Competitive Analysis of Multi-Objective Online Algorithms

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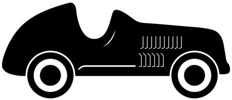
Who gets your antique car?



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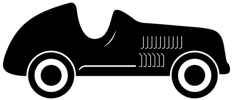
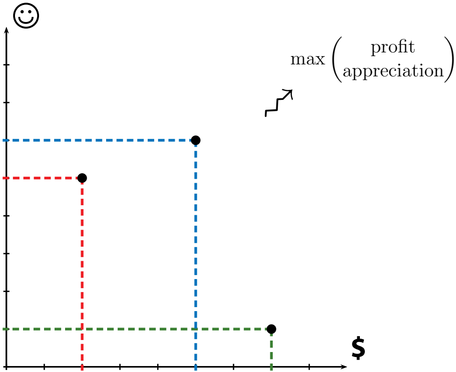




Online Optimization

In online optimization, an algorithm has to make decisions based on a sequence of incoming bits of information without knowledge of future inputs.

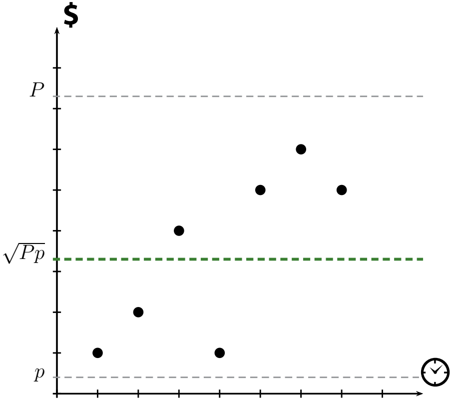
Competitive Analysis

- An algorithm $\textsc{alg}$ is called *c-competitive*, if for all sequences σ

$$\textsc{alg}(\sigma) \geq \frac{1}{c} \cdot \textsc{opt}(\sigma) + \alpha.$$

- The infimum over all values c such that $\textsc{alg}$ is c-competitive is called *the competitive ratio* of $\textsc{alg}$.
- An algorithm is called *competitive* if it attains a “constant” competitive ratio.

Online Optimization (contd.)



for $t = 1, \ldots, T$ **do**

Accept a price p_t if

$$p_t \geq \sqrt{Pp}$$

end

$\rightsquigarrow$ ALG is $\sqrt{\frac{P}{p}}$-competitive.

Multi-Objective Optimization

Consider a multi-objective optimization problem $\mathcal{P}$ for a given feasible set $\mathcal{X} \subseteq \mathbb{R}^n$, and objective vector $f : \mathcal{X} \mapsto \mathbb{R}^k$:

$$\begin{aligned} \mathcal{P} \quad & \max f(x) \\ & \text{s.t. } x \in \mathcal{X} \end{aligned}$$

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Efficient Solutions

- A feasible solution $\hat{x} \in \mathcal{X}$ is called *efficient* if there is no other $x \in \mathcal{X}$ such that $f(x) \preceq f(\hat{x})$, where $\preceq$ denotes a componentwise order, i.e., for $x, y \in \mathbb{R}^n$, $x \preceq y :\Leftrightarrow x_i \leq y_i$, for $i = 1, \ldots, n$, and $x \neq y$.

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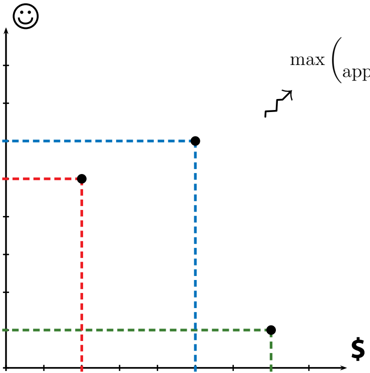
- If $\hat{x}$ is an efficient solution, $f(\hat{x})$ is called non-dominated point.

Multi-Objective Optimization (contd.)



$$\max \begin{pmatrix} \text{profit} \\ \text{appreciation} \end{pmatrix}$$

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  - feasible solution $\textsc{alg}[I] \in \mathcal{X}(I)$
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- algorithm $\textsc{alg}$
  - feasible solution $\textsc{alg}[I] \in \mathcal{X}(I)$
  - associated objective $\textsc{alg}(I) = f(I, \textsc{alg}[I])$
- optimal algorithm $\textsc{opt}$
  - $\textsc{opt}[I] = \{\mathbf{x} \in \mathcal{X}(I) \,|\, \mathbf{x} \text{ is an efficient solution to } \mathcal{P}\}$
  - objective associated with $\mathbf{x} \in \textsc{opt}[I]$ is denoted by $\textsc{opt}(\mathbf{x})$

Multi-Objective Approximation Algorithms

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$$f_i(x') \leq \rho \cdot f_i(x) \text{ for } i = 1, \ldots, n$$

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ρ-approximation of a set of efficient solutions

for every feasible solution x, X' contains a feasible solution x' that is a ρ-approximation of x

Multi-Objective Online Algorithms

c-competitive

A multi-objective online algorithm ALG is *c-competitive* if for all finite input sequences I there exists an efficient solution $\mathbf{x} \in \text{OPT}[I]$ such that $ALG(I) \leqq c \cdot \text{OPT}(\mathbf{x}) + \alpha$, where $\alpha \in \mathbb{R}^n$ is a constant vector independent of I.

Multi-Objective Online Algorithms

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strongly c-competitive

A multi-objective online algorithm $\textsc{alg}$ is *strongly c-competitive* if for all finite input sequences I and all efficient solutions $\mathbf{x} \in \textsc{opt}[I]$, $\textsc{alg}(I) \leqq c \cdot \textsc{opt}(\mathbf{x}) + \alpha$, where $\alpha \in \mathbb{R}^n$ is a constant vector independent of I.

Bi-Objective Online Search

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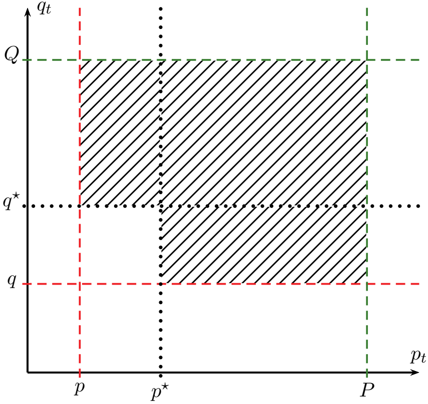
Reservation Price Policy



for $t = 1, \dots$ **do**

Accept a request r_t if

$$p_t \geq p^\star \textbf{ or } q_t \geq q^\star$$

end

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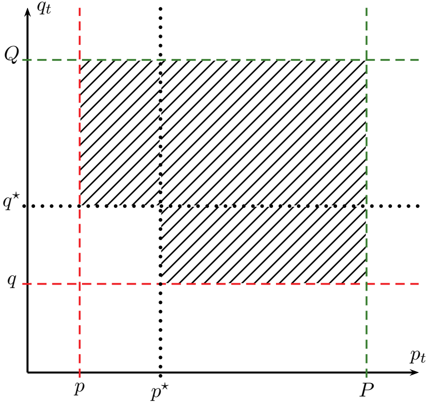

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$$c = \max\left\{\frac{P}{p}, \frac{Q}{q^\star}, \frac{P}{p^\star}, \frac{Q}{q}\right\}$$

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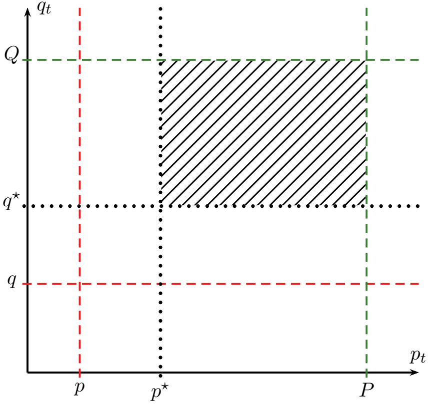

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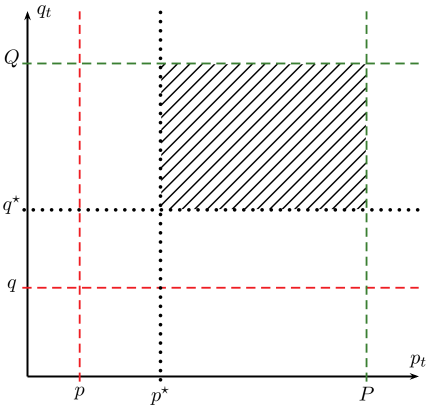

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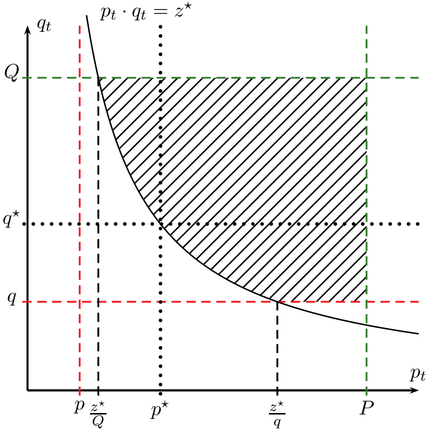

for $t = 1, \dots$ **do**

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$$p_t \cdot q_t \ge z^\star$$

end

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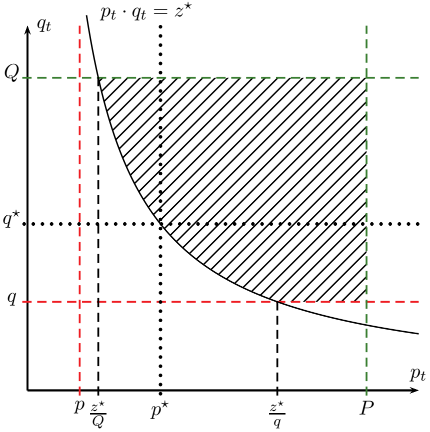

for $t = 1, \ldots$ **do**
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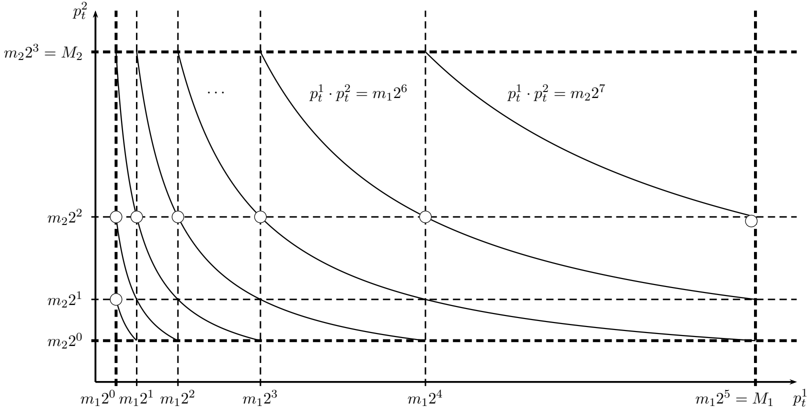
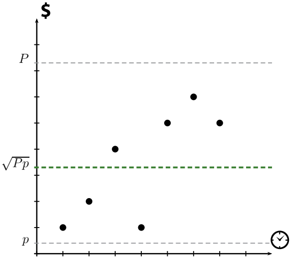
end

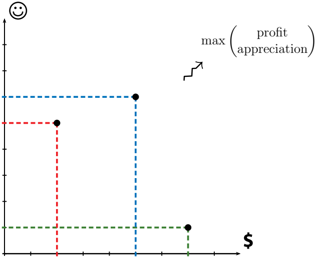
$$c = \sqrt{\frac{PQ}{pq}}$$

Randomization



Conclusion & Further Research





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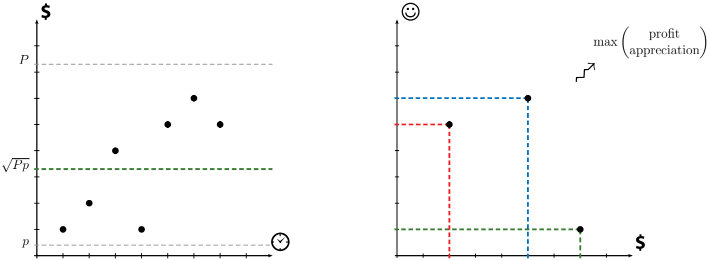

- application to multi-objective versions of classical online problems
- relations between single- and multi-objective online optimization
- alternative concepts

The Multi-Objective k-Canadian Traveller Problem