes of locally semicomplete digraphs and several classes of decomposable digraphs including quasi-transitive digraphs and directed cographs. Then in the second half of the paper we concentrate on locally semicomplete digraphs and prove that every $(2k-1)$-strong locally semicomplete digraph which is also round-decomposable (defined in Section 2.2) is k-linked. This is best possible. Finally we prove that every 5-strong locally semicomplete digraph is 2-linked. This result is best possible already for tournaments [1] and settles a conjecture from [3].

2 Further terminology and preliminaries

We use $[n]$ to denote the set of integers $\{1, 2, \dots, n\}$ and $[n]_i$ to denote the set of integers $\{i, i+1, \dots, n\}$. Let $D = (V, A)$ be a digraph. If $X \subseteq V$ then we denote by $D\langle X\rangle$ the subdigraph of D induced by X. We also use the notation $D - S$, where $S \subset V$, for the digraph $D\langle V - S\rangle$.

If there is an arc from a vertex x to a vertex y in D, then we say that x ***dominates*** y. Paths and cycles in a digraph are always meant to be directed paths and cycles. An (s,t)-path is a path whose initial (terminal) vertex is s (t), that is, a path from s to t. An (s,t)-path P is **minimal** if $D\langle V(P)\rangle$ has no shorter (s,t)-path than P.

If X and Y are disjoint subsets of vertices of D such that there is no arc from Y to X and xy is an arc for all $x \in X$ and $y \in Y$, then we say that X ***completely dominates*** Y and denote this by $X \Rightarrow Y$. We shall use the same notation when X and Y are subdigraphs of D.

For any non-strong digraph D, we can label its strong components $D_1, D_2, \dots, D_p$, $p \geq 2$, in such a way that there is no arc from D_j to D_i when $j > i$. We call this an ***acyclic ordering*** of the strong components of D. Such an ordering is not unique in general, but it is so for non-strong locally

[1] In fact, Chudnovsky et al proved this for a more general class of digraphs, but we only need the result as stated.

semicomplete digraphs, where we have $D_i \Rightarrow D_{i+1}$ for $i = 1, 2, \dots, p-1$ (see e.g. [5, Theorem 2.10.6]). We call D_1 the ***initial*** and D_p the ***terminal*** strong component of D.

If D is strong and $S \subset V(D)$ such that $D - S$ is not strong, then S is a ***separator*** of D. A separator S is ***minimal*** if no proper subset of S is a separator of D.

Lemma 2.1. *Let $D = (V, A)$ be a digraph containing distinct vertices s, t, u, v such that $D - \{u, v\}$ has 3 internally vertex-disjoint (s, t)-paths P_1, P_2, P_3, each of length at least 3 and such that the predecessor β_i of t on P_i dominates the successor α_j of s on P_j for all $i, j \in [3]$ and $D - \{s, t\}$ has a (u, v)-path. Then D contains a pair of disjoint (s, t)-, (u, v)-paths.*

Proof. Let P_1, P_2, P_3 be as in the lemma and let R be a (u, v)-path in $D - \{s, t\}$. Let x, y be chosen on R such that x (y) is the first (last) vertex on R which is also on some P_j when we traverse R from u towards v. Let $a, b \in [3]$ be such that $x \in V(P_a), y \in V(P_b)$ (possibly $a = b$). If $a = b$ and y occurs after (or is equal to) x on P_a, then let $Q = R[u, x]P_a[x, y]R[y, v]$ and otherwise let $Q = R[u, x]P_a[x, \beta_a]\alpha_b P_b[\alpha_b, y]R[y, v]$. Then Q is a (u, v)-path which is disjoint from P_c whenever $c \notin \{a, b\}$. □

Theorem 2.2. *[11] For every fixed k, there exists a polynomial algorithm for the k-linkage problem on acyclic digraphs.*

Theorem 2.3. *[9] For every fixed k there exists a polynomial algorithm for the k-linkage problem on semicomplete digraphs.*

Theorem 2.4. *[10] For every fixed pair of positive integers c, k, there exists a polynomial algorithm for the k-linkage problem on digraphs whose vertex set is partitionable into c sets inducing semicomplete digraphs.*

2.1 Decomposable digraphs

Let R be a digraph on r vertices $v_1, \dots, v_r$ and let $L_1, ..., L_r$ be a collection of distinct (but possibly isomorphic) digraphs. Then $R[L_1, ..., L_r]$ is the new digraph obtained from R by replacing v_i with L_i and adding an arc from every vertex of L_i to every vertex of L_j if and only if $v_i v_j$ is an arc of R ($1 \le i \ne j \le r$). Note that if $D = R[L_1, ..., L_r]$, then $R, L_1, ..., L_r$ are induced subdigraphs of D and we say that D is ***decomposable*** (into $R, L_1, ..., L_r$).

Let Φ be a class of digraphs. We say that a digraph D is ***totally*** Φ***-decomposable*** if either $D \in \Phi$ or $D = Q[M_1, ..., M_q]$, with $Q \in \Phi$ and M_i totally Φ-decomposable, for $i = 1, ..., q$. The ***total*** Φ***-decomposition*** of D is inductively defined as the sequence

$$\begin{cases} D, \text{ if } D \in \Phi \\ Q, L_1, ..., L_q, \text{ where } L_i \text{ is the total } \Phi\text{-decomposition of } M_i, \text{ otherwise.} \end{cases}$$

The **first layer** of the total Φ-decomposition of $D = Q[M_1, ..., M_q]$, namely $Q, M_1, ..., M_q$ is called the Φ***-decomposition*** of D.

Let $\Phi_1 :=$ { Semicomplete digraphs } $\bigcup$ { Acyclic digraphs }. The following theorem describes the structure of quasi-transitive digraphs and shows that quasi-transitive digraphs are totally Φ_1-decomposable.

Theorem 2.5. *[6] Let D be a quasi-transitive digraph.*

1. *If D is not strong, then there exist a transitive acyclic digraph T on t vertices and strong quasi-transitive digraphs $H_1, ..., H_t$ such that $D = T[H_1, ..., H_t]$*

2. *If D is strong, then there exist a strong semicomplete digraph S on s vertices and quasi-transitive digraphs $Q_1, ..., Q_s$ such that each Q_i is either a single vertex or is non-strong and $D = S[Q_1, ..., Q_s]$.*

Moreover one can find the above decompositions in polynomial time.

2.2 Structure of locally semicomplete digraphs

In this section we recall a useful structural classification of locally semicomplete digraphs (see Theorem 2.11) which will play an essential role in our proofs. To do so we need a number of definitions.

A digraph D is ***round*** if its vertices can be labelled $v_0, v_1, \ldots, v_{n-1}$ so that for each $i \in [n]$, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \ldots, v_{i-1}\}$ (subscripts are modulo n). Note that every round digraph is locally semicomplete.

Theorem 2.6. *[2] A local tournament is round if and only if each of $N^+(v)$ and $N^-(v)$ induces a transitive tournament for every vertex $v \in V(D)$.* □

It follows from Theorem 2.6 that if a local tournament D is round then there exists a unique (up to cyclic permutations) labelling of vertices of D which satisfies the properties in the definition. We refer to this as the ***round labelling*** of D.

A locally semicomplete digraph D is ***round decomposable*** if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[D_1, \ldots, D_r]$, where each D_i is a strong semicomplete digraph. We call $R[D_1, \ldots, D_r]$ a ***round decomposition*** of D.

Theorem 2.7. *[12] Let D be a non-strong locally semicomplete digraph and let $D_1, D_2, ..., D_p$ be the acyclic order of the strong components of D. Then D can be decomposed into $r \geq 2$ disjoint subdigraphs $D'_1, D'_2, ..., D'_r$ as follows:*

$$D'_1 = D_p, \quad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\{\, j \mid N^+(D_j) \cap V(D'_i) \neq \emptyset\},$$

$$\text{and} \quad D'_{i+1} = D\langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \cdots \cup V(D_{\lambda_i - 1})\rangle.$$

The subdigraphs $D'_1, D'_2, ..., D'_r$ satisfy the properties below:

(a) D'_i consists of some strong components that are consecutive in the acyclic ordering of the strong components of D and is semicomplete for $i = 1, 2, ..., r$;

(b) D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, ..., r-1$;

(c) if $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|j - i| \geq 2$. □

The unique sequence $D'_1, D'_2, ..., D'_r$ defined in Theorem 2.7 will be referred to as the ***semicomplete decomposition*** of D. For an illustration of this, see Figure 2.18 in [5].

We now turn to the structure of strong locally semicomplete digraphs.

Theorem 2.8. *[4] If a locally semicomplete digraph D is round decomposable, then it has a unique round decomposition $D = R[D_1, D_2, ..., D_r]$, $r = |V(R)|$. There exists a polynomial algorithm to decide if a given locally semicomplete digraph D has a round decomposition and to find such a decomposition if it exists.* □

Lemma 2.9. *[4] Let S be a minimal separator of the locally semicomplete digraph D. Then either $D\langle S\rangle$ is semicomplete, or $D\langle V - S\rangle$ is semicomplete.*

Let us call a separator S of a locally semicomplete digraph D ***good*** if S is a minimal separator and $D - S$ is not semicomplete.

By an ***evil*** locally semicomplete digraph we mean a locally semicomplete digraph which is not semicomplete and not round decomposable. This name illustrates that often this is structurally the most difficult class of non-semicomplete locally semicomplete digraphs.

Theorem 2.10. *[4] Let D be an evil locally semicomplete digraph. Then D is strong and satisfies the following properties.*

(a) There is a good separator S such that the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3 (and $D\langle S\rangle$ is semicomplete by Lemma 2.9);

(b) *Furthermore, for each such S, there are integers α, β, μ, ν with $\lambda_2 \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that*

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \ \ \textit{and} \ \ N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset,$$

$$\text{or} \quad N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \ \ \textit{and} \ \ N^+(D_\mu) \cap V(D_\beta) \neq \emptyset,$$

where $D_1, D_2, ..., D_p$ and $D_{p+1}, ..., D_{p+q}$ are the strong decomposition of $D - S$ and $D\langle S \rangle$, respectively, and D_{λ_2} is the initial component of D'_2 (See Figure 1). □

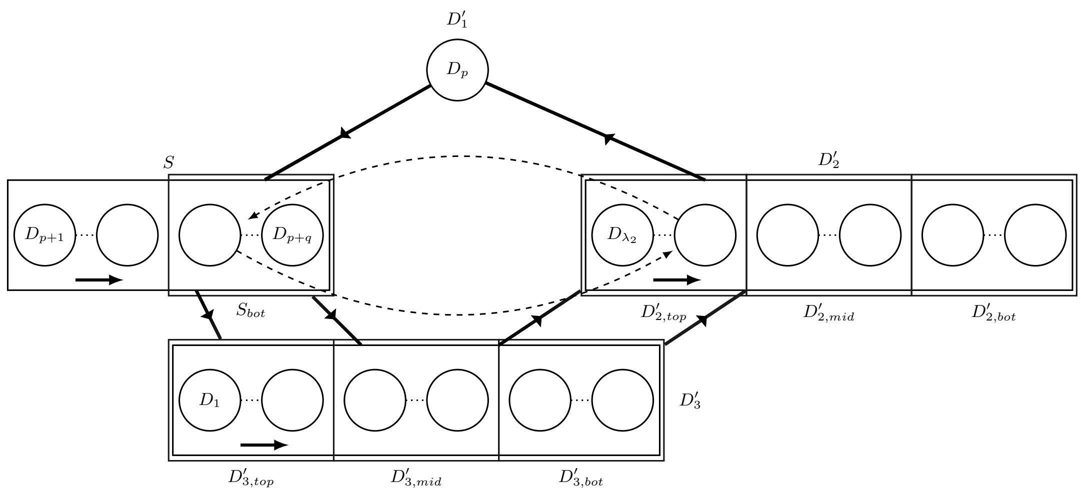


Figure 1: The good separator S and the three components D'_1, D'_2, D'_3 in the semicomplete decomposition of $D - S$. The further refinement indicated is explained in Section 6.

We can now state a full classification of locally semicomplete digraphs.

Theorem 2.11. *[4] Let D be a locally semicomplete digraph. Then exactly one of the following possibilities holds. Furthermore, there is a polynomial algorithm which decides which of the properties hold and gives a certificate for this.*

(a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, ..., D_r]$, where R is a round local tournament on $r \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, ..., r$.*

(b) *D is evil.*

(c) *D is a semicomplete digraph which is not round decomposable.* □

The following lemma gives important information about the arcs in an evil locally semicomplete digraph.

Lemma 2.12. *[4] Let D be an evil locally semicomplete digraph and let S be a good separator of D. Let $D_1, D_2, \ldots, D_p$, $p \geq 2$ be the acyclic ordering of the strong components of $D - S$ and let $D_{p+1}, \ldots, D_{p+q}$, $q \geq 1$ be the acyclic ordering of the strong components of $D\langle S \rangle$. Then the following holds:*

(i) *$D_p \Rightarrow S \Rightarrow D_1$.*

(ii) *If sv is an arc from S to D'_2 with $s \in V(D_i)$ and $v \in V(D_j)$, then*

$$D_i \cup D_{i+1} \cup \ldots \cup D_{p+q} \Rightarrow D_1 \cup \ldots \cup D_{\lambda_2 - 1} \Rightarrow D_{\lambda_2} \cup \ldots \cup D_j.$$

(iii) $D_{p+q} \Rightarrow D'_3$ *and* $D_f \Rightarrow D_{f+1}$ *for* $f \in [p+q]$*, where* $p+q+1=1$.

(iv) *If there is any arc from* D_i *to* D_j *with* $i \in [\lambda_2 - 1]$ *and* $j \in [p-1]_{\lambda_2}$*, then* $D_a \Rightarrow D_b$ *for all* $a \in [\lambda_2 - 1]_i$ *and* $b \in [j]_{\lambda_2}$.

(v) *If there is any arc from* D_k *to* D_ℓ *with* $k \in [p+q]_{p+1}$ *and* $\ell \in [\lambda_2 - 1]$*, then* $D_a \Rightarrow D_b$ *for all* $a \in [p+q]_k$ *and* $b \in [\ell]$.

3 Complexity of the k-linkage problem for decomposable digraphs

Let $D = S[M_1, ..., M_s]$ be a decomposable digraph and let P be a path in D. We say that P is ***D-internal*** if $P \subseteq M_i$ for some i, we say that P is ***D-external*** otherwise. When D is clear from the context we just call the path ***internal*** or ***external***.
Similarly we say that a pair $(s,t) \in V(D) \times V(D)$ is internal if $s, t \in V(M_i)$ for some i, and is external otherwise.

Let $\Pi = \{(s_1, t_1), ..., (s_k, t_k)\}$ be a set of k pairs of distinct terminals. A **Π-*linkage*** is a collection L of k disjoint paths P_i, $i \in [k]$ such that P_i is an (s_i, t_i)-path. If a Π-linkage L exists in the digraph D we say that L is a linkage for (D, Π)

Lemma 3.1. *Let* $D = S[M_1, ..., M_s]$ *be a decomposable digraph and* Π *a set of pairs of terminals. Then* (D, Π) *has a linkage if and only if it has a linkage whose external paths do not use any arc of* $D\langle M_i \rangle$ *for* $i \in [s]$

Proof. One implication is obvious. So let us assume that (D, Π) has a linkage and consider a linkage L that uses the least number of vertices of D.
We claim that the external paths of L have the desired property: if not consider an external path P such that $uv \in A(P) \cap A(D\langle M_i \rangle)$ for some u, v, i. Since P is external we may assume without loss of generality that there exists $z \in V(P) - V(M_i)$ such that $vz \in A(P)$ and let P' be the path obtained from P by removing the arcs uv, vz and adding the arc uz. Then $L' = L - P + P'$ is a linkage for (D, Π) with one less vertex than L, a contradiction. □

Let D be a digraph with vertex set $v_1, v_2, \ldots, v_n$ and let K be another digraph. By ***blowing up*** $\mathbf{v_i}$ ***into*** $\mathbf{K}$ ***in*** D we mean the operation that substitutes the digraph K for the vertex v_i in D, that is, creates the digraph $D' = D[\{v_1\}, \ldots, \{v_{i-1}\}, K, \{v_{i+1}\}, \ldots, \{v_n\}]$. We say that a class of digraphs Φ is ***closed with respect to blow-up*** if for any $D \in \Phi$, for every integer m and for every $v \in V(D)$, there exists a digraph K on m vertices such that blowing up of v into K results in a digraph which is still in Φ.

Lemma 3.2. *If the class* Φ *is closed with respect to the blowing-up operation,* $S \in \Phi$ *and* $D = S[M_1, ..., M_s]$*, then it is possible to replace the arcs inside the modules* M_i *with other arcs, so that the resulting digraph is in* Φ.

Proof. Starting from the digraph $S = S[v_1, ..., v_s]$, it is possible to blow up v_1 into a digraph M'_1 of size $|M_1|$ so that $S_1 := S[M'_1, v_2, ..., v_s] \in \Phi$. By iteratively repeating the blowing-up operation on $v_2, ..., v_s$ one gets a digraph $S_s = S[M'_1, ..., M'_s] \in \Phi$, where M'_i can be obtained from M_i by replacing some arcs of $D\langle M_i \rangle$ by other arcs. □

We say that a class of digraphs Φ is ***a linkage ejector*** if

1. There exists a polynomial algorithm $\mathcal{A}_\Phi$ to find a total Φ-decomposition of every totally Φ-decomposable digraph.
2. There exists a polynomial algorithm $\mathcal{B}_\Phi$ for solving the k-linkage problem on Φ. The running time depends (possibly exponentially) on k but the algorithm is polynomial when k is fixed.

3. The class Φ is closed with respect to blow-up and there exists a polynomial algorithm $\mathcal{C}_\Phi$ that given a totally Φ-decomposable digraph $D = S[M_1, ..., M_s]$, constructs a digraph of Φ by replacing the arcs inside each of the M_i's as in Lemma 3.2.

Theorem 3.3. *Let Φ be a linkage ejector. For every fixed k, there exists a polynomial algorithm to solve the k-linkage problem on totally Φ-decomposable digraphs.*

Proof. Let D be totally Φ-decomposable and let Π be a set of pairs of terminals. The following algorithm $\mathcal{M}$ decides whether (D, Π) has a linkage.

1. If $\Pi = \emptyset$, then output YES, otherwise
2. Run $\mathcal{A}_\Phi$ to find a Φ-decomposition $D = S[M_1, ..., M_s]$. If this decomposition is trivial, namely $D \in \Phi$, call $\mathcal{B}_\Phi$ to solve the problem.
3. Assume without loss of generality that $M_1, ..., M_l$ are the modules containing internal pairs from Π and let $\Pi^e \cup \Pi^i = \Pi$ be the partition of the terminal pairs into external pairs and internal pairs respectively. For every partition $\Pi_1 \cup \Pi_2 = \Pi^i$, look for two vertex disjoint linkages: one made of external paths linking the pairs in $\Pi^e \cup \Pi_1$ and one made of internal paths linking the pairs in Π_2. This is done in the following way:
   (a) If $\Pi^e \cup \Pi_1 = \emptyset$, run recursively $\mathcal{M}$ on $(D\langle M_1\rangle, \Pi_2 \cap (V(M_1) \times V(M_1))), ..., (D\langle M_l\rangle, \Pi_2 \cap (V(M_l) \times V(M_l)))$, if all of them are linked output YES
   (b) If $\Pi^e \cup \Pi_1 \neq \emptyset$, then for every choice of non-negative integers $n_1, ..., n_l \leq k$ and for every choice of $(V_1, ..., V_l)$ such that $|V_i| = n_i$ and $V(\Pi^e \cup \Pi_1) \cap V(M_i) \subseteq V_i \subseteq V(M_i) - V(\Pi_2)$, do the following:
      (i) Let $S' \in \Phi$ be the result of running the algorithm $\mathcal{C}_\Phi$ on $S[I_{n_1}, ..., I_{n_l}, M_{l+1}, ..., M_s]$, where I_r denotes the digraph on r vertices with no arcs.
      (ii) Run the algorithm $\mathcal{B}_\Phi$ on $(S', \Pi^e \cup \Pi_1)$; if this instance is linked, then run recursively the algorithm $\mathcal{M}$ on $(D\langle V(M_1) - V_1\rangle, \Pi_2 \cap (V(M_1) \times V(M_1))), ..., (D\langle V(M_l) - V_l\rangle, \Pi_2 \cap (V(M_l) \times V(M_l)))$. If these pairs are all linked output YES.
4. If all the choices of Π_1, Π_2 have been examined output NO.

We prove by induction on $|V(D)| + k$ that this algorithm is correct.
If $k = 0$ the correctness of the algorithm follows trivially, if $D \in \Phi$ the correctness follows from the correctness of the algorithm $\mathcal{B}_\Phi$. Therefore assume that we are in none of these cases.
If the algorithm outputs a YES at Step 3 it means, by induction hypothesis, that it has found linkages in a number of vertex disjoint subinstances and these linkages form altogether a linkage for (D, Π).
Now assume there exists a linkage for (D, Π) and consider a linkage L minimizing the total number of vertices. By Lemma 3.1 the external paths do not use any arcs inside the modules $M_1, ..., M_s$. Moreover by a similar argument as the one in the proof of the above lemma it can be seen that each path uses at most one non-terminal vertex per module, thus no more than k non-terminal vertices per module are used by the external paths. It follows that there exists a choice of Π_1, Π_2 and possibly $V_1, ..., V_l$ such that all the subinstances of Step 3 are linked and thus, by induction hypothesis, the algorithm outputs YES.

Let $T(n, k)$ be the running time of the main algorithm $\mathcal{M}$ on an input digraph of size n with k pairs of terminals. We show by induction on $n + k$ that the running time is $O(n^{d(k)})$, for some non decreasing fixed function $d(k)$.
If $n = 1$ or $k = 0$, the algorithm runs in constant time, so suppose this is not the case.
Let $a(k)$ be such that finding a decomposition of D and finding external and internal pairs takes time $O(n^{a(k)})$. Let $b(k)$ be such that running the algorithm $\mathcal{B}_\Phi$ at Step 2 takes time $O(n^{b(k)})$. Let $c(k)$ be such that in time $O(n^{c(k)})$ is possible to first run the algorithm $\mathcal{C}_\Phi$ and then $\mathcal{B}_\Phi$, so that finding S' and then solving the k-linkage on $(S', \Pi^e \cup \Pi_1)$ during step 3b takes time $O(n^{c(k)})$.

The running time T_a of Step 3a is $\sum_{i=1}^{l} T(n_i, k_i)$, where $n_i = |V(M_i)| < n$ (the decomposition of D is not trivial) and $k_i = |\Pi_2 \cap (V(M_l) \times V(M_l))| \leq k$ and by induction hypothesis $T_a \leq \sum_{i=1}^{l} n_i^{d(k_i)}$ which is $O(n^{d(k)})$. The running time T_b of Step 3b is at most $\left(\sum_{j=1}^{k} \binom{n}{j}\right)^k \left(\sum_{i=1}^{l} T(n_i, k_i) + O(n^{c(k))}\right)$. We define $d(k) = k^3 + a(k) + c(k)$, so T_b is $O(n^{k^2})\left(T(n, k-1) + O(n^{c(k)})\right)$ which is $O(n^{d(k)})$, as $k^2 + d(k-1) \leq d(k)$ and $k^2 + c(k) \leq d(k)$, for every $k \geq 1$.
Therefore Step 3 takes time $2^k \cdot O(n^{d(k)})$ which is $O(n^{d(k)})$ as k is constant. Thus we can conclude that $T(n, k)$ is $O(n^{a(k)}) + O(n^{b(k)}) + O(n^{d(k)})$, that is, $O(n^{d(k)})$.

□

3.1 Quasi-transitive digraphs

Recall that, by Theorem 2.5 quasi-transitive digraphs are totally Φ_1-decomposable where Φ_1 is the union of all semicomplete and all acyclic digraphs. Other classes of totally Φ_1-decomposable digraphs are extended semicomplete digraphs and directed cographs (see e.g. [7]). Moreover we have the following

Lemma 3.4. *The class Φ_1 is a linkage ejector*

Proof. We can get a polynomial algorithm for the total Φ_1-decomposition easily from a result in [5, Section 2.11], where a polynomial algorithm is given for the class of all acyclic and all semicomplete multipartite digraphs.
A polynomial algorithm to solve the k-linkage problem on semicomplete digraphs is given by Theorem 2.3. A polynomial algorithm to solve the k-linkage problem on acyclic digraphs is given by [11]. To fulfil the last condition note that if in a digraph of Φ_1 we blow up a vertex into a transitive tournament of any size we stay in the class Φ_1. Therefore, given a totally Φ_1-decomposable digraph $D = S[M_1, ..., M_s]$, the arcs of $M_1, ..., M_s$ can be replaced in order to form transitive tournaments so that the resulting digraph is in Φ_1. □

We can thus get the following corollary of Theorem 3.3

Theorem 3.5. *For every fixed k, there exists a polynomial algorithm to solve the k-linkage problem on directed cographs, quasi-transitive digraphs and extended semicomplete digraphs.*

4 Complexity of the k-linkage problem for locally semicomplete digraphs

Define

$$\Phi_2 := \{ \text{ Semicomplete digraphs } \} \bigcup \{ \text{ Round digraphs } \}$$

Round decomposable digraphs are clearly totally Φ_2-decomposable.

Theorem 4.1. *For every fixed k, there exists a polynomial algorithm to solve the k-linkage problem on round digraphs.*

Proof. Let D be a round digraph with round ordering $v_1, ..., v_n$ and let $\Pi = \{(s_1, t_1), ..., (s_k, t_k)\}$ be a set of pairs of vertices of D for which we seek a Π-linkage. Given $j \in [n-1]$, we say that an arc $v_a v_b \in A(D)$ is ***over*** $v_j v_{j+1}$ if $v_b \in \{v_{j+1}, v_{j+2}, ..., v_{a-1}\}$. Note that the removal of all the arcs over $v_j v_{j+1}$ from D leaves an acyclic digraph. We show that if (D, Π) has a Π-linkage, then there exists a linkage such that each of the paths uses at most one arc over any $v_j v_{j+1}$, namely the linkage that minimizes the total number of used vertices.
Suppose, by contradiction, that an (s_i, t_i)-path P uses two arcs over $v_j v_{j+1}$ and call them $u_1 w_1$ and $u_2 w_2$. Assume without loss of generality that the arc $u_1 w_1$ precedes $u_2 w_2$ on the path P. There are four possibilities for the relative positions of the four vertices in the round ordering: (u_1, u_2, w_1, w_2), (u_2, u_1, w_1, w_2), (u_1, u_2, w_2, w_1), (u_2, u_1, w_2, w_1). In all these cases the path P can be shortened by using, for instance, the arc $u_1 u_2$ in the first case and $u_1 w_2$ in the other cases (such

arcs exist by the round property). It follows that P uses at most one arc over v_jv_{j+1}.
A polynomial algorithm is obtained by selecting a $j \in [n-1]$, then for every choice of an ordered h-tuple of pairs $((s_{i_1}, t_{i_1}), ..., (s_{i_h}, t_{i_h}))$ (with $0 \leq h \leq k$) and every choice of arcs $u_1w_1, ..., u_hw_h$ over v_jv_{j+1} we do the following: construct the digraph D' by deleting all the arcs over v_jv_{j+1} from D and run the algorithm for k-linkage on acyclic digraphs (from Theorem 2.2) with input D' and terminals $(s_{i_1}, u_1), (w_1, t_{i_1}), ..., (s_{i_h}, u_h), (w_h, t_{i_h})$ plus the remaining original pairs. If a solution is found, construct a solution for the original instance by using the selected arcs $u_1w_1, ..., u_hw_h$. If there is no solution for each of the possible choices, it means there is no linkage using at most k arcs over v_jv_{j+1}, and hence no linkage at all.
The above algorithm results in running a polynomial number of times the polynomial algorithm from Theorem 2.2 and hence is polynomial. □

We are going to use the following result.

Theorem 4.2. *[5] There exists a polynomial algorithm for the total Φ_2-decomposition of totally Φ_2-decomposable digraphs.*

We are now ready to prove the following theorem.

Theorem 4.3. *For every fixed k, there exists a polynomial algorithm to solve the k-linkage problem on round decomposable digraphs.*

Proof. Round decomposable digraphs are totally Φ_2-decomposable, hence by Theorem 3.3 we only need to prove that Φ_2 is a linkage ejector:
By Theorem 4.2 there exists a polynomial algorithm for the total Φ_2-decomposition. By Theorems 2.3 and 4.1 there exists a polynomial algorithm to solve the k-linkage problem on Φ_2.
Given $D \in \Phi_2$ any blow up of a vertex of D into a transitive tournament will result in a digraph of Φ_2 and given a totally Φ_2-decomposable digraph $D = S[M_1, ..., M_s]$ one gets a digraph of Φ_2 by substituting the internal arcs of the M_i's with the arcs of a transitive tournament on the same number of vertices as M_i, therefore also the third condition of a linkage ejector is fulfilled.

□

Theorem 2.10 implies that if D is an evil locally semicomplete digraph, then D can be covered by 3 disjoint semicomplete subdigraphs of D (e.g. the digraphs $D_3', D_2', D\langle V(S) \cup V(D_1')\rangle$). In fact two semicomplete digraphs always suffice [13] but we only need the weaker version below. By Theorem 2.8 it is possible to decide in polynomial time whether a locally semicomplete digraph D is round decomposable. By running the algorithm from Theorem 4.3 if D is round-decomposable and the algorithm from Theorem 2.4 if D is not round-decomposable, we get the following theorem.

Theorem 4.4. *For every fixed k, there exists a polynomial algorithm to solve the k-linkage problem on locally semicomplete digraphs.*

5 k-linked round decomposable locally semicomplete digraphs

Lemma 5.1. *Every digraph D which is decomposable as $D = R[M_1, ..., M_r]$, with R round, such that $d^+(M_i) \geq 2k-1$ for $i = 1, ..., r$ is k-linked.*

Proof. We use induction on k.
For $k = 1$, the above condition, together with the round property of R, implies strong connectivity for D, so there is a path between each pair of vertices.
Assume that the statement is true for k, we prove that every digraph decomposable as $D = R[M_1, ..., M_r]$, with R round such that $d^+(M_i) \geq 2k+1$ for $i = 1, ..., q$ is $k+1$-linked. Suppose that we want a linking between $s_1, ..., s_{k+1} \in V(D)$ and $t_1, ..., t_{k+1} \in V(D)$ respectively. We construct an (s_1, t_1)-path P whose removal leaves a digraph $D' = Q[M_1', ..., M_q']$, with Q round and $d^+_{D'}(M_i') \geq 2k-1$ for $i = 1, ..., q$. Thus, by the induction hypothesis, D' is k-linked, so we are done.
The path P starts from $s_1 \in M_i$ and uses an available widest arc: an arc s_1v such that
$v \notin \{s_2, ..., s_{k+1}, t_2, ..., t_{k+1}\}$ and $v \in M_j$, with M_j maximizing the distance from M_i in the round ordering of R, namely for every l such that $M_i < M_j < M_l$ in the round ordering s_1 has no arc to M_l;

the path P keeps using widest available arcs until a vertex adjacent to t_1 is reached, in which case the path continues to t_1. Now for $i = 1, ..., r$, define $M'_i := M_i - V(P)$, let r' be the number of nonempty sets of the form M'_i, and R' be the round digraph obtained from R by removing the vertices v_i such that $M'_i = \emptyset$. The digraph $D' = R'[M'_1, ...M'_{r'}]$ is as desired. Indeed for every i there do not exist three vertices x, y, z of P inside $N^+_D(M_i)$, since, by the fact that $x, y, z \in N^+_D(M_i)$ and by the round property of R, one of the vertices dominates the other two or the three vertices belong to the same module in the decomposition. In both cases one of the arcs of P would not be the widest available or will not be directed to the target. It follows that for every i, $N^+_{D'}(M'_i)$ has size at least $2k - 1$, so D' has the desired property. □

Corollary 5.2. *Let D be a digraph on $n \geq 2k$ vertices that is not semicomplete and is decomposable as $D = R[M_1, ..., M_r]$, with R round and $M_1, ..., M_r$ semicomplete. The digraph D is k-linked if and only if it is $(2k - 1)$-strong.*

Proof. Suppose that D is $(2k-1)$-strong. Given that $D = R[M_1, ..., M_r]$ is not semicomplete, we have $r \geq 3$ and for every i, $D - N^+(M_i)$ is non-empty. It follows that for every i, $N^+(M_i)$ is a separator and hence must be of size at least $2k - 1$. Therefore D satisfies the hypothesis of Lemma 5.1 and thus D is k-linked.
Vice versa a k-linked digraph on $n \geq 2k$ vertices must necessarily be $(2k - 1)$-strong, otherwise a set of size at most $2k - 2$ would separate two vertices s, t of the digraph, so if these vertices formed the first $k - 1$ pairs and s, t the k-th pair, there is no good linkage. □

Corollary 5.2 immediately applies to round decomposable digraphs.

Theorem 5.3. *Let D be a round decomposable digraph on $n \geq 2k$ vertices that is not semicomplete. The digraph D is k-linked if and only if it is $(2k - 1)$-strong.*

Note that the decomposition of Lemma 5.1, $D = R[M_1, ..., M_r]$, need not be a proper decomposition (that is, $R \neq D$), indeed even if $|M_i| = 1$ for every i, the proof holds. Therefore the previous results hold for round digraphs too.

Theorem 5.4. *Let k be an integer. A round digraph on $n \geq 2k$ vertices is $(2k-1)$-strong if and only if it is k-linked.*

6 Every 5-strong locally semicomplete digraph is 2-linked.

We will prove the following theorem.

Theorem 6.1. *Let D be a 5-strong locally semicomplete digraph. Then D is 2-linked.*

In [1] the first author proved that every 5-strong semicomplete digraph is 2-linked and gave an example to show that this is best possible even for tournaments. Hence to prove this theorem we need only prove it for round decomposable locally semicomplete digraphs and for evil locally semicomplete digraphs. The case of round decomposable locally semicomplete digraphs is covered by Theorem 5.3 so it remains to prove the theorem for locally semicomplete digraphs that are evil.

Throughout this section we assume that D is a 5-strong evil locally semicomplete digraph. We start by a number of observations on the structure of evil locally semicomplete digraphs. These will play an important role in our proof of Theorem 6.1. Notice that given a good separator S, Lemma 2.10 implies that the semicomplete decomposition of $D - S$ has three components D'_1, D'_2, D'_3. We will introduce a more detailed refinement of D'_2, D'_3, S. In order to simplify notation we will often use D'_i as well as D_j to denote both the corresponding semicomplete digraphs as well as their vertex sets.

Define the following indexes, both of which are well-defined due to Theorem 2.10 (b)):

- $\mu \in [q]$ is the smallest index such that there is an arc from $D_{p+\mu}$ to D'_2
- $\gamma \in [p - 1]_{\lambda_2}$ is the largest index such that there is an arc from S to D_γ.

Based on these indices we now define a refinement of the semicomplete decomposition of $D - S$ which plays an important role in our proof.

The blocks of D'_2.

- $\boldsymbol{D'_{2,top}}$ is the union of the strong components of D'_2 that are dominated by all vertices of D'_3. So by Lemma 2.12 (ii) we have that $D_{\lambda_2}, \dots, D_\gamma$ are all in $D'_{2,top}$.
- $\boldsymbol{D'_{2,mid}}$ is the (possibly empty) union of those strong components of $D'_2 - D'_{2,top}$ that are dominated by some vertex of D'_3. By Lemma 2.12 (iv) we have $D_{\lambda_2 - 1} \Rightarrow D'_{2,top} \cup D'_{2,mid}$.
- $\boldsymbol{D'_{2,bot}}$ is the (possibly empty) union of those strong components of D'_2 that have no neighbour in D'_3

The blocks of S. Only one part of S plays a special role, namely $\boldsymbol{S_{bot}}$ which is the union of the strong components $D_{p+\mu}, \dots, D_{p+q}$. So, by Lemma 2.12 (ii) every vertex of S_{bot} dominates all of D'_3.

The blocks of D'_3.

- $\boldsymbol{D'_{3,top}}$ the set of strong components of D'_3 that are dominated by all vertices of S.
- $\boldsymbol{D'_{3,bot}}$ the set of strong components of D'_3 that dominate all vertices of $D'_{2,top} \cup D'_{2,mid}$. Note that, by Lemma 2.12 (iv), $D_{\lambda_2 - 1}$ is contained in $D'_{3,bot}$.
- $\boldsymbol{D'_{3,mid}}$ the (possibly empty)[2] set of strong components of $D'_3 - D'_{3,top} - D'_{3,bot}$.

With these definitions we see from Theorem 2.10 that

There is at least one arc sv from S_{bot} to $D'_{2,top}$ and at least one arc from $D'_{2,top}$ to S_{bot}. (1)

Notice that, by Lemma 2.12 and the definition of $D'_{2,bot}$, there is no arc from D'_3 to $D'_{2,bot} \cup D'_1 \cup S$, so

$$\text{The set } S^* = D'_{2,top} \cup D'_{2,mid} \text{ is also a good separator.} \tag{2}$$

Furthermore, if we reverse all arcs, obtaining the 5-strong evil locally semicomplete digraph $\overleftarrow{D}$ and interchange the names of s_i, t_i we get an equivalent instance. We shall use this fact several times. In particular, whenever convenient, we may consider the semicomplete decomposition of any of the four choices $D - S, D - S^*, \overleftarrow{D} - S, \overleftarrow{D} - S^*$. See Figures 2 and 3. We first describe the structure of the semicomplete decomposition of the last three digraphs above. Below $D_1, D_2, \dots, D_p$, respectively, $D_{p+1}, \dots, D_{p+q}$ always denote the acyclic orderings of the strong components of $D - S$, respectively $D\langle S \rangle$.

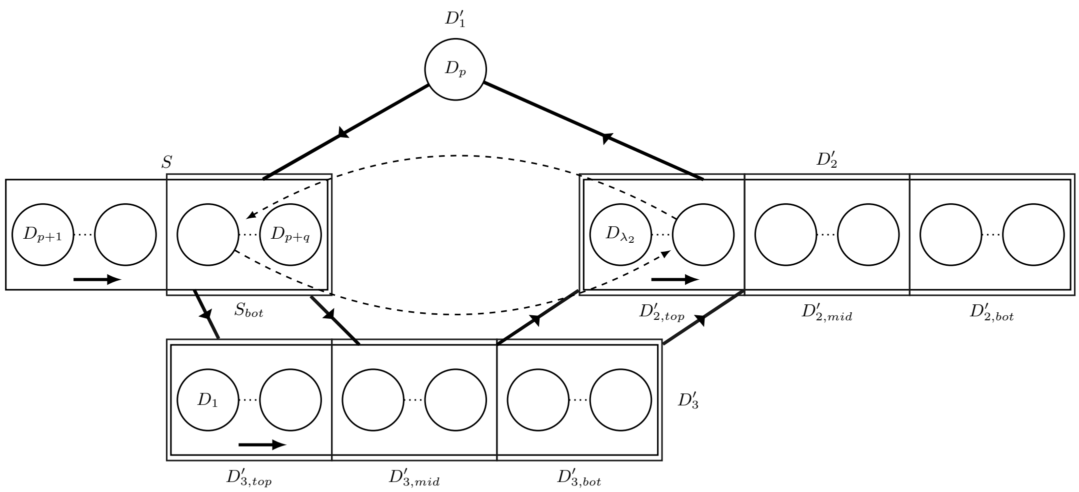
The semicomplete decomposition after reorienting all arcs

Clearly S is a minimal separator of D if and only if S is a minimal separator of the digraph $\overleftarrow{D}$.

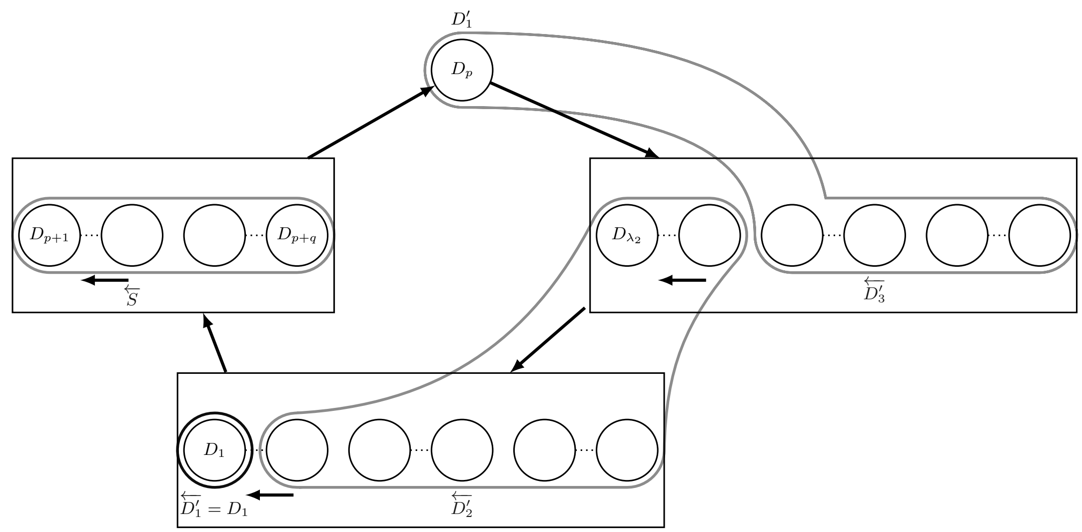
Lemma 6.2. *Let the labelling of D be as described in Theorem 2.10, then reorienting the arcs of D and letting S be the good separator, the following will be the semicomplete decomposition $\overleftarrow{D} - S$.*

1. $\overleftarrow{D'_1} = D_1$*, that is, the first strong component of* $D'_{3,top}$.
2. $\overleftarrow{D'_2} = \overleftarrow{D'}_{2,top} \cup \overleftarrow{D'_3} - \overleftarrow{D_1}$.
3. $\overleftarrow{D'_3} = D'_{2,mid} \cup D'_{2,bot} \cup D'_1$.

[2] Note that we may have $D'_{3,top} \cap D'_{3,bot} \neq \emptyset$ (in which case $D'_{3,mid} = \emptyset$) but this has no influence on our proof.



(a) D with the blocks marked and the pair of evil arcs showed as dotted arcs.

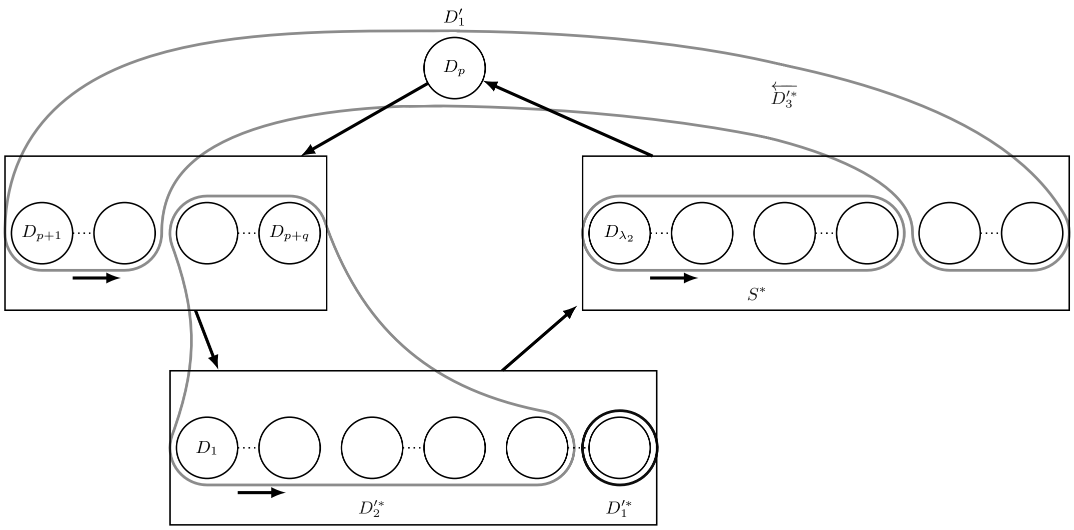


(b) The semicomplete decomposition of $\overleftarrow{D}$ with respect to the separator $\overleftarrow{S}$.The sets indicate the four parts.

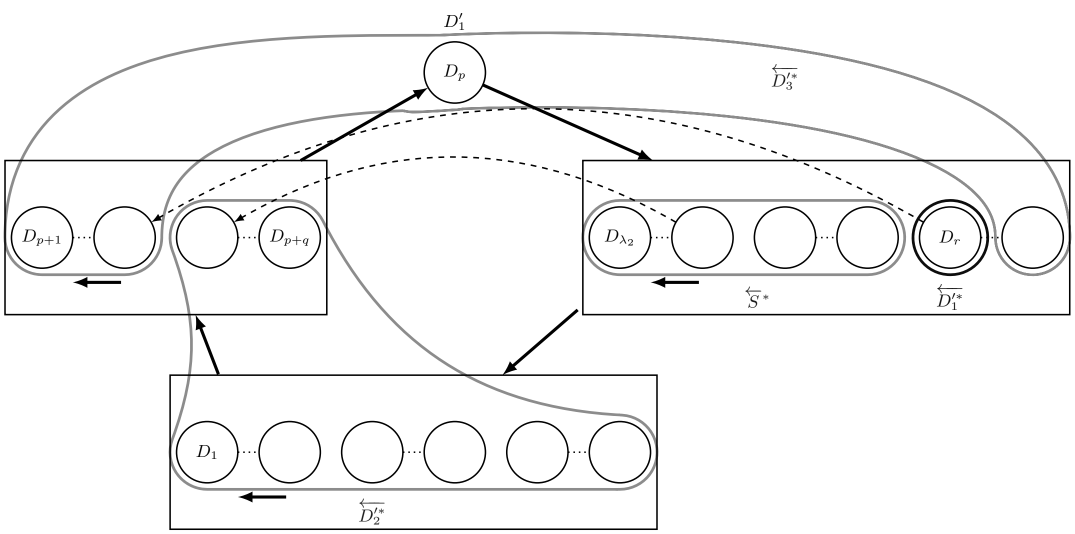
Figure 2

Proof. By the definition of the semicomplete decomposition, $\overleftarrow{D'_1}$ must be a strong component of $\overleftarrow{D} - S$ and this dominates S in $\overleftarrow{D}$, so $\overleftarrow{D'_1} = D_1$. For 2. notice first that, by the definition of the semicomplete decomposition, $\overleftarrow{D'_2}$ is formed by those strong components of $\overleftarrow{D}$ that dominate $\overleftarrow{D'_1}$. Thus, as the first component of D'_3 dominates all other vertices of D'_3 in D, clearly when reorienting the arcs, all other vertices of D'_3 dominate $\overleftarrow{D'_1}$. Secondly, by Lemma 2.12 (ii) $D'_{2,top}$ is dominated by all vertices of D'_3 implying that $\overleftarrow{D'}_{2,top}$ dominates $\overleftarrow{D'_1}$ in $\overleftarrow{D}$. By the definition of $D'_{2,mid}$ and Lemma 2.12 (iv), no vertex in $D'_{2,mid}$ is dominated by a vertex in D_1 in D. 3. follows from Theorem 2.10 since the semicomplete decomposition of an evil locally semicomplete digraph has exactly three components. $\square$

Decomposition based on the minimal separator $S^* = D'_{2,top} \cup D'_{2,mid}$ **in** D



(a) The semicomplete decomposition of D with respect to S^*. The sets indicate the four parts.



(b) The semicomplete decomposition of $\overleftarrow{D}$ with respect to the separator $\overleftarrow{S^*}$. The sets indicate the four parts. The dotted arcs are in D and indicate the arcs with head y respectively y', where y,y' are as defined before Lemma 6.4

Figure 3

Lemma 6.3. *Let the labelling of D be as described above. With S^* as the good separator the semicomplete decomposition of $D - S^*$ is given by*

1. $D'^*_1 = D_{\lambda_2 - 1}$ *is the last strong component of* $D'_{3,bot}$.

2. $D_2'^{*} = \widetilde{S} \cup D_3' - D_1'^{*}$.

3. $D_3'^{*} = D'_{2,bot} \cup D'_1 \cup S - \widetilde{S}$.

where $\widetilde{S} \subseteq S$ is the set of vertices of S that dominate all vertices of $D'_{3,bot}$. Notice that, by Lemma 2.12 and the definition of S_{bot} we have $S_{bot} \subseteq \widetilde{S}$. See Figure 3.

Proof. The proof of this lemma is again based on the structural information from Lemma 2.12 and very similar to the proof above. □

Decomposition based on the minimal separator S^* in $\overleftarrow{D}$

If $D'_{2,bot} \neq \emptyset$, then let $r \in [p-1]_{\lambda_2}$ be the smallest index such that $D_r \subseteq D'_{2,bot}$ and let $X = D_{r+1} \cup \ldots \cup D_{p-1}$. Now, since every vertex of $D'_{2,top}$ dominates every vertex of D_r we get that $D_r \Rightarrow D_y$ for every $y \in [p+q]_{p+1}$ such that there is an arc from $D'_{2,top}$ to D_y and there is at least one such y by (1). Let $y' \in [p+q]_{p+1}$ be the largest index such that there is an arc from D_r to $D_{y'}$ and let $Y = D_{p+1} \cup \ldots \cup D_{y'}$. If r is not defined above, then let $X = \emptyset$ (note that X is also empty if $r = p-1$) and $Y = \emptyset$.

Now it is easy to see that the following holds.

Lemma 6.4. *The semicomplete decomposition $\overleftarrow{D}_1^*, \overleftarrow{D}_2^*, \overleftarrow{D}_3^*$ of $\overleftarrow{D} - S^*$ has the following form. If $D'_{2,bot} = \emptyset$ ($S^* = D'_2$) then we have $\overleftarrow{D}_1^* = D_p$, $\overleftarrow{D}_2^* = S$ and $\overleftarrow{D}_3^* = D'_3$. Otherwise, r is defined and we have*

1. $\overleftarrow{D}_1^* = D_r$.

2. $\overleftarrow{D}_2^* = X \cup D_p \cup Y$.

3. $\overleftarrow{D}_3^* = D'_3 \cup S - Y$.

See Figure 3. □

6.1 Proof of Theorem 6.1

Proof. Suppose for a contradiction that D is a 5-strong evil locally semicomplete digraph which is not 2-linked and let s_1, s_2, t_1, t_2 be distinct vertices such that D has no pair of disjoint (s_1,t_1)-,(s_2,t_2)-paths. We will prove a series of claims that eventually lead to a contradiction. We denote by $\boldsymbol{D^i} = D\langle V - \{s_{3-i}, t_{3-i}\}\rangle$ for $i \in [2]$.

Claim 1. *D^i has no (s_i,t_i)-path of length less than 4.*

Proof of claim. If there exists an (s_i,t_i)-path P in D^i of length at most 3 for $i \in [2]$, then $D - V(P)$ is strong and hence contains an (s_{3-i}, t_{3-i})-path, a contradiction. ◊

Claim 2. *D^i is an evil locally semicomplete digraph for $i \in [2]$*

Proof of claim. Suppose first that D^i is semicomplete for $i = 1$ or $i = 2$. As D is 5-strong each D^i is 3-strong and hence contains 3 internally disjoint (s_i,t_i)-paths for $i = 1, 2$. By Claim 1 these paths all have length at least 3. Then Lemma 2.1 and Claim 1 implies that D has the desired paths, a contradiction. Hence, by Theorem 2.11, we may assume w.l.o.g. that D^1 is round decomposable and not semicomplete. Let $s_2'(t_2')$ be an out-neighbour of s_2 (an in-neighbour of t_2) in $V - \{s_1, t_1\}$. As D^1 is 3-strong it follows from Theorem 5.3 that D^1 has disjoint paths P, Q such that P is an (s_1,t_1)-path and Q is an (s_2', t_2')-path. But now P and s_2Qt_2 are the desired paths, a contradiction. ◊

Claim 3. *$t_is_i \in A(D)$ for $i \in [2]$*

Proof of claim. By Claim 1 D contains none of the arcs s_1t_1, s_2t_2. Assume there is no arc between s_i and t_i and consider the shortest (s_i, t_i)-path P in D^i. We know by Claim 1 that this path has length at least 4, so let $s_iv_2v_3v_4$ be the beginning of this path. As the independence number of D^i is 2 (Every evil locally semicomplete digraph has independence number 2, see Corollary 3.14 in [2]) and $s_it_i \notin A$, v_3 either dominates s_i or is dominated by t_i. Assume v_3 dominates s_i (the proof is similar in the other case). Since P is shortest possible, there is no arc v_jv_r where $r > j + 1$ and now we get, using that D is locally semicomplete, that $v_4s_i, v_5s_i, ...$ are all arcs, implying that t_is_i is an arc, a contradiction. ◊

Claim 4. *D^i contains an (s_i, t_i)-path P of length 4 such that $D\langle V(P)\rangle$ is not semicomplete for $i \in [2]$.*

Proof of claim. By Claim 2, D^i is an evil locally semicomplete digraph. Let $\tilde{S}$ be a good separator of D^i and let $\tilde{D}'_1, \tilde{D}'_2, \tilde{D}'_3$ be the semicomplete decomposition of $D^i - \tilde{S}$. First observe that Claim 1 and Lemma 2.12 imply that s_i, t_i both belong to the same subset among $\tilde{S}, \tilde{D}'_1, \tilde{D}'_2, \tilde{D}'_3$. Using Claim 1 this is easy to see for $\tilde{S}, \tilde{D}'_2, \tilde{D}'_3$ so we give the argument only when $s_i \in \tilde{D}'_1$. If $t_i \notin \tilde{D}'_1$, then, by Claim 3, $t_i \in \tilde{D}'_2$. If $t_i \in \tilde{D}'_{2,top} \cup \tilde{D}'_{2,mid}$, then s_isdt_i is a path violating Claim 1, where $s \in \tilde{S}_{bot}$ and $d \in \tilde{D}'_{3,bot}$. Finally, if $t_i \in D'_{2,bot}$ then s_isvt_i, where sv is an arc from S_{bot} to $D'_{2,top}$, violates Claim 1.

We now describe how to find the desired path P of length 4 such that $D\langle V(P)\rangle$ is not semicomplete in each of the 4 cases

- If $s_i, t_i \in \tilde{S}$, then $P = s_iabct_i$ where $a \in \tilde{D}'_{3,top}, b \in \tilde{D}'_{2,top}, c \in \tilde{D}'_1$ and a, c are not adjacent.
- If $s_i, t_i \in \tilde{D}'_3$, then $P = s_iabct_i$ where $a \in \tilde{D}'_{2,top}, b \in \tilde{D}'_1, c \in \tilde{S}_{bot}$ and s_i, b are not adjacent.
- If $s_i, t_i \in \tilde{D}'_1$, then $P = s_iabct_i$ where $a \in S, b \in \tilde{D}'_{3,top}, c \in \tilde{D}'_{2,top}$ and s_i, b are not adjacent.
- If $s_i, t_i \in \tilde{D}'_2$ and t_i is dominated by some vertex c of $\tilde{D}'_3$, then $P = s_iabct_i$ where $a \in \tilde{D}'_1, b \in \tilde{S}_{bot}$, c is any in-neighbour of t_i in $\tilde{D}'_3$ and a, c are not adjacent. So we may assume that t_i is not adjacent to any vertex of $\tilde{D}'_3$ and hence $t_i \in \tilde{D}'_{2,bot}$. Let uv be an arc from $\tilde{D}'_{2,top}$ to $\tilde{S}_{bot}$ (this exists by Theorem 2.10). Now we have that uv, ut_i are arcs so Claim 1 (or Lemma 2.12 and the definition of the blocks of $\tilde{D}'_2$) implies that t_i dominates v. Thus, by Claim 3, s_i and v are adjacent and since $t_i \notin \tilde{D}'_{2,top}$, implying that $s_i \notin \tilde{D}'_{2,top}$, we have that s_i dominates v. Thus $P = s_ivaut_i$ where $a \in \tilde{D}'_{3,top}$ and a and t_i are not adjacent.

◊

Claim 5. *For each $i = 1, 2$ and for every (s_i, t_i)-path P of length 4 such that $D\langle(V(P)\rangle$ is not semicomplete, the digraph $D - V(P)$ is semicomplete. In particular, s_i, t_i are adjacent to all but at most 3 vertices in D^i for $i = 1, 2$.*

Proof of claim. The first part of the claim follows from Lemma 2.9 and the second part follows from Claim 4: consider an (s_i, t_i)-path P in D^i such that $D\langle V(P)\rangle$ is not semicomplete. This means that s_{3-i}, t_{3-i} are adjacent to all vertices of $V - V(P)$. ◊

Claim 6. *(a) $V(D) - \{s_1, s_2, t_1, t_2\}$ does not contain two vertices u, v such that s_1u, s_1v, s_2u, s_2v are all arcs or ut_1, ut_2, vt_1, vt_2 are all arcs.*

(b) $V(D) - \{s_1, s_2, t_1, t_2\}$ does not contain three vertices u, v, w such that $s_iu, s_iw, wv, s_{3-i}u, s_{3-i}v$ are all arcs or $uw, wt_i, vt_i, ut_{3-i}, vt_{3-i}$ are all arcs.

Proof of claim. This follows easily from Menger's theorem since D is 5-strong. For example, if $s_iu, s_iw, wv, s_{3-i}u, s_{3-i}v$ are all arcs, then any pair of two disjoint paths joining $\{u, v\}$ to $\{t_1, t_2\}$ in $D - \{s_1, s_2, w\}$ can be extended to the desired paths. ◊

Now we are ready for the core of the proof. We first show a number of results on the distribution of the terminals with respect to any good separator. These observations then allow us to prove that we may select a good separator S so that D'_1 has no terminals and then use this to obtain further structure, eventually leading to the desired contradiction.

Claim 7. *For every good separator S we have $|V(D_3') \cap \{s_1, s_2\}| \leq 1$ and $|V(D_3') \cap \{t_1, t_2\}| \leq 1$.*

Proof of claim. Suppose we have $s_1, s_2 \in D_3'$. Then Claim 6 implies that $|D'_{2,top}| = 1$ and that at least one of s_1, s_2, w.l.o.g. s_1 is not in $D'_{3,bot}$ (recall that $D'_{3,bot} \Rightarrow S^*$). Furthermore, s_1, s_2 have no common out-neighbour in D_3' so s_2 must be in $D'_{3,bot}$. Thus s_1 dominates s_2 and every out-neighbour of s_1 in D_2' is also an out-neighbour of s_2. Claim 6 and Lemma 2.12 (iv) now implies that s_1 has an out-neighbour w in $D_3' - \{s_2, t_2\}$ which dominates a vertex in $D'_{2,mid}$ but then (b) in Claim 6 is violated. This proves the first part of the claim and the second part follows by considering $\overleftarrow{D}$ and the good minimal separator $S^* = D'_{2,top} \cup D'_{2,mid}$ (See Figure 3). ◊

Claim 8. *For every good separator S we have $|V(D_1') \cap \{s_1, s_2\}| \leq 1$ and $|V(D_1') \cap \{t_1, t_2\}| \leq 1$.*

Proof of claim. This follows from Claim 7 by considering the separator S^* in D. ◊

Claim 9. *For every good separator S we have $|V(D_3') \cap \{s_i, t_i\}| \leq 1$ for $i \in [2]$.*

Proof of claim. Suppose w.l.o.g that $s_1, t_1 \in D_3'$ and let P be a minimal (s_2, t_2)-path of length 4 such that $D\langle V(P)\rangle$ is not semicomplete and hence $D - V(P)$ is semicomplete (Claims 4 and 5). This implies that $V(D_1') \subseteq V(P)$. Note that P is not contained in D_1' as this is a semicomplete digraph. We start by showing that $D - V(P)$ contains an arc from $D_2' - V(P)$ to $S - V(P)$. Assume this is not the case. As $D - V(P)$ is semicomplete and $D_3' - V(P) \neq \emptyset$ we can conclude (by minimality of P) that $|D_1'| = 1$ and P uses this vertex of D_1'. Hence, as D is 5-strong, there are 4 disjoint arcs from D_2' to S. By our assumption, all 4 arcs are incident to $V(P)$, implying that the remaining four vertices of P are in $D_2' \cup S$. We will denote the two vertices of $P - (\{s_2, t_2\} \cup D_1')$ by x_1, x_2 and denote the two arcs (among the 4 disjoint arcs) adjacent to these a_1, a_2, respectively.

Observe that if $x_i \in D'_{2,top}$ for $i = 1$ or $i = 2$, then we conclude by Claim 1 that $t_1 \notin D'_{3,top}$, as otherwise we have the path $s_1 a_i t_1$.

We have $|V(P) \cap S| \leq 3$ as otherwise $D_1' = \{s_2\}$ as s_2 is the only vertex of P without an in-neighbour, but this contradicts that t_2 dominates s_2. Similarly $|V(P) \cap D_2'| \leq 3$. Now P will use at least one arc uv from S_{bot} to $D'_{2,top}$: if P does not use the arc we either find that P is not using D_1' (contradicting that $D - V(P)$ is semicomplete) or P is not minimal (by the easy observation that no minimal path can use both an arc from D_2' to S and a vertex in D_1').

If $t_2 = v$, then $s_2 \in S_{bot}$ as otherwise one of the paths $s_2 u t_2$, $s_2 d u t_2$, where $D_1' = \{d\}$, violates Claim 1. As P uses the vertex of D_1' and has length 4 it will use another arc $s_2 x_1$ from S_{bot} to $D'_{2,top}$ implying, by the remark above, that $t_1 \notin D'_{3,top}$. But then we have the (s_2, t_2)-path $s_2 q t_2$ with $q \in D'_{3,top}$ a contradiction. If $s_2 = u$ then we conclude similarly that $t_2 \in D'_{2,top}$ and obtain a contradiction.

Hence we have (possibly after relabelling x_1, x_2) that $x_1 = u, x_2 = v$ and $x_1 x_2 \in A(P)$. As $D - V(P)$ is semicomplete the tail $t(a_1)$ of a_1 is adjacent to the head $h(a_2)$ of a_2 and as we have assumed that $D - V(P)$ does not contain an arc from D_2' to S we have the arc $h(a_2)t(a_1)$. But this implies that $t(a_1) \in D'_{2,top}$ contradicting Claim 1 as $s_1 a_1 t_1$ is a short path in D^1.

Thus there exists an arc $u_3 v_3$ from $D_2' - V(P)$ to $S - V(P)$. As $D - V(P)$ is semicomplete there also exists an arc $u_1 v_1$ from the terminal strong component of $S - V(P)$ to the initial component of $D_3' - V(P)$ and an arc $u_2 v_2$ from the terminal component of $D_3' - V(P)$ to the initial component of $D_2' - V(P)$. Let Q be a (v_3, u_1)-path in $S - V(P)$ and let R be a (v_2, u_3)-path in $D_2' - V(P)$. Then $D\langle (V(D_3') - V(P)) \cup V(Q) \cup V(R)\rangle$ is strong and contains s_1, t_1, contradicting that $D - V(P)$ has no (s_1, t_1)-path. ◊

Claim 10. *For every good separator S we have $|V(D_3') \cap \{s_1, s_2, t_1, t_2\}| \leq 1$*

Proof of claim. Assume this is not the case, then by Claim 9 and Claim 7 we may assume w.l.o.g. that $D_3' \cap \{s_1, s_2, t_1, t_2\} = \{s_2, t_1\}$. This implies that $s_1 \in D_2'$ and $t_2 \in S$. By Claim 1 we have $S_{bot} = \{t_2\}$ and $D'_{2,top} = \{s_1\}$ (otherwise we would have an (s_i, t_i)-path of length 3 in D^i for $i = 1$, respectively $i = 2$). But now $\overleftarrow{D}$ and S^* contradict Claim 9. ◊

Claim 11. *For every good separator S we have $|V(D_1') \cap \{s_1, s_2, t_1, t_2\}| \leq 1$*

Proof of claim. This follows from Claim 10 applied to $\overleftarrow{D}$ and S. ◊

Claim 12. *For every good minimal separator S we have $|V(S) \cap \{s_1, s_2\}| \leq 1$ and $|V(S) \cap \{t_1, t_2\}| \leq 1$.*

Proof of claim. Suppose we have $s_1, s_2 \in S$. As in the proof of Claim 7 we may assume that $S_{bot} = \{s_2\}$, $D'_{3,top} = \{a\}$ for some $a \in V$ and a is the only out-neighbour of s_1 in D'_3. If $D'_3 \neq \{a\}$ we consider a path Q from s_1 to $D'_3 - \{a\}$ in $D - \{s_2, t_2, a, t_1\}$. By Lemma 2.12, Q will start with an arc $s_1 w$ inside S and then an arc from w to D'_3 but then (b) in Claim 6 is violated for s_1, s_2. Hence $D'_3 = \{a\}$ and now Claim 1 implies that $t_1, t_2 \notin D'_1 \cup D'_2$ (if $t_i \in D'_1 \cup (D'_2 - D'_{2,top})$ then we get a path $s_i avt_i$ where $v \in D'_{2,top}$). Hence we have $t_1, t_2 \in S$. Now consider a path $s_2 abct_2$ where $b \in D'_{2,top}$ and $c \in D'_1$. In $D - \{s_2, t_2, a\}$ there are two internally disjoint (s_1, t_1)-paths R_1, R_2 and they both contain an arc $u_j v_j$ with $u_j \in S, v_j \in D'_{2,top}$ and $u_j, v_j \in V(R_j)$, $j \in [2]$. Without loss of generality c is not in R_1 so R_1 is disjoint from $s_2 av_2 ct_2$, a contradiction. The second part of the claim follows by considering $\overleftarrow{D}$ and S. ◊

Claim 13. *For every good separator S we have $|V(D'_{2,mid} \cup D'_{2,bot}) \cap \{s_1, s_2, t_1, t_2\}| \leq 1$*

Proof of claim. This follows from Claim 10 by considering $\overleftarrow{D}$ and S. ◊

Claim 14. *For every good separator S we have $|V(D'_2) \cap \{s_1, s_2\}| \leq 1$*

Proof of claim. Suppose $s_1, s_2 \in D'_2$. Then Claim 6 implies that $D'_1 = \{p\}$ for some $p \in V$. By Claim 12 applied to S^* in D we have w.l.o.g that $s_2 \in D'_{2,bot}$ and then Claim 3 implies that $t_2 \in D'_2$. Claim 7 applied to D and S^* implies that $s_1 \in D'_{2,top} \cup D'_{2,mid}$. Now Lemma 2.12 and Claims 1, 3 imply that $t_1 \in D'_{2,top} \cup D'_{2,mid}$ holds. Then $t_2 \in D'_{2,bot}$ or we contradict Claim 12 with S^*. However this contradicts Claim 13. ◊

Claim 15. *For every good separator S we have $|V(D'_2) \cap \{t_1, t_2\}| \leq 1$*

Proof of claim. Suppose $t_1, t_2 \in D'_2$. Claim 3 implies that $V(D'_3) \cap \{s_1, s_2\} = \emptyset$ and Claims 12, 13 applied to D, S^* implies that precisely one of t_1, t_2 is in S^*. W.l.o.g. $t_2 \in S^*$ and $t_1 \in D'_{2,bot}$. Now Claim 10 (applied to D, S^*) and Claim 12 imply that $|S \cap \{s_1, s_2\}| = 1$. Then Claim 1 implies that we have $S \cap \{s_1, s_2\} = \{s_1\}$ and $D'_{2,top} \subseteq \{s_2, t_2\}$. Thus we have 3-internally disjoint (s_1, t_1)-paths in $D - \{s_2, t_2\}$ each of which use a vertex in D'_3 so $|D'_3| \geq 3$. By Claim 4 there exists an (s_2, t_2)-path P of length 4 in D^2 which does not induce a semicomplete digraph and hence, by Lemma 2.9, t_1 is adjacent to all vertices of $V - V(P)$. However, this is impossible since P cannot use all vertices of D'_3. ◊

Claim 16. *There exists a good separator such that, possibly after reversing all arcs and renaming the terminals, we have $|V(D'_1) \cap \{s_1, s_2, t_1, t_2\}| = 0$.*

Proof of claim. Assume this is not the case. Then considering the separators S, S^* in both D and $\overleftarrow{D}$ (four possibilities) we get that there are terminals in D_1 and in $D_{\lambda_2 - 1}$ and if $S^* \neq D'_2$ then there is also a terminal in $D'_{2,bot}$. Now it follows from the fact that t_i dominates s_i that we may choose a good separator and one of $D, \overleftarrow{D}$ such that s_i is the only terminal in D'_1. W.l.o.g. $s_2 \in D'_1$ and then $t_2 \in D'_2$. By Claim 10 we have $D'_3 = D_1$ and exactly one of s_1, t_1 is in D_1.

Consider first the case where $s_1 \in D_1$ then $t_1 \in S$ (Claim 3) and $D'_1 = \{s_2\}$ (by Claim 1 and the fact that $D_1 \Rightarrow S^*$). By Claim 10 applied to $\overleftarrow{D}$ and S^* $t_1 \notin S_{bot}$ as otherwise $\overleftarrow{D^*_3}$ contains two terminals. This and Claim 1 implies that $t_2 \in D'_{2,top}$ and $D_1 = \{s_1\}$. Hence by the choice of S, $D'_{2,bot} = \emptyset$. Let P_1, P_2, P_3 be internally disjoint minimal (s_1, t_1)-paths in D^1 and let H_1, H_2, H_3 be internally disjoint minimal (s_2, t_2)-paths in D^2. Then each P_i has the form $P_i = s_1 u_i Q_i[v_i, t_1]$, where $u_i \in D'_2, v_i \in S$ and Q_i is a path in S and each H_i has the form $H_i = s_2 p_i R_i[q_i, t_2]$, where $p_i \in S_{bot}$, $q_i \in D'_2$ and R_i is a path in D'_2. Since $t_1 \notin S_{bot}$ the path Q_i avoids all of $\{p_1, p_2, p_3\}$ and hence P_i will be disjoint from two of the paths H_j, $j \in [3]$, a contradiction.

Thus we may assume that $t_1 \in D_1$ and thus, by Claim 3, $s_1 \in S^*$. Suppose first that $S^* \neq D'_2$. Then, by the choice we made for S in the beginning of the proof of the claim, we have $t_2 \in D'_{2,bot}$. Now we obtain a contradiction to Claim 10 applied to D and S^*.

Consider the remaining case $S^* = D'_2$. We must have $D_1 = \{t_1\}$ as otherwise $s_2 spt_2$ with $s \in S_{bot}, p \in D_1 - \{t_1\}$ violates Claim 1. Consider 3 internally disjoint minimal (s_2, t_2)-paths Z_1, Z_2, Z_3

in D^2. They have the form $Z_i = s_2\alpha_i F_i[\beta_i, t_2]$, where $\alpha_i \in S$ and F_i is a path in D'_2. By Claim 1, each Z_i has length at least 4. Consider an (s_1,t_1)-path R in D^1 and let x be the first vertex of R which is also on some Z_i. W.l.o.g. $x \in Z_1$. If Z_1 has length at least 5 then the predecessor ϵ_1 of t_2 on F_1 dominates β_1 (as F_1 is a minimal (β_1, t_2)-path in the semicomplete digraph D'_2). But then the minimality of Z_1 implies that $\epsilon_1\alpha_1$ is an arc and now the path $R[s_1,x]F_1[x,\epsilon_1]\alpha_1 t_1$ intersects none of Z_2, Z_3, a contradiction. Hence Z_1 has length precisely 4 and there is no arc between ϵ_1 and α_1. Since D has no pair of disjoint (s_1,t_1)-,(s_2,t_2)-paths, R must also intersect Z_2 and Z_3 and w.l.o.g. it intersects Z_2 before Z_3. Using the same argument as above, we conclude that Z_2 has length 4 and there is no arc between α_2 and ϵ_2, the predecessor of t_2 on F_2. Now Claim 4 applied to Z_2 implies that there is an arc between α_1 and ϵ_1, contradicting what we just concluded above. This completes the proof of Claim 16. $\diamondsuit$

Now choose S and either D or $\overleftarrow{D}$ (and call the result D) so that D'_1 has no terminal and D'_3 has the minimum number of terminals. We will show in the next claim that this number will be zero.

Claim 17. *With the choice of separator above we have* $|V(D'_1) \cap \{s_1,s_2,t_1,t_2\}| = 0$ *and* $|V(D'_3) \cap \{s_1,s_2,t_1,t_2\}| = 0$

Proof of claim. Suppose $|V(D'_3) \cap \{s_1,s_2,t_1,t_2\}| = 1$. Then w.l.o.g. precisely one of s_1,t_1 is in D'_3. Consider first the case where $s_1 \in D'_3$ and then $t_1 \in S$, as t_1 dominates s_1. Claim 9 implies that $t_1 \notin S_{bot}$. Claim 12 then implies that $t_2 \in D'_2$ and by Claim 1 we have $D'_{2,top} \subseteq \{s_2,t_2\}$ and s_1 has no (out-)neighbour in $D'_2 - \{s_2,t_2\}$. Then $|D'_3| \geq 4$ as D is 5-strong and now $s_2 \in D'_2$ by Claim 1. Now $Q = s_2abct_2$ with $a \in D'_1$, $b \in S_{bot}$ and $c \in D'_3 - s_1$ is a path of length 4 such that $D\langle V(Q)\rangle$ is not semicomplete. Hence $D - V(Q)$ is semicomplete. This implies that s_1 dominates all of $D'_2 - \{s_2,t_2\}$, contradicting our conclusion above. Hence we have $V(D'_3) \cap \{s_1,s_2,t_1,t_2\} = \{t_1\}$. The claims already established now imply that $s_1 \in D'_2$ and $s_2 \in S$. Claim 1 implies that $S_{bot} \subseteq \{s_2,t_2\}$ but then we have at least two terminals in $\overleftarrow{D^*_3}$ contradicting Claim 10.

$\diamondsuit$

Now we are ready for the conclusion of the proof. So far we have established that we may choose a good minimal separator S and a possible reorientation of D so that with respect to this S (and orientation) we have (possibly after exchanging the indices) that $t_1, s_1 \in D'_2$ and $t_2, s_2 \in S$. Then Claim 1 implies that there is no arc from s_1 to $S - \{s_2,t_2\}$ and no arc from s_2 to $D'_2 - \{s_1,t_1\}$. As D is 5-strong, each of the digraphs D^i, $i \in [2]$ are 3-strong and hence have 3 internally disjoint (u,v)-paths for every choice of distinct vertices u, v.

Case 1. $D_1 \Rightarrow D'_2$

Pick $w \in D'_2 - \{s_1,t_1\}, s \in S - \{s_2,t_2\}, q \in D'_1$ and $p \in D'_{3,top}$. Then $P = s_2pwqt_2$ and $Q = s_1qspt_1$ are both paths that induce non-semicomplete digraphs. Let P_1,P_2,P_3 be internally disjoint minimal (s_1,t_1)-paths in D^1 and Q_1,Q_2,Q_3 be internally disjoint minimal (s_2,t_2)-paths in D^2. Then each P_i intersects P and each Q_j intersects Q. Consider first P_1,P_2,P_3 and let α_i, β_i denote, respectively, the successor of s_1 and the predecessor of t_1 on P_i. W.l.o.g. we have $q \in P_1, p \in P_2$. Then, by the minimality of P_1,P_2, $\alpha_1 = q$, $\beta_2 = p$ and $\alpha_2 \in D'_1$ since otherwise we would have $\alpha_2 \in D'_2$ and then $\alpha_2 q$ and $p\alpha_2$ would be arcs and the path $s_2p\alpha_2qt_2$ would be disjoint from P_3. This shows that $|D'_1| \geq 2$. Next consider the paths Q_1,Q_2,Q_3 and let ϵ_i, γ_i denote, respectively, the successor of s_2 and the predecessor of t_2 on Q_i. W.l.o.g we have $p \in Q_1, q \in Q_2$. As above we see that $p = \epsilon_1, q = \gamma_2$ and $\epsilon_2 \in D'_3$ since otherwise the path $s_1q\epsilon_2pt_1$ is disjoint from Q_3. This implies that $|D'_3| \geq 2$. Now we see that $D - P$ is not semicomplete, contradicting Claim 5.

Case 2. *There exists* $p \in D_1, x \in D'_2$ *so that* p *and* x *are not adjacent.*

Suppose first that $t_1 \in D'_{2,top}$. Then, as above, considering the paths $Q = s_1qspt_1$ and Q_1,Q_2,Q_3, we conclude that $|D'_3| > 1$ and since $D\langle V(Q)\rangle$ is not semicomplete we have, by Lemma 2.9 that $D'_3 - \{p\}$ dominates $D'_2 - \{s_1,t_1\}$ and s_2 dominates $D'_3 - \{p\}$. Then considering $P' = s_2p'wqt_2$ we reach the same contradiction as above (by showing that we also have $|D'_1| \geq 2$). Hence none of t_1, s_1 are in $D'_{2,top}$. As we could choose the vertex w to be in $D'_{2,top}$, we see that here is no (s_1,t_1)-path in D'_2. Suppose first that there is an arc uv from S_{bot} to $D'_{2,top}$ such that $u \neq t_2$ (we cannot have $u = s_2$

since there are no arcs from s_2 to $D'_2 - \{s_1, t_1\}$ as we argued above). Now $W = s_1quvt_1$ is a path and there is no arc between u and t_1 as such an arc would be t_1u, implying that us_1 is an arc, but then $s_1 \in D'_{2,top}$ would hold. So $D\langle W\rangle$ is not semicomplete. Now considering the path $Z = s_2pvqt_2$ and three internally disjoint minimal (s_1, t_1)-paths in D^1 we conclude as above that $|D'_1| \geq 2$. But then $D - W$ is not semicomplete as p is non-adjacent to all vertices of D'_1, a contradiction. The only remaining possibility is that every arc from S to D'_2 starts in t_2 and hence we also have $s_2 \in S_{bot}$. This contradicts Claim 9 applied to $\overleftarrow{D}$ and S^*.

The last contradiction completes the proof of the theorem. $\square$

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Paper: Degree constrained 2-partition of semicomplete digraphs

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Degree constrained 2-partitions of semicomplete digraphs

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Abstract

A 2-partition of a digraph D is a partition (V_1, V_2) of $V(D)$ into two disjoint non-empty sets V_1 and V_2 such that $V_1 \cup V_2 = V(D)$. A semicomplete digraph is a digraph with no pair of non-adjacent vertices. We consider the complexity of deciding whether a given semicomplete digraph has a 2-partition such that each part of the partition induces a (semicomplete) digraph with some specified property. In [4] and [5] Bang-Jensen, Cohen and Havet determined the complexity of 120 such 2-partition problems for general digraphs. Several of these problems are NP-complete for general digraphs and thus it is natural to ask whether this is still the case for well-structured classes of digraphs, such as semicomplete digraphs. This is the main topic of the paper. More specifically, we consider 2-partition problems where the set of properties are minimum out-, minimum in- or minimum semi-degree. Among other results we prove the following:

- For all integers $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 3$ it is NP-complete to decide whether a given digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i\rangle$ has out-degree at least k_i for $i = 1, 2$.
- For every fixed choice of integers $\alpha, k_1, k_2 \geq 1$ there exists a polynomial algorithm for deciding whether a given digraph of independence number at most α has a 2-partition (V_1, V_2) such that $D\langle V_i\rangle$ has out-degree at least k_i for $i = 1, 2$.
- For every fixed integer $k \geq 1$ there exists a polynomial algorithm for deciding whether a given semicomplete digraph has a 2-partition (V_1, V_2) such that $D\langle V_1\rangle$ has out-degree at least one and $D\langle V_2\rangle$ has in-degree at least k.
- It is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i\rangle$ is a strong tournament.

Keywords: Semicomplete digraph, Tournament, 2-partition, minimum semi-degree, minimum out-degree, minimum in-degree, NP-complete, digraphs of bounded independence number.

1 Introduction

A 2-***partition*** of a (di)graph G is a partition (V_1, V_2) of $V(G)$ into two disjoint non-empty sets. Let $\mathcal{P}_1, \mathcal{P}_2$ be two graph properties, then a $(\mathcal{P}_1, \mathcal{P}_2)$-***partition*** is a 2-partition (V_1, V_2) where V_1 induces a graph with property $\mathcal{P}_1$ and V_2 a graph with property $\mathcal{P}_2$. For example a $(\delta^+ \geq k, \delta^+ \geq k)$-partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least k. It is natural to ask when properties such as high (edge)-connectivity or minimum degree can be maintained by both parts of some 2-partition of a (di)graph G. As an example of this, Alon [1] and independently Stiebitz [14] posed the following problem.

Problem 1.1. *[1, 14] Does there exist a function $h(k_1, k_2)$ such that every digraph $D = (V, A)$ with minimum out-degree $h(k_1, k_2)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition?*

It is easy to see that the answer to Problem 1.1 is yes if and only if there exists a function $h'(k_1, k_2) \geq k_1 + k_2 + 1$ such that every digraph with minimum out-degree $h'(k_1, k_2)$ contains disjoint induced subdigraphs D_1, D_2 such that D_1 has minimum out-degree at least k_1 and D_2 has minimum

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out-degree at least k_2. The lower bound comes from the complete digraph on $k_1 + k_2 + 1$ vertices in which every vertex has out-degree $k_1 + k_2$. Clearly this does not have the desired 2-partition since there will be too few vertices in one of the sets of any 2-partition.

It is known that high minimum out-degree guarantees many disjoint cycles in a digraph [1, 15], in particular minimum out-degree 3 is enough to guarantee two disjoint cycles. Using this, one can easily show that $h(1,1) = 3$ (see the beginning of Section 3). Already the existence of $h(1,2)$ is open and we will show in Theorem 3.1 that it is NP-complete to decide if a digraph has a $(\delta^+ \geq 1, \delta^+ \geq 2)$-partition.

The following two problems are natural variations of Problem 1.1. For definitions of semi-degree etc, see the next section.

Problem 1.2. *Do there exist functions $w_1(k_1,k_2)$, $w_2(k_1,k_2)$ such that every digraph $D = (V,A)$ with minimum out-degree $\delta^+(D) \geq w_1(k_1,k_2)$ and minimum in-degree $\delta^-(D) \geq w_2(k_1,k_2)$ has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-partition.*

Problem 1.3. *Does there exist a function $z(k_1,k_2)$ such that every digraph $D = (V,A)$ with minimum semi-degree $\delta^0(D) \geq z(k_1,k_2)$ has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition.*

In [10] Lichiardopol answered Problems 1.1 to 1.3 in the affirmative for tournaments. It can easily be seen that his proofs can be generalized to semicomplete digraphs.

Theorem 1.4. *[10] Let T be a tournament (semicomplete digraph) with minimum out-degree at least $\frac{k_1^2+3k_1+2}{2} + k_2$, then T has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.*

Theorem 1.5. *[10] Let T be a tournament (semicomplete digraph) with minimum semi-degree at least $k_1^2 + 3k_1 + 2 + k_2$, then T has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition.*

In [5] Bang-Jensen, Havet and Cohen determined the complexity of 120 partition problems for general digraphs. Among these are the following.

Theorem 1.6. *[5] The following 3 decision problems are NP-complete for general digraphs:*

- *deciding whether D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition,*
- *deciding whether D has a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition and*
- *deciding whether D has a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition.*

We show in the beginning of Section 3 that the $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition problem is polynomially solvable for general digraphs. From these results two natural questions emerge. What is the complexity of the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem when $k_1 + k_2 \geq 3$ and what is the complexity of the three problems in Theorem 1.6 if we restrict the input to a well-structured class of digraphs such as semicomplete digraphs? This paper will answer these questions as well as several related ones. In Section 3 we prove that as soon as $k_1 + k_2 \geq 3$ the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem becomes NP-complete for general digraphs. Then we prove that for all fixed pairs of integers $k_1, k_2 \geq 1$ there exists a polynomial algorithm for the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem for semicomplete digraphs. In Section 4 we prove that all three problems from Theorem 1.6 are polynomially solvable for semicomplete digraphs and in Section 5 we prove that for every fixed integer $k \geq 1$ the $(\delta^+ \geq 1, \delta^- \geq k)$-problem is polynomially solvable for semicomplete digraphs. In Section 6 we prove that, even for semicomplete digraphs, if we require several properties for each part of a 2-partition, we may obtain problems that are NP-complete, by showing that it is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) so that $D\langle V_i\rangle$ is a stong tournament for $i = 1, 2$. Finally in Section 7 we conclude with some remarks and open problems. In particular we outline why one of our proofs generalizes to digraphs of bounded independence number.

2 Notation, Definitions and Preliminary results

Notation follows [3] all digraphs considered have neither loops nor parallel arcs. We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$ and $[i,k]$ for the set $\{i, i+1, \ldots, k\}$. Let $D = (V,A)$ be a digraph with vertex set V and arc set A and let $v \in V$. If $xy \in A$ is an arc, then we say that x **dominates** y. The ***in-degree*** of v, denoted by $d^-_D(v)$, is the number of arcs from $V - v$ to v. Similarly the ***out-degree***, denoted by $d^+_D(v)$, is the number of arcs from v to $V - v$. Furthermore $N^+(v)$ is the out-neighbours of v, for a set $X \subseteq V$, $N^+[X]$ is the union of the the set X and all out-neighbours of the vertices of X and $N^+(X) = N^+[X] - X$ denotes the set of out-neighbours of X that do not belong to X. Definitions for the in-neighbours of vertices and sets are similar. Finally the ***minimum out-degree***, respectively ***minimum in-degree*** of a digraph D is denoted by $\delta^+(D)$, respectively $\delta^-(D)$ and the ***minimum semi-degree*** of D, denoted by $\delta^0(D)$ is defined as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$.

The ***subdigraph induced*** by a set of vertices X in a digraph D, denoted $D\langle X\rangle$, is the digraph with vertex set X and which contains those arcs from D that have both end-vertices in X.

A ***path*** of a digraph is a sequence of distinct vertices $x_1, x_2, \dots, x_l$ such that $x_i x_{i+1}$ is an arc for every $i \in [l-1]$. A ***cycle*** is defined as a path except that $x_1 = x_l$. If C is a cycle of k vertices we say that C is a k***-cycle***. A digraph D is ***acyclic*** if it does not contain any cycles and a ***feedback vertex set*** of D is a set $Z \subset V$ such that $D - Z$ is acyclic.

A **strong component** of a digraph $D = (V, A)$ is a maximal induced subdigraph $D\langle X\rangle$ with the property that there exists a path from u to v for every ordered pair of vertices $u, v \in X$. A digraph $D = (V, A)$ is ***strongly connected*** or just ***strong*** if it has exactly one strong component. If D is not strongly connected, then we can order its strong components $D_1, \dots, D_k$, $k \geq 2$ such that there is no arc in D which goes from a vertex in D_j to a vertex in D_i where $i < j$. A strong component is **trivial** if it consists on just one vertex. A digraph on at least $k+1$ vertices is k***-strong*** if the digraph $D - X$, obtained by deleting all vertices of X and their incident arcs, remains strongly connected for every subset $X \subseteq V$ with $|X| \leq k - 1$. Furthermore if $Y \subset V$ such that $D\langle V - Y\rangle$ is not strong, then Y is called a ***separator*** of D and it is a ***minimal separator*** if $D - Y'$ is strong for every proper subset Y' of Y.

A ***complete digraph*** is a digraph in which every pair of distinct vertices induce a directed 2-cycle. A ***semicomplete digraph*** is a digraph where there is an arc between every pair of vertices and a ***tournament*** is a semicomplete digraph without 2-cycles. A ***transitive tournament*** or ***acyclic tournament***, is a tournament without any cycles. For such tournaments there exists an unique ordering of the vertices $v_1, \dots, v_n$ such that $v_i v_j$ is an arc if and only if $i < j$. We call the vertex v_1 (v_n) the ***source*** (***sink***) of T. It is easy to see that for non-strong semicomplete digraphs there exists a unique ordering of its strong components $D_1, \dots, D_r$ such that each vertex of D_i dominates all vertices of D_j if and only if $i < j$. Strong semicomplete digraphs have many cycles as indicated by the following classical result of Moon[1].

Theorem 2.1. *[12] Every vertex of a strong semicomplete digraph on n vertices is contained in a k-cycle for every $k \in [3, n]$*

Below we let $D = (V, A)$ be a given digraph and let k be a fixed integer. D is said to be ***out-critical*** (with respect to k) if $\delta^+(D) = k$ and no subset of its vertices can be removed without decreasing the minimum out-degree of the resulting digraph. Let D be a digraph with minimum out-degree at least k and let X be a subset of its vertices. A set $X' \subseteq V$ is called X***-out-critical*** if $X \subseteq X'$, $\delta^+(D\langle X'\rangle) \geq k$ and $\delta^+(D\langle X' - Z\rangle) < k$ for every $\emptyset \neq Z \subseteq X' - X$. Note that if $\delta^+(D\langle X\rangle) \geq k$, then X is the only X-out-critical set in D. By definition, a digraph of minimum out-degree at least k contains at least one X-out-critical set for every subset X of vertices (including the empty set).

3 The complexity of the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem

As mentioned in the introduction, the $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition problem is polynomial for general digraphs. In fact, such a partition exists if and only if D has two vertex disjoint cycles. One direction is clear and if we have a pair of disjoint cycles C_1, C_2 in D, then put all vertices with a directed path to $V(C_1)$ in $D\langle V - V(C_2)\rangle$ together with $V(C_1)$ and the rest together with $V(C_2)$. By a result of McCuaig [11] one can test the existence of two vertex disjoint cycles, and find such a pair if they exist,

[1] As every strong semicomplete digraph contains a spanning strong tournament, Moon's original theorem also holds for semicomplete digraphs

in a given digraph in polynomial time, implying that $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition is polynomial for D.

However already for $k_1 + k_2 \geq 3$ the problem becomes NP-complete.

Theorem 3.1. *Let k_1, k_2 be positive integers such that $k_1 + k_2 \geq 3$. It is NP-complete to decide whether a given input digraph $D = (V, A)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.*

Proof. Without loss of generality we have $k_1 \leq k_2$. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an instance of monotone 1-IN-3-SAT with boolean variables $x_1, x_2, \ldots, x_n$ (monotone means that no clauses contain negated variables). That is, we seek a truth assignment $t : \{x_1, \ldots, x_n\} \to \{T, F\}^n$ such that each clause C_i will have exactly one variable true (T). This problem is NP-complete [16]. It is easy to see that given an instance $\mathcal{F}$ of monotone 1-IN-3-SAT we can extend it by adding clauses to an equivalent instance $\mathcal{F}'$ in which each variable occurs in at least k_2 clauses. So below we assume that already $\mathcal{F}$ has this property. We construct the digraph $D = D(\mathcal{F})$ from $\mathcal{F}$ as follows: First we take m disjoint copies $H_1^{(1)}, \ldots, H_m^{(1)}$ of the complete digraph on k_1 vertices and m copies $H_1^{(2)}, \ldots, H_m^{(2)}$ of the complete digraph on k_2 vertices all disjoint and disjoint from the first complete digraphs. Then we add the following new vertices $\{c_i^{(1)} | i \in [m]\} \cup \{c_i^{(2)} | i \in [m]\} \cup \{w_i | i \in [m]\} \cup \{z_i | i \in [m]\} \cup \{v_j | j \in [n]\}$. The vertices $c_i^{(1)}, c_i^{(2)}, w_i, z_i$ as well as the two complete digraphs $H_i^{(1)}, H_i^{(2)}$ are associated with the clause C_i for each $i \in [m]$ and the vertex v_j is associated with the variable x_j for each $j \in [n]$. Now we add the following arcs:

- $\{w_i c_i^{(1)} | i \in [m]\} \cup \{c_i^{(2)} z_i | i \in [m]\}$.
- $\max\{k_1 - 2, 0\}$ non-parallel arcs from w_i to $V(H_i^{(1)})$ and an arc from each vertex of $V(H_i^{(1)})$ to w_i.
- $k_1 - 1$ non-parallel arcs from $c_i^{(1)}$ to $V(H_i^{(1)})$.
- $k_2 - 2$ non-parallel arcs from z_i to $V(H_i^{(2)})$.
- $k_2 - 2$ non-parallel arcs from $c_i^{(2)}$ to $V(H_i^{(2)})$ and an arc from each vertex of $V(H_i^{(2)})$ to $c_i^{(2)}$.
- The arcs of the cycle $c_1^{(2)} c_2^{(2)} \ldots c_m^{(2)} c_1^{(2)}$
- If $k_1 > 1$ then add the arcs of the cycle $w_1 w_2 \ldots w_m w_1$.
- for each $i \in [m]$, if C_i is given by $C_i = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$, then we add the 12 arcs

$$c_i^{(1)} v_{i_1}, c_i^{(1)} v_{i_2}, c_i^{(1)} v_{i_3}, v_{i_1} c_i^{(1)}, v_{i_2} c_i^{(1)}, v_{i_3} c_i^{(1)}, v_{i_1} c_i^{(2)}, v_{i_2} c_i^{(2)}, v_{i_3} c_i^{(2)}, z_i v_{i_1}, z_i v_{i_2}, z_i v_{i_3}.$$

Clearly D can be constructed in polynomial time, given $\mathcal{F}$. We claim that there exists a truth assignment to $x_1, \ldots, x_n$ such that each clause C_i contains exactly one true literal if and only if D has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.

We first note some properties of any $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition (V_1, V_2) of D. As all the vertices w_i, $i \in [m]$ have out-degree exactly k_1 in D, they, the vertices $c_i^{(1)}$, $i \in [m]$ and all the vertices of $H_1^{(1)}, \ldots, H_m^{(1)}$ must all belong to the same set V_q and if $k_1 < k_2$ this must be V_1. Similarly, since each vertex $c_i^{(2)}$ has out-degree exactly k_2 in D, all the vertices $c_i^{(2)}$, $i \in [m]$, all the vertices z_i, $i \in [m]$ and all vertices of the digraphs $H_1^{(2)}, \ldots, H_m^{(2)}$ must belong to the same set V_p. It is easy to see that we must have $p \neq q$ as otherwise only the variable vertices v_j can be in the set V_{3-q} but there are no arcs between the variable vertices. Now the fact that the vertices $z_i, c_i^{(1)}$ belong to the different sets of the partition and the fact that z_i has exactly $k_2 + 1$ out-neighbours in D implies that exactly two of the vertices $v_{i_1}, v_{i_2}, v_{i_3}$ must belong to V_p to give z_i out-degree k_2 in $D\langle V_p \rangle$ and the last will belong to V_q to give $c_i^{(1)}$ out-degree k_1 in $D\langle V_q \rangle$.

Now we can finish the proof easily. Suppose first that D has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition (V_1, V_2). If $k_1 < k_2$ we have all $c_i^{(1)}$'s in V_1. If $k_1 = k_2$ then by renaming if necessary, we again have that V_1 contains all $c_i^{(1)}$'s and in both cases each such vertex has exactly one of its neighbours among the variable vertices in V_1. Thus if we assign the value true to x_r, $r \in [n]$ if the vertex v_r is in V_1, then we obtain a truth assignment that sets exactly one literal true for each clause. Conversely, given a truth assignment ϕ that sets exactly one literal true for each clause, we obtain the desired partition by letting V_1 consist of all vertices of $H_1^{(1)}, \dots, H_m^{(1)}$, all vertices $c_i^{(1)}$, $i \in [m]$, all vertices w_i, $i \in [m]$ and all those variable vertices v_r for which the corresponding variable x_r is set true by ϕ. It follows from the observations above and the fact that each variable vertex v_h, $h \in [n]$ has at least k_2 out-neighbours in each of the sets $\{c_1^{(1)}, \dots, c_m^{(1)}\}$, $\{c_1^{(2)}, \dots, c_m^{(2)}\}$ that $(V_1, V - V_1)$ is a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition of V. □

As mentioned earlier in Theorem 1.4, Lichiardopol proved in [10] that $\delta^+(T) \geq \frac{k_1^2+3k_1+2}{2} + k_2$ is sufficient to guarantee a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition for tournaments. This is also true for semicomplete digraphs. We now describe a polynomial algorithm to find such a partition if one exists. We need a few lemmas. For fixed integers k_1, k_2 a vertex v of a semicomplete digraph D is said to be ***out-dangerous*** (with respect to k_1 and k_2) if $d^+(v) < (k_1 + k_2) - 1$.

Lemma 3.2. *Let k_1, k_2 be fixed integers and let S be a semicomplete digraph. Then the number of out-dangerous vertices of S is at most $2(k_1 + k_2) - 3$.*

Proof. Let X be the set of out-dangerous vertices of S. Then the number of arcs in the semicomplete digraph $S\langle X\rangle$ is at most $|X|(k_1 + k_2 - 2)$ and at least $\frac{|X|(|X|-1)}{2}$ implying that $|X| \leq 2(k_1 + k_2) - 3$. □

In [10] it was shown that for every fixed k any out-critical set is of bounded size. With a small modification of this proof we can bound the size of X-out-critical sets for any fixed set X.

Lemma 3.3. *Let S be a semicomplete digraph with minimum out-degree at least k and let $X \subseteq V(S)$. Then every X-out-critical set X' of S will have size at most $\frac{k^2+3k+2}{2} + |X|$.*

Proof. Suppose that for some set $X \subset V$ there is an X-out-critical set X' of size at least $\frac{k^2+3k+2}{2} + |X| + 1$. Consider the semicomplete digraph $S' = S\langle X'\rangle$. Let M be the set of vertices that have out-degree exactly k in S' and let $m = |M|$.

As each $v \in M$ has out-degree k in the semicomplete digraph $S'\langle M\rangle$ we have

$$|N_{S'}^+[M]| \leq m + mk - \frac{m(m-1)}{2} = -\frac{m^2}{2} + \left(\frac{3}{2} + k\right) m =: P(m).$$

Now $P(m)$ has global maximum at $(3/2 + k)$ and maximum for m integer at $k + 1$ and $k + 2$ with $P(k+1) = P(k+2) = \frac{k^2+3k+2}{2}$. Hence as $|X'| > \frac{k^2+3k+2}{2} + |X|$ there exists a vertex $u \in X' - (N_{S'}^+[M] \cup X)$ such that $\delta^+(S'\langle X' - u\rangle) \geq k$. But then the set $Z = \{u\}$ is contained in $X' - X$ and $\delta^+(S\langle X' - Z\rangle) \geq k$, contradicting the fact that X' is an X-out-critical set in S. □

We are now ready to prove the existence of a polynomial algorithm for deciding whether a given semicomplete digraph has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.

Theorem 3.4. *For every fixed pair of integers k_1, k_2 there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition of a given semicomplete digraph S or correctly outputs that none exist.*

Proof. Let (O_1, O_2) be a given partition of the out-dangerous vertices of S. Let $X \subseteq V - O_2$ be a set containing O_1 such that $|X| \leq \frac{k_1^2+3k_1+2}{2} + |O_1|$ and $\delta^+(S\langle X\rangle) \geq k_1$ (if no such set exists, we stop considering the pair (O_1, O_2)). The following subalgorithm $\mathcal{B}$ will decide whether there exists a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition (V_1, V_2) with $X \subseteq V_1, O_2 \subseteq V_2$: Starting from the partition $(V_1, V_2) = (X, V - X)$, and moving one vertex at a time, the algorithm will move vertices of $V_2 - O_2$ which have $d^+_{S\langle V_2\rangle}(v) < k_2$ to V_1. If, at any time, this results in a vertex $v \in O_2$ having $d^+_{S\langle V_2\rangle}(v) < k_2$, or $V_2 = \emptyset$, then there is no $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition with $X \subseteq V_1$ and $O_2 \subseteq V_2$ and the algorithm $\mathcal{B}$ terminates. Otherwise $\mathcal{B}$ will terminate with $O_2 \subseteq V_2 \neq \emptyset$ and hence it has found an $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition (V_1, V_2) with $O_i \subseteq V_i$, $i = 1, 2$.

The correctness of $\mathcal{B}$ follows from the fact that we only move vertices that are not dangerous and each such vertex has at least $k_1 + k_2 - 1$ out-neighbours in S. Hence, as the vertex that we move does not have k_2 out-neighbours in V_2, it must have at least k_1 out-neighbours in V_1, so $\delta^+(S\langle V_1\rangle) \geq k_1$ will hold throughout the execution of $\mathcal{B}$.

By Lemma 3.2, the number of out-dangerous vertices is at most $2(k_1 + k_2) - 3$ and hence the number of (O_1, O_2)-partitions of the set of out-dangerous vertices is at most $2^{2(k_1+k_2)-3}$ which is a constant because k_1, k_2 are fixed. Furthermore, by Lemma 3.3, the size of every O_1-critical set is also bounded by a function of k_1 and hence for each (O_1, O_2)-partition there is only a polynomial number of O_1-critical sets that are disjoint from O_2. Thus we obtain the desired polynomial time algorithm by running the subalgorithm $\mathcal{B}$ for all possible partitions (O_1, O_2) of the out-dangerous vertices and all possible choices of sets X with $O_1 \subseteq X$ and $|X| \leq \frac{k_1^2+3k_1+2}{2} + |O_1|$. Note that we do not need to check whether X is O_1-out-critical, we just check all possible supersets of O_1 of size at most $\frac{k_1^2+3k_1+2}{2} + |O_1|$. □

As k_1, k_2 are fixed, the running time of the algorithm above is $O(n^{g(k_1,k_2)})$ for some (quadratic) polynomial g. We made no attempt to improve the running time above and it is natural to ask whether there exists an FPT algorithm for the problem.

Problem 3.5. *Does there exist a function $f(k_1, k_2)$ and a constant c such that one can decide, for a given semicomplete digraph S and pair of integers k_1, k_2 whether S has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition in time $O(f(k_1, k_2)n^c)$?*

We saw above that we could solve the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem in polynomial time using the fact that the number of out-dangerous vertices are bounded for semicomplete digraphs. It is natural to ask whether a similar approach can be used for the $(\delta^+ \geq k_1, \delta^- \geq k_2)$-, $(\delta^0 \geq k_1, \delta^- \geq k_2)$- and $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition problem. This however is not the case. There is no natural way to define dangerous vertices in these cases, as it depends on whether we consider a vertex with respect to the first property or with respect to the second property. Given a partition (V_1, V_2) of a digraph D where $\delta^+(D\langle V_1\rangle) \geq k_1$ and $\delta^-(D\langle V_2\rangle) < k_2$, a vertex with in-degree less that k_2 in V_2 might not have k_1 out-neighbours in V_1 and hence cannot be moved directly to V_1 without decreasing the minimum out-degree of V_1.

4 2-partitions where both constants are one

In this section we will consider the three problems from Theorem 1.6 and prove that there is a polynomial algorithm for each of these when the input is a semicomplete digraph. It is clear that for all three problems the existence of a pair of disjoint cycles is a necessary condition. It is easy to check whether a semicomplete digraph has a pair of disjoint cycles: D has such cycles if and only if it has disjoint cycles C_1, C_2 where $|V(C_i)| \leq 3$ for $i \in [2]$ (every cycle of length 4 or more in a semicomplete digraph induces a semicomplete digraph with a shorter cycle).

Lemma 4.1. *A semicomplete digraph S with $\delta^-(S) \geq 1$ has a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition if and only if it has a pair of disjoint cycles. Furthermore, such a partition can be found in polynomial time when it exists.*

Proof. Let S be a semicomplete digraph with $\delta^-(S) \geq 1$ and disjoint cycles C_1 and C_2. If $S_1 = S\langle V - V(C_1)\rangle$ has minimum in-degree at least 1 then $(V(C_1), V - V(C_1))$ is a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition so let x_1 be a vertex in $V - C_1$ with in-degree 0 in S_1. Similarly either $(V(C_2), V - V(C_2))$ is a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition or there is a vertex x_2 in $S_2 = S\langle V - C_2\rangle$ with in-degree 0. As $\delta^-(S) \geq 1$ the vertex x_i must have an in-neighbour on C_i for $i = 1, 2$, implying that $x_1 \neq x_2$. It follows from the choice of x_1, x_2 above that $x_1, x_2 \notin V(C_1) \cup V(C_2)$ and hence both have in-degree 0 in the semicomplete digraph $S\langle V - V(C_1) - V(C_2)\rangle$, contradiction. It follows from the proof that it will always be the case that $(V(C_i), V - V(C_i))$ is a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition for $i = 1$ or 2, implying that we can find the desired partition in polynomial time. □

Theorem 4.2. *A semicomplete digraph S has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition if and only if it has a pair of disjoint cycles. Furthermore, such a partition can be found in polynomial time when it exists.*

Proof. We prove by induction on the number of vertices that the presence of two disjoint cycles guarantee the existence of the desired partition. In the base case S consists of two disjoint induced cycles so they each have length at most 3 and they themselves form the desired partition. Hence we proceed to the induction step and assume that S is a semicomplete digraph with disjoint cycles C_1 and C_2. By Lemma 4.1, it suffices to consider the case when S contains a vertex x of in-degree 0. Clearly $S - x$ also has two disjoint cycles (as x is not on any cycle) so by induction it has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition (V_1', V_2') and now $(V_1' + x, V_2)$ is the desired partition of S. From the argument above it is easy to obtain a polynomial algorithm to find the desired partition: continue to remove vertices of in-degree 0 until the remaining semicomplete digraph S' has $\delta^-(S') \geq 1$. Then find a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition (V_1', V_2') of S' using the algorithm from Lemma 4.1 and finally return the $(\delta^+ \geq 1, \delta^- \geq 1)$-partition $(V - V_2', V_2')$. □

Theorem 4.3. *There exists a polynomial algorithm that either finds a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition of a semicomplete digraph or correctly outputs that none exist.*

Proof. If D has a vertex of in-degree 0, then it has no $(\delta^0 \geq 1, \delta^- \geq 1)$-partition and otherwise it follows from Lemma 4.1 that the partition exists and can be found in polynomial time.

□

For the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition problem the existence of disjoint cycles is not sufficient in general, but the existence of a pair of complementary cycles is. Two cycles C_1, C_2 of a digraph D are **complementary** if they are disjoint and cover all vertices of D. Reid [13] proved that every 2-strong tournament on at least 8 vertices has a pair of complementary cycles. In [7] Guo and Volkmann proved that every 2-strong semicomplete digraph of at least 8 vertices has a pair of complementary cycles. Bang-Jensen and Nielsen [6] proved that checking the existence of complementary cycles of semicomplete digraphs and finding such a pair if they exist can be done in polynomial time. Notice that if C_1, C_2 is a pair of complementary cycles of a semicomplete digraph, then C_1 (or C_2) is allowed to be a 2-cycle. Now we are ready to prove the following.

Theorem 4.4. *There exists a polynomial algorithm that either finds a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition of a semicomplete digraph or correctly outputs that none exists.*

Proof. Suppose first that S is not strong and let $D_1, \ldots, D_r$, $r \geq 2$, be the strong components. If D_1 or D_r is a trivial component, then clearly there is no $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition, so we may assume that $\min\{|D_1|, |D_r|\} \geq 2$ and that $r \geq 3$ or we are done. If $|D_i| \geq 2$ for some $i \in [2, r-1]$, then $(V(D_i), V - V(D_i))$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. Hence we may assume that there are distinct vertices $d_2, \ldots, d_{r-1}$ such that $D_i = \{d_i\}$ for $i \in [2, r-1]$. Now it is easy to see that there is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if and only if at least one of the following holds:

- D_1 has a $(\delta^- \geq 1, \delta^- \geq 1)$-partition and D_r has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition.
- D_1 has a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition
- D_r has a $(\delta^+ \geq 1, \delta^0 \geq 1)$-partition.

For each of these problems we already established polynomial algorithms so from now on we may assume that S is strong. If $n < 8$ we just check all possible partitions, so assume $n \geq 8$. First check whether S has a pair of complementary cycles, using the algorithm of [6] and output the 2-partition induced by these if they exist. Hence we may now assume that S does not contain a pair of complementary cycles and that it is not 2-strong by the aforementioned results of [7, 13]. Note that, because every strong semicomplete digraph is hamiltonian (by Theorem 2.1), the fact that S has no pair of complementary cycles implies that S must have at least 3 disjoint cycles if it has a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition.

Let x be a separating vertex of S and let $D_1, \ldots, D_r$, $r \geq 2$ be the strong components of $S - x$. As S is strong the vertex x dominates at least one vertex in D_1 and is dominated by at least one vertex in D_r. Now it is easy to either find a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition or deduce the following (in each case, if the claim does not hold, then a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition can be constructed easily):

(i) If $r \geq 3$ then there are distinct vertices $d_2, \ldots, d_{r-1}$ such that $D_i = \{d_i\}$ for all $i \in [2, r-1]$.

(ii) If D_1 is non-trivial, then x only dominates vertices y of D_1 that are separators of D_1 and only the initial strong component of $D_1 - y$ (denoted D_{11}) can be non-trivial. Furthermore, either $r = 2$ or $xd_2 \notin A$, implying that $d_2x \in A$.

(iii) Similarly, if D_r is non-trivial, then x is only dominated by vertices z of D_r that are separators of D_r and only the terminal strong component of $D_r - z$ (denoted D_{rs}) can be non-trivial. Furthermore, either $r = 2$ or $d_{r-1}x \notin A$, implying that $xd_{r-1} \in A$.

Case 1) $r = 2$ and $|D_1|, |D_2| \geq 2$.
If x has arcs to and from D_i for $i = 1$ or $i = 2$ then $(V(D_i) \cup \{x\}, V(D_{3-i}))$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition so we can assume that x dominates all vertices of D_1 and is dominated by all vertices of D_2. As $n \geq 8$ we have $\max\{|D_1|, |D_r|\} \geq 4$, so, by Theorem 2.1, for some $i \in [2]$ there exists a vertex $v \in D_i$ such that $D_i - v$ is strong. Now we see that (V_1, V_2) is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if we let $V_1 = V(D_i) - v$ and $V_2 = V - V_1$.

Case 2a) $r \geq 2$ and $|D_1| = |D_r| = 1$. In this case the vertex x is on all cycles of S so it is a 'no'-instance.

Case 2b) $r \geq 2$, $\min\{|D_1|, |D_2|\} = 1$ and $\max\{|D_1|, |D_r|\} \geq 2$.
By reversing all arcs if necessary, we may assume $D_1 = \{d_1\}$ and $|D_r| \geq 2$. Assume that (V_1, V_2) is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition of S with $x \in V_1$. Then as x is the only in-neighbour of d_1, d_1 also belongs to V_1. Continuing this way we see that $\{x, d_1, \ldots, d_{r-1}\} \subseteq V_1$. As $xd_{r-1} \in A$ any $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition (V_1, V_2) will have vertices of D_r in both V_1 and V_2, and $V_2 \subset D_r$. Suppose first that x has no in-neighbour among $\{d_1, d_2, \ldots, d_{r-1}\}$. Then it is easy to see that the semicomplete digraph S' obtained by deleting $d_1, \ldots, d_{r-1}$ and adding an arc from x to each vertex of D_r will have a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if and only if S does. Thus we can solve the problem by calling the algorithm recursively on S'. Hence we may assume that x has at least one in-neighbour among $\{d_1, d_2, \ldots, d_{r-1}\}$. Note that if D_r does not have two disjoint cycles, then D has no set of 3-disjoint cycles and hence is a no-instance as we already know it has no pair of complementary cycles. Thus we may assume that D_r has a pair of disjoint cycles and now it follows from Theorem 4.3 (applied to D_r with all arcs reversed) that, in polynomial time we can find a $(\delta^+ \geq 1, \delta^0 \geq 1)$-partition (V'_1, V'_2) of D_r. Now it is easy to check that $(V - V'_2, V'_2)$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition of S.

Case 3) $r > 2$ and $|D_1|, |D_r| \geq 2$.
If there are indices $1 < i < j < r$ so that $xd_i, d_jx \in A$, then $(\{x, d_i, d_{i+1}, \ldots, d_j\}, V - \{x, d_i, d_{i+1}, \ldots, d_j\})$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. Hence, by (ii) and (iii) we may assume that there is an index $2 \leq f < r-1$ such that x has no arc to $\{d_2, \ldots, d_f\}$ and $\{d_{f+1}, \ldots, d_{r-1}\}$ has no arc to x. Fix a vertex $y \in V(D_1)$ such that $xy \in A$ and a vertex $z \in V(D_r)$ such that $zx \in A$. By (ii) and (iii) S contains the 3-cycles $C_3 = \{x, y, d_2\}$ and $C'_3 = \{x, d_{r-1}, z\}$. If the initial component D_{11} of $D_1 - y$ satisfies $|D_{11}| \geq 2$, then we have the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition $(V(C_3), V - V(C_3))$. Similarly if the terminal component D_{rs} of $D_r - z$ satisfies $|D_{rs}| \geq 2$, then we have a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition $(V(C'_3), V - V(C'_3))$. So assume $|D_{11}| = |D_{rs}| = 1$ and hence by (ii) and (iii) that the strong components of $D_1 - y$, respectively $D_r - z$ all have size one. Denote the vertices of these by $d_{11}, \ldots, d_{1p}$ and $d_{r1}, \ldots, d_{rs}$, respectively. Since $D_1 - d_{1j}$ is strong when $j \notin \{1, p\}$, it follows from (ii) that the only possible out-neighbours of x in $V(D_1) - y$ are d_{11} and d_{1p}. Similarly, (iii) implies that the only possible in-neighbours of x in $V(D_r) - z$ are d_{r1} and d_{rs}. If $xd_{1p} \in A$ and $d_{1j}y \in A$ for some $1 \leq j < p$, then $(\{x, d_{1p}, d_2\}, V - \{x, d_{1p}, d_2\})$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. If $xd_{11} \in A$ and $yd_{1i} \in A$ for some $1 < i \leq p$, then again we easily get a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. So either we find the desired partition or conclude that that following holds: if $xd_{1p} \in A$ then d_{1p} is the only in-neighbour of y in D_1 and if $xd_{11} \in A$, then d_{11} is the only out-neighbour of y in D_1. By similar observations we either find the desired partition or conclude that if $d_{r1}x \in A$, then d_{r1} is the only out-neighbour of z in D_r and if $d_{rs}x \in A$, then d_{rs} is the only in-neighbour of z in D_r. This implies that S does not have 3 disjoint cycles, because $D - \{x, y, z\}$ is acyclic and none of $D - D_1$, $D - D_r$ have two disjoint cycles. But then it cannot have a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition, since we have already assumed that S has no pair of complementary cycles.

This completes the description of the algorithm. For Case 2b, notice that each of the recursive calls (if any) will be on semicomplete digraphs which have at least 2 vertices less than the current one. Hence in at most $O(n)$ calls the algorithm we will terminate and thus the algorithm runs in polynomial time.

□

5 2-partitions when one constant is 1 and the other at least 2

When one of the two sides of the partition must have in-degree or out-degree at least k for some $k \geq 2$, we need more work to establish a polynomial algorithm. We begin with a Lemma which could be of independent interest. Below we use the shorthand notation $d^+_X(v)$ ($d^-_X(v)$) for $d^+_{D\langle X\rangle}(v)$ ($d^-_{D\langle X\rangle}(v)$), where X is a subset of the vertices of D and $v \in X$.

Lemma 5.1. *Let $k \geq 1$ be a fixed integer. Then there exists a polynomial algorithm for the following problem: let $S = (V, A)$ be a semicomplete digraph and X_1, X_2 disjoint subsets of V such that*

(a) $V - X_1 - X_2$ induces a transitive tournament.

(b) If there is a vertex v of X_1 such that $d^+_{X_1}(v) = 0$ then v is dominated by at most $k - 1$ vertices of $V - X_1 - X_2$.

decide whether S has a $(\delta^+ \geq 1, \delta^- \geq k)$-partition (V_1, V_2) with $X_i \subset V_i$ for $i \in [2]$ and find such a partition when it exists.

Proof. We start by setting $V'_1 = X_1$ and $V'_2 = X_2$. Throughout this proof we let $Z = V - V'_1 - V'_2$ and say that a $(\delta^+ \geq 1, \delta^- \geq k)$-partition (V_1, V_2) is ***good*** if $V'_i \subset V_i$ where V'_1, V'_2 are the current sets obtained by the algorithm. The algorithm will move vertices of Z to V'_1 and V'_2 until we either find such a good partition or we can conclude that none exists.

Assume first that there is a vertex $u \in V'_1 \cup Z$ ($u \in V'_2 \cup Z$) such that $d^+_{V'_1\cup Z}(u) = 0$ ($d^-_{V'_2\cup Z}(u) < k$). If $u \in V'_i$ then clearly there is no good $(\delta^+ \geq 1, \delta^- \geq k)$-partition and we can stop. If $u \in Z$ then there can only exist a good $(\delta^+ \geq 1, \delta^- \geq k)$-partition if $u \in V'_{3-i}$ and we may move u to V'_{3-i}. Continue moving vertices from the current Z which are forced into one of the sides of any good partition (as above) until we either have $Z = \emptyset$, in which case we just check whether (V'_1, V'_2) is a $(\delta^+ \geq 1, \delta^- \geq k)$-partition, or we have $Z \neq \emptyset$, $V'_1 \cup Z$ induces a semicomplete digraph with $\delta^+ \geq 1$ and $V'_2 \cup Z$ induces a semicomplete digraph with $\delta^- \geq k$. If $(V'_1, V'_2 \cup Z)$ is a $(\delta^+ \geq 1, \delta^- \geq k)$-partition we are done so let v be the unique vertex of out-degree zero in $S\langle V'_1\rangle$.

So far we have only moved a vertex to $V'_i - X_i$ if it was forced to belong to that set in any good partition. Assume now that there exists a good $(\delta^+ \geq 1, \delta^- \geq k)$-partition (V_1, V_2) and let $W = Z \cap V_1$. Furthermore let $w_1, \ldots, w_m$ be the unique acyclic ordering of the vertices of W. Then w_m must dominate at least one vertex of V'_1 and the vertex v of V'_1 must dominate a vertex of W. Let w_j be the vertex of W with the highest index such that v dominates w_j. If $j > 1$ then $(V_1 \backslash \{w_1, \ldots, w_{j-1}\}, V_2 \cup \{w_1, \ldots, w_{j-1}\})$ is also a good $(\delta^+ \geq 1, \delta^- \geq k)$-partition since $\delta^-(S\langle V'_2 \cup (Z - W)\rangle) \geq k$ and $w_1, \ldots, w_{j-1}$ have no in-neighbours among $w_j \ldots, w_m$. Hence we can restrict the search for a good $(\delta^+ \geq 1, \delta^- \geq k)$-partition (V_1, V_2) to one where v only dominates the source of the transitive tournament $V_1 \cap Z$. Furthermore v is either the (original) sink of X_1 or a vertex added to V'_1 because it had less that k in-neighbours in $V'_2 \cup Z$. In any case, by (b), the vertex v is dominated by at most $k - 1$ vertices of Z and hence for any subset of Z of size at least $k + 1$, v will dominate at least two vertices. This implies that if there is any good partition (V_1, V_2) then there is one where $|V_1| \leq |V'_1| + k$. We can check for such a partition by looking at all possible subsets W of Z of size at most k with the further condition that v only dominates the unique vertex of in-degree 0 in $S\langle W\rangle$. There are $O(n^{k+1})$ such subsets, implying that our algorithm is polynomial.

□

Theorem 5.2. *There exists a polynomial algorithm that either finds a $(\delta^+ \geq 1, \delta^- \geq 2)$-partition of a semicomplete digraph S or correctly outputs that none exist.*

Proof. Below we make no attempt to optimize the running time. There are $O(n^3)$ cycles of length at most 3 in S. Let h denote the number of such cycles and order them as $C_1, \ldots, C_h$. For each $i \in [h]$ or until we find a solution we proceed as follows. Start by letting $V_1 = V(C_i)$, where C_i is the next cycle to consider, and let $V_2 = V - V_1$. Now move vertices of in-degree at most one in $D\langle V_2 \rangle$ to V_1 until either $V_2 = \emptyset$ in which case there is no $(\delta^+ \geq 1, \delta^- \geq 2)$-partition (V_1, V_2) with $C_i \subseteq V_1$ (and we go to the next cycle C_{i+1}) or V_2 induces a semicomplete digraph with minimum in-degree at least 2. If $\delta^+(S\langle V_1 \rangle) \geq 1$ then we have found a $(\delta^+ \geq 1, \delta^- \geq 2)$-partition (V_1, V_2), so assume that v has out-degree zero in $S\langle V_1 \rangle$.

Let B be the set of vertices in V_2 that have in-degree at most 4 in $S\langle V_2 \rangle$. If there exists a cycle C' of length at most 3 in $S\langle V_2 \rangle$ such that $V_2 - C'$ still induces a semicomplete digraph with minimum in-degree 2, then $(V_1 \cup C', V_2 - C')$ is a $(\delta^+ \geq 1, \delta^- \geq 2)$-partition, because v has at most one in-neighbour on C' (as it was moved at some point) so it will dominate at least one vertex on C'. Hence we may assume that for each 3-cycle C' of $S\langle V_2 \rangle$ there is a vertex of B that has an in-neighbour in C'. But then it follows from Theorem 2.1 that every cycle of $S\langle V_2 \rangle$ contains an in-neighbour of B. This implies that $F = N^-[B]$ is a feedback vertex set of $S\langle V_2 \rangle$, that is, $S\langle V_2 \rangle - F$ is a transitive tournament.

If T has any $(\delta^+ \geq 1, \delta^- \geq 2)$-partition $(\widehat{V_1}, \widehat{V_2})$ with $V_1 \subset \widehat{V_1}$, then $F_i = \widehat{V}_i \cap F$, $i = 1, 2$, induces a partition of the vertices of F. Hence to find a $(\delta^+ \geq 1, \delta^- \geq 2)$-partition we need only check if S has a partition $(\widehat{V_1}, \widehat{V_2})$ where $V_1 \cup F_1 \subset \widehat{V_1}$ and $F_2 \subset \widehat{V_2}$ for every partition F_1, F_2 of F (possibly with $F_i = \emptyset$ for $i = 1$ or $i = 2$).

To realize that this can be done in polynomial time, notice that there are at most 9 vertices in B and since each of these has in-degree at most 4 in V_2 the size of F is at most[2] 45. Hence there are at most 2^{45} partitions of F to check. For each partition (F_1, F_2) of F we can use the algorithm of Lemma 5.1 with $X_1 = F_1 \cup V_1$ and $X_2 = F_2 \cup V_2$. If none of these partitions of F result is a solution, we move to the next cycle C_{i+1}.

□

With a bit more effort we can extend the theorem to any fixed lower bound on the in-degree in $S\langle V_2 \rangle$.

Theorem 5.3. *For every fixed integer $k \geq 1$ there exists a polynomial algorithm that either constructs a $(\delta^+ \geq 1, \delta^- \geq k)$-partition of a semicomplete digraph S or correctly outputs that none exist.*

Proof. Again we order the set of cycles of length at most 3 as $C_1, \ldots, C_h$, where $h \in O(n^3)$ and consider these one by one until we either find a solution or there are no more cycles to try. When considering C_i we start by letting $V_1 = C_i$ and $V_2 = V - V_1$. Then we move vertices of in-degree less than k in $D\langle V_2 \rangle$ to V_1 until either $V_2 = \emptyset$, in which case there is no partition with $C_i \subseteq V_1$, or the process stops when V_2 induces a semicomplete digraph with minimum in-degree at least k. Now if V_1 induces a semicomplete digraph with minimum out-degree at least 1, we have found a $(\delta^+ \geq 1, \delta^- \geq k)$-partition (V_1, V_2), so assume that v has out-degree zero in $S\langle V_1 \rangle$.

Let $p = \lceil \frac{k}{2} \rceil$ and let B be the set of vertices of V_2 that have in-degree at most $k + 3p - 1$. If $S\langle V_2 \rangle$ has a collection of p disjoint cycles $C'_1, \ldots, C'_p$, each of length at most 3, such that $S\langle V_2 - \cup_{i=1}^{p} V(C'_i) \rangle$ has minimum in-degree at least k, then we obtain a $(\delta^+ \geq 1, \delta^- \geq k)$-partition by adding the vertices of $C'_1, \ldots, C'_p$ to V_1 and removing them from V_2 (as in the previous proof the vertex v will have at least one out-neighbour among the newly added vertices). The existence of such a collection of cycles can be determined by trying all subsets of V_2 on at most $3p$ vertices. Hence we may assume below that $Y = N^-[B]$ intersects all sets of p disjoint cycles in V_2. Now there are at most $p - 1$ disjoint cycles of length at most 3 in $V_2 - Y$ and hence removing some subset Y' of at most $3(p-1)$ extra vertices of $V_2 - Y$ we obtain a transitive tournament. Thus $F = Y \cup Y'$ is a feedback vertex set of $S\langle V_2 \rangle$.

The rest of the proof is similar to that of Theorem 5.2. The only difference is that instead of moving one cycle of length at most 3 we move sets of at most p disjoint cycles, each of which have length at most 3. If no good set of p cycles was found, then we found a small feedback vertex set F and we try each partition of the feedback vertex set F. To finish the proof we only need to argue that the size of F is bounded by a function in k. This follows from the following crude estimate

[2]This estimate is not precise. In fact $|F| \leq 30$, but the crude estimate suffices for our argument.

which suffices for our needs: the in-degree of the vertices in B is at most $\frac{5k}{2}+2$ so there are at most $5k+5$ vertices in B and hence $|Y| \leq (5k+5)+(5k+5)(\frac{5k}{2}+2) \leq \frac{25k^2+55k+30}{2}$. We also have that $|Y'| \leq 3(p-1) \leq \frac{3k}{2}$ so $|F| \leq \frac{25k^2+58k+30}{2}$.

□

We cannot directly use the same approach if we want a $(\delta^0 \geq 1, \delta^- \geq k)$-partition. This is because, after moving vertices that are forced to be in V_1 the semicomplete digraph $S\langle V_1\rangle$ may have both a vertex v with out-degree 0 and another vertex v' with in-degree 0. For v we still know that it has at most $k-1$ in-neighbours in V_2, but for v' we have no control of its number of out-neighbours in V_2, so we cannot guarantee that we will add an in-neighbour of v' when we add any set of p cycles of length at most 3.

Consider the case where we want a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-partition when both k_1 and k_2 are at least 2. The following would be natural generalization of the proof technique used above: first construct the (polynomial) list of all k_1-out-critical subgraphs $X_1, X_2, \ldots, X_q$. Then starting from $V_1 = X_i$ and $V_2 = V - X_i$ first move all vertices with in-degree less than k_2 in $S\langle V_2\rangle$ to V_1 and then try to move a small (as a function of k_1) subset of V_2 to V_1 so that we obtain a solution. Unfortunately this approach does not work as, already for $k_1 = k_2 = 2$, there exist infinitely many tournaments with minimum in-degree 2 that contain no subtournament of minimum out-degree 2.

Despite the seeming need for new proof techniques, based on the evidence from the results of this paper, we believe that the following holds.

Conjecture 5.4. *For every pair of fixed integers $k_1, k_2 \geq 2$ there exist polynomial algorithms for deciding the following for a given semicomplete digraph S:*

- *whether S has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-partition,*
- *whether S has a $(\delta^+ \geq k_1, \delta^0 \geq k_2)$-partition,*
- *whether S has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition*

In the proof of Lemma 5.1 we used the fact that k is fixed to obtain a polynomial algorithm $\mathcal{A}$. If k is part of the input the running time of $\mathcal{A}$ will no longer be polynomial in the size of the input.

Problem 5.5. *What is the complexity of the $(\delta^+ \geq 1, \delta^- \geq k)$-partition problem when the input is a semicomplete digraph and a positive integer k?*

6 2-partitions of semicomplete digraphs into tournaments

We now show that even for semicomplete digraphs we may obtain very difficult 2-partition problems if we pose the extra condition that each part of the 2-partition must induce a tournament. It follows from the polynomial algorithm from [6] for finding complementary cycles in semicomplete digraphs that without the requirement that each V_i induces a tournament, the problem below is polynomially solvable.

Theorem 6.1. *It is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i\rangle$ is a strong tournament.*

Proof. [3]
Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of not-all-equal 3-SAT (NAE-3-SAT) over the set of n boolean variables $x_1, \ldots, x_n$. That is, we seek a truth assignment $t : \{x_1, x_2, \ldots, x_n\} \rightarrow \{T, F\}^n$ so that each clause has at least one true literal and at least on false literal. This problem is NP-complete [16].

We construct a semicomplete digraph $D = D(\mathcal{F})$ which has a 2-partition (V_1, V_2) such that $D\langle V_i\rangle$ is a strong tournament if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.

[3]We would like to thank Anders Yeo for his help in correcting our first incomplete proof of Theorem 6.1

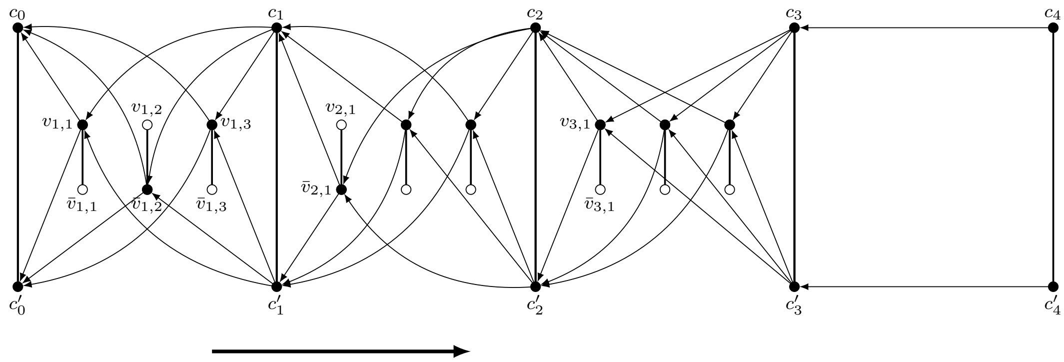

Figure 1: An example of the digraph $D = D(\mathcal{F})$ where $\mathcal{F} = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$. For clarity only the most important arcs are shown. The big arrow indicates that, except for the 2-cycles between clause vertices and between vertices corresponding to literals over the same variable, all arcs not shown go from left to right.

The vertex set of D is given by

$$V(D) = \{c_0, c_1, \ldots, c_{m+1}\} \cup \{c'_0, c'_1, \ldots, c'_{m+1}\} \cup \bigcup_{j=1}^{m} \{v_{j,1}, \bar{v}_{j,1}, v_{j,2}, \bar{v}_{j,2}, \ldots, v_{j,n}, \bar{v}_{j,n}\}$$

Here the vertices c_j, c'_j correspond to the clause C_j for $j \in [m]$ and the vertices $\{v_{1,i}, \ldots, v_{m,i}\}$, respectively $\{\bar{v}_{1,i}, \ldots, \bar{v}_{m,i}\}$ correspond to the literal x_i, respectively the literal $\bar{x}_i$.

We first define the arc set A' of a semicomplete digraph D' on the same vertex set as D and then describe how to obtain the arc set A of D by reversing certain arcs that will correspond closely to the clauses of $\mathcal{F}$.

The arc set $A' = A(D')$ is defined as follows:

- For all $0 \le j < j' \le m+1$, A' contains the arcs $c_j c_{j'}, c'_j c'_{j'}$, except when $j = m$ (and $j' = m+1$) where we have $c_{m+1} c_m, c'_{m+1} c'_m \in A'$.
- There is a 2-cycle between c_i and c'_j for all $0 \le i, j \le m+1$.
- For every $i \in [n]$ the vertices $\{v_{1,i}, \ldots, v_{m,i}\} \cup \{\bar{v}_{1,i}, \ldots, \bar{v}_{m,i}\}$ induce a complete bipartite digraph with bipartition $\{v_{1,i}, \ldots, v_{m,i}\}$, $\{\bar{v}_{1,i}, \ldots, \bar{v}_{m,i}\}$, that is, there is a 2-cycle between $v_{j,i}$ and $\bar{v}_{j',i}$ for all $j, j' \in [m], i \in [n]$.
- For all $j, j' \in [m]$ and all $i, i' \in [n]$ such that $j < j'$ and $i \neq i'$ or $j = j'$ and $i < i'$ A' contains the arcs $v_{j,i} v_{j',i'}, v_{j,i} \bar{v}_{j',i'}, \bar{v}_{j,i} v_{j',i'}, \bar{v}_{j,i} \bar{v}_{j',i'}$.
- For all $0 \le j < j' \le m$ and every $i \in [n]$ A' contains the arcs $c_j v_{j',i}, c_j \bar{v}_{j',i}, c'_j v_{j',i}, c'_j \bar{v}_{j',i}$.
- For all $1 \le j \le j' \le m+1$ and all $i \in [n]$, A' contains the arcs $v_{j,i} c_{j'}, v_{j,i} c'_{j'}, \bar{v}_{j,i} c_{j'}, \bar{v}_{j,i} c'_{j'}$

Now we describe how to obtain D from $D' = (V, A')$ by performing $12m$ arc-reversals. For each $j \in [m]$: let $\ell_{i_1}, \ell_{i_2}, \ell_{i_3}$ be the literals of C_j and let $u_{j,1}, u_{j,2}, u_{j,3}$ be those three vertices of $\{v_{j,1}, \bar{v}_{j,1}, v_{j,2}, \bar{v}_{j,2}, \ldots, v_{j,n}, \bar{v}_{j,n}\}$ which correspond to these literals (e.g. if $C_j = (x_1 \vee \bar{x}_5 \vee x_8)$ then $u_{j,1} = v_{j,1}, u_{j,2} = \bar{v}_{j,5}, u_{j,3} = v_{j,8}$). Now we reverse the 12 arcs between $\{c_{j-1}, c'_{j-1}, c_j, c'_j\}$ and $\{u_{j,1}, u_{j,2}, u_{j,3}\}$. This concludes the construction of D.

It is easy to check that D is a semicomplete digraph. We first make some observations about 2-partitions (V_1, V_2) of $V(D)$ such that $D\langle V_i \rangle$ is a tournament for $i \in [2]$.

(a) The vertices $\{c_0, c_1, \ldots, c_m, c_{m+1}\} \cup \{c'_0, c'_1, \ldots, c'_m, c'_{m+1}\}$ induce a complete bipartite digraph which implies that we have $\{c_0, c_1, \ldots, c_m, c_{m+1}\} \subset V_i$ and we have $\{c'_0, c'_1, \ldots, c'_m, c'_{m+1}\} \subset V_{3-i}$ for $i = 1$ or $i = 2$.

(b) for each $i \in [n]$, the vertices $\{v_{1,i}, \ldots, v_{m,i}\} \cup \{\bar{v}_{1,i}, \ldots, \bar{v}_{m,i}\}$ induce a complete bipartite digraph which implies that we have $\{v_{1,i}, \ldots, v_{m,i}\} \subset V_p$ and $\{\bar{v}_{1,i}, \ldots, \bar{v}_{m,i}\} \subset V_{3-p}$ for $p = 1$ or $p = 2$.

(c) If we delete any vertex c_j (c'_j), $j \in [m]$ from the set V_i which contains c_j (c'_j), then the resulting semicomplete digraph $D\langle V_i - c_j\rangle$ ($D\langle V_i - c'_j\rangle$) has no (c_p, c_q)-path when $m+1 \geq p > j > q \geq 0$.

Suppose that $\phi : \{x_1, \ldots, x_n\} \to \{T, F\}^n$ is a truth assignment such that each clause C_j, $j \in [m]$ has at least one true literal and at least one false literal. Define the 2-partition (V_1, V_2) so that V_1 consists of precisely the vertices $\{c_0, c_1, \ldots, c_m, c_{m+1}\}$ and all those literal vertices which correspond to true literals. Because of the arcs we reversed when going from D' to D we have that $D\langle V_1\rangle$ contains a cycle $H = c_{m+1}c_m u_{m,j_m} c_{m-1} u_{m-1,j_{m-1}} \ldots c_1 u_{1,j_1} c_0 c_{m+1}$, where u_{j_q} is one of the vertices corresponding to a true literal of C_j, $j \in [m]$. If C_1 contains two true literals, then let u_{i,j'_1} be the other and add the path $c_1 u_{1,j'_1} c_0$ to H. Because c_0 dominates all vertices of $V_1 - V(H)$ and c_{m+1} is dominated by all of these, we see that $D\langle V_1\rangle$ is strong. A similar argument shows that $D\langle V_2\rangle$ is strong (because at least one literal of each clause is false under ϕ).

Suppose now that (V_1, V_2) is a 2-partition of $V(D)$ such that $D\langle V_i\rangle$ is a strong tournament for $i \in [2]$. By (a) we can assume w.l.o.g. that $\{c_0, c_1, \ldots, c_m, c_{m+1}\} \subset V_1$ and $\{c'_0, c'_1, \ldots, c'_m, c'_{m+1}\} \subset V_2$. Now (c) implies that for each $j \in [m]$ V_1 must contain at least one and at most two of the vertices corresponding to the literals of C_j. Thus if we construct a truth assignment where variable x_i is true if and only if all the vertices (by (b)) $\{v_{1,i}, \ldots, v_{m,i}\}$ are in V_1, then we obtain a truth assignment which satisfies at least one literal per clause and also has at least one false literal per clause. Thus $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.

□

7 Remarks and open problems

In this paper we have considered 2-partition problems on semicomplete digraphs. These are also the digraphs of independence number[4] $\alpha = 1$. It is well-known and easy to show that for digraphs with bounded independence number $\alpha \leq r$ we also have that the number of vertices of in-, out- or semi-degree at most k is bounded by a function $g(k, r)$. In particular, it follows from Turan's theorem that in a digraph with independence number at most α there are at most $\alpha(2k + 1)$ vertices of out-degree at most k. Furthermore, it is easy to check that if a digraph D with independence number at most r has a cycle C, then $D\langle V(C)\rangle$ contains a cycle of length at most $2r + 1$. Using these observations it is not hard to see that we can extend Theorem 3.4 to the following. We leave the details to the interested reader.

Theorem 7.1. *For every choice of positive integers r, k_1, k_2 there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition of a given digraph with independence number at most r or correctly outputs that none exist.*

We cannot directly extend our proof of Theorem 5.3 to digraphs of bounded independence number: in the proof of Lemma 5.1 we use the fact that if we take any set of $k + 1$ vertices from Z, the vertex v will dominate at least two of these. The corresponding property does not necessarily hold even if we take a set of some $f(k)$ vertices from Z when we have independence number at most α.

Conjecture 7.2. *For every choice of positive integers r, k there exists a polynomial algorithm that either constructs a $(\delta^+ \geq 1, \delta^- \geq k)$-partition of a given digraph with independence number at most r or correctly outputs that none exist.*

Problem 7.3. *Determine the complexity every pair of fixed integers $r, k_1, k_2 \geq 1$ of deciding the following for a given digraph D with independence number r:*

[4] The independence number α denotes the maximum cardinality of set of vertices such that there are no arcs between vertices in the set

- *whether D has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-partition,*
- *whether D has a $(\delta^+ \geq k_1, \delta^0 \geq k_2)$-partition,*
- *whether D has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition*

Kühn et al. proved the following result about 2-partitions and tournaments into highly connected tournaments. We formulate it for semicomplete digraphs, since every $3r-2$ strong semicomplete digraph contains an r-strong spanning tournament, see e.g. [3, Theorem 11.10.4].

Theorem 7.4. *[8] There exists a constant c such that every ck^7-strong semicomplete digraph S has a 2-partition (V_1, V_2) such that $S\langle V_i\rangle$ is k-strong for $i = 1, 2$.*

Instead of demanding high strong connectivity inside each set of the partition (V_1, V_2) we may also ask for a partition of a semicomplete digraph S into strongly connected semicomplete digraphs S_1, S_2, each of which have out-degree at least a specified number k_i, $i = 1, 2$. Note that, since every strong semicomplete digraph is hamiltonian, when $k_1 = k_2 = 1$ we just ask for a pair of complementary cycles.

Problem 7.5. *Does there exist a function $g(k_1, k_2)$ such that every strong semicomplete digraph S with $\delta^+(S) \geq g(k_1, k_2)$ has a 2-partition (V_1, V_2) such that $S\langle V_i\rangle$ is strong and $\delta^+(S\langle V_i\rangle) \geq k_i$ for $i = 1, 2$?*

It was shown in [9] that every tournament with minimum out-degree at least 3 has a 2-partition into two strong tournaments. As every mixed graph has an orientation whose out-degree at every vertex is at least half of the original out-degree, this implies that $g(1, 1)$ exists (it is at most 6). The problem is open for all values $k_1, k_2 \geq 1$ with $k_1 + k_2 \geq 3$.

The following related result of [2] shows that the bounds in Theorems 1.4 and 1.5 are far from being best possible when k becomes large.

Theorem 7.6. *There exists an absolute constant c_1 so that every semicomplete digraph S with minimum out-degree at least $2k + c_1\sqrt{k}$ has a 2-partition (V_1, V_2) so that $\delta^+(S\langle V_i\rangle) \geq k$ for $i = 1, 2$.*

Theorem 7.7. *There exists an absolute constant c_2 so that every semicomplete digraph S with minimum semi-degree at least $2k + c_2\sqrt{k}$ has a 2-partition (V_1, V_2) so that $\delta^0(S\langle V_i\rangle) \geq k$ for $i = 1, 2$.*

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